MULTIPRESSURE REGULARIZATION FOR MULTIPHASE FLOW

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Received 3 October 1984

The standard theory of ideal single-pressure multiphase fluid dynamics, which is known to be ill-posed, is regularized via the hamiltonian formalism by extending the noncanonical Poisson brackets for the standard single-pressure equations to the case of multiple pressures. This formalism is used to find Lyapunov stability conditions for the regularized system.

1. Introduction. Multiphase flow involves interpenetration of various material species. Hydrodynamic models describing such systems by using multiple velocity and density fields at a single, common pressure are known to be ill-posed and possess various types of instabilities [1]. The ultimate cause for these difficulties can be traced to the assumption of equal pressures for all the different species and phases. We propose to regularize this theory by dispensing with the constraint of common pressure, introducing multiple pressures, as well as material quantities associated with interface dynamics and inertia.

The idea of regularizing multiphase flow by introduction of additional pressures is not new. However, our approach and results obtained by reasoning via the hamiltonian formalism differ from others, which introduce, e.g., viscous dissipation [2], numerical filtering [3], surface tension [4], bubble inertia [5], or phenomenological interfacial pressure jumps [6]. See also ref. [7] for a recent review.

2. The single pressure model. The single-pressure model of ideal multiphase flow in R^n is described by the equations [7]

$$\partial_t \bar{\rho}^s + \operatorname{div} \bar{\rho}^s \mathbf{v}^s = 0\,, \quad \partial_t v_i^s + v_j^s v_{i,j} = -\theta^s (\bar{\rho}^s)^{-1} P_{,i} - \Phi_{,i}\,; \qquad \partial_t \eta^s + \mathbf{v}^s \cdot \nabla \eta^s = 0\,, \tag{1, 2}$$

where $\bar{\rho}^s = \rho^s \theta^s$ is the macroscopic density of the sth species, ρ^s is its microscopic density, θ^s is its volume fraction, $\mathbf{v}^s$ is its velocity, η^s is its specific entropy, P is the pressure, Φ is the potential of an external field, summation is assumed over all repeated indices except the species label s. In terms of the momentum density $\mathbf{M}^s = \bar{\rho}^s \mathbf{v}^s$, the motion equation (1b) can be written as

$$\partial_t \mathbf{M}^s + (\mathbf{M}^s M_j^s / \bar{\rho}^s)_{,j} = -\theta^s \nabla P - \bar{\rho}^s \nabla \Phi\,. \tag{3}$$

The variables θ^s, $s = 1, \ldots, N$, are considered as given functions of $\{\bar{\rho}^s, \eta^s\}$ through the relations $\Sigma_{s=1}^N \theta^s = 1$, $P^1(\bar{\rho}^1/\theta^1, \eta^1) = \ldots = P^N(\bar{\rho}^N/\theta^N, \eta^N) = P$ where $P^s = (\rho^s)^2 \partial e^s/\partial \rho^s$, with $e^s = e^s(\rho^s, \eta^s)$ being the specific internal energy of the sth species.

Eqs. (1)–(3) can be written in the hamiltonian form $\partial_t F = \{H_1, F\}_1$ with $F \in \{\bar{\rho}^s, \eta^s, \mathbf{M}^s\}$ and Poisson bracket $\{\ ,\ \}_1$ given in terms of these variables by [8]

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(North-Holland Physics Publishing Division)

$$\{J, I\}_1 = -\sum_s \int \mathrm{d}^n x \left[\frac{\partial I}{\delta \bar{\rho}^s} \partial_j \bar{\rho}^s \frac{\delta J}{\delta M_j^s} + \frac{\partial I}{\delta \eta^s} \eta^s_{,j} \frac{\delta J}{\partial M_j^s} + \frac{\delta I}{\delta M_i^s} \left(\bar{\rho}^s \partial_i \frac{\delta J}{\delta \bar{\rho}^s} - \eta^s_{,i} \frac{\delta J}{\delta \eta^s} + (\partial_j M_i^s + M_j^s \partial_i) \frac{\delta J}{\delta M_j^s} \right) \right], \tag{4}$$

and the hamiltonian being the total energy, H_1,

$$H_1 = \sum_s \int \mathrm{d}^n x [|M^s|^2/2\bar{\rho}^s + \bar{\rho}^s e^s + \bar{\rho}^s \Phi(x)] . \tag{5}$$

Although this model is hamiltonian, there are two difficulties associated with it. First, eqs. (1), (2) are well known to be ill-posed even in the simplest case $n = 1, N = 2$, where the equations are not hyperbolic since they possess complex-valued characteristics [1,4,6]. Second, for arbitrary n and N, the second variation of functions whose extremal points are stationary flows of (1), (2) is indefinite due to the presence of the single pressure and, thus, the Lyapunov stability of stationary flows is prevented [8].

Both these difficulties with the single-pressure model can be overcome at once by allowing multiple pressures within the framework of the hamiltonian formalism.

3. Multipressure model. The following multipressure model provides a regularization of the corresponding single-pressure equations and is hyperbolic when specialized to the $n = 1, N = 2$ case.

$$\partial_t \bar{\rho}^s + \operatorname{div} \bar{\rho}^s \mathbf{v}^s = 0 , \quad \partial_t \eta^s + \mathbf{v}^s \cdot \nabla \eta^s = 0 ; \quad \partial_t \mathbf{v}^s + (\mathbf{v}^s \cdot \nabla)\mathbf{v}^s = -\theta^s (\bar{\rho}^s)^{-1} \nabla P^s - \nabla \Phi , \tag{6,7}$$

$$\partial_t \theta^s + \mathbf{w} \cdot \nabla \theta^s = 0 ; \quad \partial_t \mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{w} = -\sigma^{-1} \sum_s P^s \nabla \theta^s ; \quad \partial_t \sigma + \operatorname{div} \sigma \mathbf{w} = 0 , \tag{8,9,10}$$

where $\bar{\rho}^s$, η^s, $\mathbf{v}^s \theta^s$, P^s, Φ, are the same as in the previous section, while $\mathbf{w}$ is an effective interface velocity and σ is an associated interface mass density. For the two-species case, eqs. (8)–(10) become, using $\theta^1 = \theta$, $\theta^2 = 1 - \theta$,

$$\partial_t \theta + \mathbf{w} \cdot \nabla \theta = 0 ; \quad \partial_t \mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{w} = \sigma^{-1}(P^2 - P^1) \nabla \theta ; \quad \partial_t \sigma + \operatorname{div} \sigma \mathbf{w} = 0 . \tag{8',9',10'}$$

For separated flow, eq. (8′) describes transport of volume fraction θ by the interface with velocity $\mathbf{w}$, whose acceleration is given in (9′) in the form of Newton's law, with an inertial mass density σ, which by (10′) is also transported (as a density) by the interface.

In the case where the first $N - 1$ species, say, are dispersed in species number N, eq. (9) written as

$$\partial_t \mathbf{w} + (\mathbf{w} \cdot \nabla)\mathbf{w} = \sigma^{-1} \sum_{s=1}^{N-1} (P^N - P^s) \nabla \theta^s , \tag{9''}$$

has the form of the newtonian force law for acceleration of the interface. Eqs. (6)–(10) comprise a hamiltonian system with Poisson bracket

$$\{J, I\}_2 = \{J, I\}_1 - \int \mathrm{d}^n x \left[\frac{\delta I}{\delta \sigma} \partial_j \sigma \frac{\delta J}{\delta M_j} + \frac{\delta I}{\delta M_i} \left(\sigma \partial_i \frac{\delta J}{\delta \sigma} - \sum_s \theta^s_{,i} \frac{\delta J}{\delta \theta^s} + (\partial_j M_i + M_j \partial_i) \frac{\delta J}{\delta M_j} \right) + \sum_s \frac{\delta I}{\delta \theta^s} \theta^s_{,i} \frac{\delta J}{\delta M_i} \right] \tag{11}$$

and hamiltonian

$$H_2 = H_1 + \int \mathrm{d}^n x \, |M|^2/2\sigma , \tag{12}$$

where $M = \sigma \mathbf{w}$ is the effective momentum density of the interface. The Lie algebriac interpretation of both Poisson brackets (4) and (11) can be found in ref. [9].

Conserved quantities for the system (6)–(10) in three dimensions are

$$C^s = \int \mathrm{d}^3 x \, \bar{\rho}^s F^s(\eta^s, q^s) , \tag{13}$$

where

$$q^s = (\bar{\rho}^s)^{-1} \operatorname{curl} \mathbf{v}^s \cdot \nabla \eta^s \tag{14}$$

is the potential vorticity for the sth species and F^s is an arbitrary function.

Additional conserved quantities of the same form are

$$K^s = \int d^3x \, \sigma G^s(Q^s, \theta^s) , \tag{15}$$

where

$$Q^s = \sigma^{-1} \operatorname{curl} \mathbf{w} \cdot \nabla \theta^s \tag{16}$$

satisfies

$$\partial_t Q^s + \mathbf{w} \cdot \nabla Q^s = 0 , \tag{17}$$

and G^s are arbitrary functions.

One can show by the methods of refs. [10–14], that stationary flows $\{\rho_e^s, \eta_e^s, M_e^s, \theta_e^s, w_e, \sigma_e\}$ of the system (6)–(10) are extremal points of the sum $H_{C,K} = H_2 + \Sigma_s (C^s + K^s)$, i.e., $\delta H_{C,K} = 0$ for stationary flows. An analysis of the second variation $\delta^2 H_{C,K}$ evaluated at the external point shows that $\delta^2 H_{C,K}$ can be positive definite, as required for Lyapunov stability, provided hamiltonian H_2 is modified by the addition of a term $\int d^n x \, \epsilon(\sigma)$ representing energy of compressibility in σ, i.e.,

$$H_3 = H_1 + \int d^n x [|M|^2/2\sigma + \epsilon(\sigma)] . \tag{19}$$

Addition of this term modifies the $\mathbf{w}$ equation to

$$\partial_t \mathbf{w} + (\mathbf{w} \cdot \nabla) \mathbf{w} = -\sigma_s^{-1} \sum P^s \nabla \theta^s - \epsilon''(\sigma) \nabla \sigma , \tag{9'''}$$

but does not change the conservation laws (13) and (15) since these are properties of the Poisson bracket (11).

For the case of two species, $s = 1, 2$, $\theta^1 = \theta$, $\theta^2 = 1 - \theta$, the stationary flows of the new system (6)–(10) with (9) replaced by (9″) are extremal points of $H_C = H_3 + \Sigma_{s=1}^2 C^s + K$.

The second variation of H_C at equilibrium, $\delta^2 H_C$, is given by the quadratic form

$$\begin{aligned} 2\delta^2 H_C = \int d^3x \, &[\sigma_e |\delta \mathbf{w} + \sigma_e^{-1} \mathbf{w}_e \delta\sigma|^2 + (\epsilon''(\sigma_e) - \sigma_e^{-1} |\mathbf{w}_e|^2)(\delta\sigma)^2 \\ &+ 2G_\theta \delta\sigma\delta\theta + 2G_Q \operatorname{curl} \delta\mathbf{w} \cdot \nabla\delta\theta + 2\sigma_e G_{Q\theta} \delta\theta\delta Q + \sigma_e G_{\theta\theta}(\delta\theta)^2 + \sigma_e G_{QQ}(\delta Q)^2] \\ &+ \sum_s \int d^3x \, \{\bar{\rho}_e^s |\delta \mathbf{v}^s + (\bar{\rho}_e^s)^{-1} \mathbf{v}_e^s \delta\bar{\rho}^s|^2 - (\bar{\rho}_e^s)^{-1} |\mathbf{v}_e^s|^2 (\delta\bar{\rho}^s)^2 + \bar{\rho}^s(\beta^s)^2 [\delta(\bar{\rho}^s/\theta^s) + (\beta_e^s)^{-2} e_{\rho\eta}^s \delta\eta^s]^2 \\ &+ 2(e_\eta^s + F_\eta^s)\delta\bar{\rho}^s \delta\eta^s + 2F_q^s \operatorname{curl} \delta\mathbf{v}^s \cdot \nabla\delta\eta^s + 2\bar{\rho}_e^s F_{\eta q}^s \delta\eta^s \delta q^s + \bar{\rho}_e^s [e_{\eta\eta}^s - (e_{\rho\eta}^s/\beta^s)^2 + F_{\eta\eta}^s](\delta\eta^s)^2 \\ &+ \bar{\rho}_e^s F_{qq}^s (\delta q^s)^2\} , \end{aligned} \tag{20}$$

where $\delta(\bar{\rho}^s/\theta^s) = (\theta_e^s)^{-1}\delta\bar{\rho}^s - (\theta_e^s)^{-2}\bar{\rho}_e^s \delta\theta^s$, $\bar{\rho}_e^s \delta q = -q_e^s \delta\bar{\rho}^s + \operatorname{curl} \delta\mathbf{v}^s \cdot \nabla\eta_e^s + \operatorname{curl} \mathbf{v}_e^s \cdot \nabla\delta\eta^s$, with an analogous expression for $\sigma_e \delta Q$, and where, throughout, G, F^s, and their derivatives G_θ etc., are to be evaluated at equilibrium, and $(\beta^s)^2 = (\theta_e^s C_e^s/\bar{\rho}_e^s)^2$, with $(C_e^s)^2 = \partial P_e^s/\partial(\bar{\rho}_e^s/\theta_e^s)$ being the square of the sound speed for the sth species. A given flow characterized by the functions F^s and G will be linearly Lyapunov stable in the class of smooth solutions, provided F^s and G satisfy the conditions required for $\delta^2 H_C$ to be positive definite. The quadratic form (20) reduces formally to that for adiabatic flow of a single fluid in three dimensions when $\mathbf{v}^1 = \mathbf{v}^2 = \mathbf{v}$, $\theta^1 = \theta^2 = \frac{1}{2}$, $\bar{\rho}^1 = \bar{\rho}^2$

$= \frac{1}{2}\rho$, $\delta\theta = \delta w = \delta\sigma = 0$, and w, σ, Q, are absent. In that case, the conditions for Lyapunov stability are given in ref. [14].

A characteristic analysis of the two-species, one-dimensional case of the system (6)–(10) shows that this system is *hyperbolic* provided $\epsilon''(\sigma) > 0$.

Remark. In the multispecies case with $N > 2$, the notion of interface velocity $\boldsymbol{w}$ may be generalized to allow for motion involving any pair of adjacent materials, by introducing $N(N-1)/2$ interface velocities $\boldsymbol{w}^{\alpha\beta} = \boldsymbol{w}^{\beta\alpha}$, $\alpha \neq \beta$, where α and β take values 1, 2, …, N, corresponding $\sigma^{\alpha\beta} = \sigma^{\beta\alpha}$ being the mass densities of the interfaces, and variables $\phi^{\alpha\beta} = -\phi^{\beta\alpha}$, such that $\theta^s - 1/N = \Sigma_\alpha \phi^{s\alpha}$. The motion equations for the new system

$$\partial_t \bar{\rho}^s + \operatorname{div} \bar{\rho}^s \boldsymbol{v}^s = 0\,, \quad \partial_t \eta^s + \boldsymbol{v}^s \cdot \nabla \eta^s = 0\,, \quad \partial_t \boldsymbol{v}^s + (\boldsymbol{v}^s \cdot \nabla)\boldsymbol{v}^s = (\bar{\rho}^s)^{-1}\left(\frac{1}{N} + \sum_\alpha \phi^{s\alpha}\right)\nabla P^s\,, \tag{21}$$

$$\partial_t \sigma^{\alpha\beta} + \operatorname{div} \sigma^{\alpha\beta} \boldsymbol{w}^{\alpha\beta} = 0\,, \quad \partial_t \phi^{\alpha\beta} + \boldsymbol{w}^{\alpha\beta} \cdot \nabla \phi^{\alpha\beta} = 0\,, \quad \partial_t \boldsymbol{w}^{\alpha\beta} + (\boldsymbol{w}^{\alpha\beta} \cdot \nabla)\boldsymbol{w}^{\alpha\beta} = (P^\beta - P^\alpha)\nabla\phi^{\alpha\beta} - \epsilon''^{\alpha\beta}(\sigma\alpha^\beta)\nabla\sigma^{\alpha\beta}\,,$$

form a hamiltonian system with Poisson bracket

$$\{J, I\}_3 = \{J, I\}_1 - \sum_{\alpha<\beta} \int \mathrm{d}^n x \left[\frac{\delta I}{\delta\sigma^{\alpha\beta}} \partial_j \sigma^{\alpha\beta} \frac{\delta J}{\delta M_j^{\alpha\beta}} + \frac{\delta I}{\delta\phi^{\alpha\beta}} \phi^{\alpha\beta}_{,i} \frac{\delta J}{\delta M_i^{\alpha\beta}} \right.$$
$$\left. + \frac{\delta I}{\delta M_i^{\alpha\beta}} \left(\sigma^{\alpha\beta}\partial_i \frac{\delta J}{\delta\sigma^{\alpha\beta}} - \phi^{\alpha\beta}_{,i} \frac{\delta J}{\delta\phi^{\alpha\beta}} + (\partial_j M_i^{\alpha\beta} + M_j^{\alpha\beta}\partial_i) \frac{\delta J}{\delta M_j^{\alpha\beta}}\right)\right],$$

and with hamiltonian

$$H_4 = H_1 + \sum_{\alpha<\beta} \int \mathrm{d}^n x \left[|M^{\alpha\beta}|^2/2\sigma^{\alpha\beta} + \epsilon^{\alpha\beta}(\sigma^{\alpha\beta})\right],$$

where $\boldsymbol{M}^{\alpha\beta} = \boldsymbol{w}^{\alpha\beta}\sigma^{\alpha\beta}$ (no sum on α, β) and $\epsilon^{\alpha\beta}$ is the internal energy density of the $\alpha\beta$-interface.

For the $N = 2$ species case, both systems (21) and (6)–(10) reduce to the same set of equations when σ^{12} is identified with σ^{12}, ϕ with $\frac{1}{2}(\theta^1 - \theta^2)$, and $\boldsymbol{M}^{12}$ with $\sigma\boldsymbol{w}$.

We are very grateful to J.M. Hyman and B.B. Wendroff for helpful discussions of multiphase flow. This work is partly supported by DOE and NSF.

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