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Location Algorithms and Errors in Time-Of-Arrival Systems

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LOCATION ALGORITHMS AND ERRORS IN TIME-OF-ARRIVAL SYSTEMS

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Abstract

This report describes least squares solution methods and linearized estimates of solution errors caused by data errors. These methods are applied to event locating systems which use time-of-arrival (TOA) data. Analyses are presented for algorithms that use the TOA data in a "direct" manner and for algorithms utilizing Time-of-arrival Squared (TSQ) methods. Location and error estimation results were applied to a "typical" satellite TOA detecting system. Using Monte Carlo methods, it was found that the linearized location error estimates were valid for random data errors with relatively large variances and relatively poor event/sensor geometries.

In addition to least squares methods, which use an L_2 norm, methods were described for L_1 and L_∞ norms. In general, these latter norms offered little improvement over least squares methods.

Reduction of the location error variances can be effected by using information in addition to the TOA data themselves by adding judiciously chosen "conditioning" equation(s) to the least squares system. However, the added information can adversely affect the mean errors. Also, conditioned systems may offer location solutions where nonconditioned scenarios may not be solvable. Solution methods and linearized error estimates are given for "conditioned" systems. It was found that for significant data errors, the linearized estimates were also close to the Monte Carlo results.

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1.0 INTRODUCTION

An event at an unknown location generates a pulse of energy at an unknown time. The pulse radiates outward and is detected by a set of sensors, each sensor at a known location. Each sensor estimates the time of arrival (TOA) of the pulse at the sensor. If the velocity of the pulse to each sensor is known, it is possible to estimate the location of the event and its time of occurrence, since the distance from the event to each sensor is proportional to the time difference from the event time to the time measured at the sensor. Data from a minimum of four sensors are required to estimate the (three-space) event location and time. This report discusses various methods for finding the event location and the errors in the event location and time associated with the data errors.

In general, the TOA data are subject to errors, so any scheme to estimate the event location and time will produce errors. The location and time errors depend on the TOA data errors and also on the geometric configuration of the sensors and the event. If, for example, the event and all sensors are coplanar, the event cannot even be located unless it is known that the event was coplanar with the sensors. We will discuss the properties of these location errors for various location algorithms and event/sensor configurations.

A very common and useful method for location estimation is by the method of least squares. A general discussion of this method is given in Section 2. A discussion of errors in general least squares solutions appears in Section 3. The application of various least squares schemes to location estimation in TOA systems and their associated error estimates are given in Section 4. Section 5 contains numerical results for location and time errors for specific “realistic” event/sensor geometries, data errors, and location algorithms. A Linear Programming approach is discussed in Section 6, and “conditioned” least squares methods are examined in Section 7. In the context of this report, “conditioning” means using some information external to the data measurements themselves to improve the location accuracy. A summary of the report is in Section 8. Most of the analyses in Sections 2, 3, and 4 is discussed in references [1] through [4].

We use **bold** for matrices and vectors. Unless otherwise noted, vectors will be column vectors. Upper-case T denotes vector or matrix transpose. Expectation is denoted by E{…}.

2.0 THE METHOD OF LEAST SQUARES

The method of least squares is usually attributed to Gauss, who towards the end of the 18th century used, or suggested the use of, the method to estimate celestial orbits from measurements subject to errors. The basic idea is to create an estimate that minimizes the sum of the squares of the data errors. In addition to references [1] through [4], discussions of the method may be found in Hamming [5], Papoulis [6], and Lawson and Hanson [7]. Least squares uses an L_2 norm in that it minimizes the sum of squares of data errors. It has the advantage over other norms in that it succumbs relatively easily to analytic solutions. It has the disadvantage, compared to the L_1 norm, in that it tends to exaggerate large data errors (see pages 225 and 226 of Hamming [5] for a brief but lucid discussion of this phenomenon).

Suppose we have a set of M unknowns, $\mathbf{x} = (x_1, x_2, ..., x_M)^T$, and a set of N independent measurements (or data), $\mathbf{d} = (d_1, d_2, ..., d_N)^T$, $M \le N$. If there are fewer measurements than unknowns, the unknowns cannot be uniquely found unless other information is provided. We assume that the unknowns are related to the data by a set of N independent "model" equations, $\mathbf{f}(\mathbf{x}, \mathbf{d}) = (f_1, f_2, ..., f_N)^T$. Also, we choose a positive definite, symmetric weighting matrix **W**, which in some sense takes into account our estimates of the accuracy of the measurements. The least squares solution is the value of **x** that minimizes the scalar

$$s = \mathbf{f}(\mathbf{x}, \mathbf{d})^T \mathbf{W} \mathbf{f}(\mathbf{x}, \mathbf{d}). \tag{2.1}$$

The functions **f** may be linear or nonlinear, or any mix of the two.

2.1 Least Squares Solutions

In general, we assume that **f** is differentiable with respect to both **x** and **d**. Minimization of equation (2.1) is usually accomplished by observing that the necessary condition for a minimum of s occurs where the partial derivatives of s with respect to **x**, $\partial s / \partial \mathbf{x}$, are zero. This method leads to M equations in M unknowns. If $M \le N$, the system is said to be "overdetermined." In this case, some or all of the model equations may not have value zero at the solution values. In fact, if there are errors in the data, the model equations need not be satisfied exactly.

Define $\mathbf{A} = \partial \mathbf{f} / \partial \mathbf{x}$. Then the "best" solution in the least squares sense, is the value of **x** that satisfies the equation

$$\mathbf{A}^T \mathbf{W} \mathbf{f} = \mathbf{0}. \tag{2.2}$$

These equations are called "normal equations" and consist of M equations in M unknowns. If all the elements of **f** are linear in **x**, then the normal equations are also linear in **x**, and the "best" solution, if it exists, is the solution of a system of M *linear* equations in M unknowns. Assume that that **f** is linear in x and that $\mathbf{f} = \mathbf{Ax} + \mathbf{c}$, where **A** and **c** are constant. Then

$$\mathbf{A}^T \mathbf{Wf} = \mathbf{A}^T \mathbf{W}(\mathbf{Ax} + \mathbf{c}) = 0. \tag{2.3}$$

The solution of equation (2.3) is

$$\mathbf{x} = -\mathbf{B}^{-1} \mathbf{A}^T \mathbf{Wc}, \tag{2.4}$$

where

$$\mathbf{B} = \mathbf{A}^T \mathbf{WA}. \tag{2.5}$$

Note that since **W** is symmetric, **B** is symmetric, as well as $\mathbf{B}^{-1}$.

If **f** is nonlinear in **x**, other methods must be used to solve equation (2.2).

2.2 Newton's Method

If the normal equations are nonlinear, as is the case with all our location algorithms, they may be solved by Newton's method, which is an iterative scheme. Let $\mathbf{x}^i$ be the i-th estimate of **x**, where $\mathbf{x}^0$ is an initial estimate. Define $\mathbf{A}^i \triangleq \partial \mathbf{f} / \partial \mathbf{x} |_{\mathbf{x}=\mathbf{x}^i}$. Then the successive estimates of **x** are

$$\mathbf{x}^{i+1} = \mathbf{x}^i - \mathbf{B}^{i^{-1}} \mathbf{A}^{iT} \mathbf{Wf}(\mathbf{x}^i, \mathbf{d}), \mathbf{B}^i = \mathbf{A}^{iT} \mathbf{WA}^i. \tag{2.6}$$

The iteration ends when

$$| \mathbf{x}^{i+1} - \mathbf{x}^i | \leq \varepsilon > 0, \tag{2.7}$$

where ε is a preassigned positive number. Note that this method requires the choice of an initial estimate of $\mathbf{x}$ and a convergence criterion, ε. If the initial guess is "poor" or if the data errors are "large" or if $\mathbf{B}^i$ is "ill conditioned," the method may not converge. By "ill conditioned," we mean that the elements of $\mathbf{B}^{i^{-1}}$ are very sensitive to minor changes in the elements of $\mathbf{B}^i$. In fact, for example, if the rows of $\mathbf{B}^i$ are linearly dependent, its inverse does not even exist!

Another difficulty with iterative methods is that the result may depend on the initial guess. The solution may go to a local, rather than a global, minimum. Also, because of data errors and problem condition, the "correct" solution may be at a local minimum, and the global solution may not be the desired one.

3.0 ERRORS IN LEAST SQUARES

In general, there are errors in the data that induce errors in the unknowns. Let the actual data be $\mathbf{d} = \mathbf{d}_o + \mathbf{d}_e$, where $\mathbf{d}_o = (d_{o1},...d_{oN})^T$ is the “exact” data measurements and $\mathbf{d}_e = (d_{e1},...,d_{eN})^T$ is the vector of data errors. Let the solution be $\mathbf{x} = \mathbf{x}_o + \mathbf{x}_e$, where the exact result is $\mathbf{x}_o = (x_{o1},...,x_{oM})^T$ and $\mathbf{x}_e = (x_{e1},...,x_{eM})^T$ is the resultant error in the unknowns.

In general, we assume that the $\mathbf{d}_e$ are random variables, so we can compute only the statistics of $\mathbf{x}_e$. Indeed, if the **f** are nonlinear, we can usually only *estimate* the statistics of the errors in **x**. The errors in **x** depend on the errors in **d**, the condition of **B** and, if an iterative method is used, perhaps on the initial guess of **x** and the convergence criterion. For the TOA problem, these phenomena correspond to the errors in the TOA measurements at the sensors, the condition of event/sensor geometry, the choice of an initial location, and the choice of the convergence criterion.

In any iterative (thereby nonlinear) least squares problem, there is always the possibility of finding local, rather than global, minima because of the initial solution choice. Even if data errors are small and the problem is well conditioned, the procedure can find a local minimum if the initial guess is in the wrong region. There is also the possibility that the global minimum is not unique. What is even more problematic is that the global minimum may *not* be the desired minimum. All TOA algorithms are nonlinear. The only way to (hopefully) guarantee that the found minimum conforms to the desired solution is to utilize some information external to the method itself as a test of the reasonableness of any solution. Because of data errors, finding the global minimum is not necessarily a sufficient guarantee. The error analyses in this report assume that solutions are “near” the desired locations, not at some other, far-away minimum.

3.1 Linear Estimation of Least Square Errors

Define

$$\mathbf{f}_o \triangleq \mathbf{f}(\mathbf{x}_o, \mathbf{d}_o)\,,\ \mathbf{A}_o \triangleq \partial\mathbf{f} / \partial\mathbf{x}\,|_{\mathbf{x}=\mathbf{x}_o,\mathbf{d}=\mathbf{d}_o}\,, \text{ and } \mathbf{G}_o \triangleq \partial\mathbf{f} / \partial\mathbf{d}\,|_{\mathbf{x}=\mathbf{x}_o,\mathbf{d}=\mathbf{d}_o}\,;$$

that is, $\mathbf{A}_o$ is the sensitivity of **f** with respect to **x**, and $\mathbf{G}_o$ is the sensitivity of **f** with respect to the data **d**, both evaluated at the “exact” solution and with errorless measurements. Expanding **f** in a Taylor series about $\mathbf{x} = \mathbf{x}_o$ and $\mathbf{d} = \mathbf{d}_o$, and retaining only the linear terms, suggests the least squares problem

$$\min(\mathbf{f}_o + \mathbf{A}_o\mathbf{x}_{el} + \mathbf{G}_o\mathbf{d}_e)^T\,\mathbf{W}(\mathbf{f}_o + \mathbf{A}_o\mathbf{x}_{el} + \mathbf{G}_o\mathbf{d}_e),$$

where $\mathbf{x}_{el}$ is the *linear* estimate of the solution errors. Since these equations are linear in $\mathbf{x}_{el}$, the solution is

$$\mathbf{x}_{el} = -\mathbf{B}_o^{-1}\mathbf{A}_o^T\mathbf{W}(\mathbf{f}_o + \mathbf{G}_o\mathbf{d}_e), \quad \mathbf{B}_o \equiv \mathbf{A}_o^T\mathbf{W}\mathbf{A}_o.$$

But $\mathbf{f}_o = 0$, hence

$$\mathbf{x}_{el} = -\mathbf{B}_o^{-1}\mathbf{A}_o^T\mathbf{W}\mathbf{G}_o\mathbf{d}_e. \tag{3.1}$$

Note that the linear estimates of the location errors are linear combinations of the data errors.

Assuming that the elements of $\mathbf{d}_e$ are random variables, we can compute the mean and variance of $\mathbf{x}_{el}$. We get

$$\overline{\mathbf{x}}_{el} \triangleq E\{\mathbf{x}_{el}\} = -\mathbf{B}_o^{-1}\mathbf{A}_o^T\mathbf{W}\mathbf{G}_o E\{\mathbf{d}_e\} = -\mathbf{B}_o^{-1}\mathbf{A}_o^T\mathbf{W}\mathbf{G}_o\overline{\mathbf{d}}_e. \tag{3.2}$$

Define the covariance matrix of the linear errors in $\mathbf{x}$ as $\mathbf{C}_{xl} \triangleq E\{(\mathbf{x}_{el} - \overline{\mathbf{x}}_{el})(\mathbf{x}_{el} - \overline{\mathbf{x}}_{el})^T\}$. From equation (3.1), we get

$$\mathbf{C}_{xl} = E\{(-\mathbf{B}_o^{-1}\mathbf{A}_o^T\mathbf{W}\mathbf{G}_o\mathbf{d}_e - \overline{\mathbf{x}}_{el})(-\mathbf{B}_o^{-1}\mathbf{A}_o^T\mathbf{W}\mathbf{G}_o\mathbf{d}_e - \overline{\mathbf{x}}_{el})^T\}, \tag{3.3}$$

and since $(\mathbf{XY})^T = \mathbf{Y}^T\mathbf{X}^T$ for any matrices $\mathbf{X}$ and $\mathbf{Y}$, and $\mathbf{B}$ is symmetric (i.e., $\mathbf{B} = \mathbf{B}^T$),

$$\mathbf{C}_{xl} = \mathbf{B}_o^{-1}\mathbf{A}_o^T\mathbf{W}\mathbf{G}_o\mathbf{C}_d\mathbf{G}_o^T\mathbf{W}\mathbf{A}_o\mathbf{B}_o^{-1} - \overline{\mathbf{x}}_{el}\overline{\mathbf{x}}_{el}^T, \tag{3.4}$$

where $\mathbf{C}_d \triangleq E\{\mathbf{d}_e\mathbf{d}_e^T\}$ is the covariance matrix of the data errors.

In general, the mean of the data errors is zero; that is, $\overline{\mathbf{d}}_e = 0$. If this is not so, the data errors are said to be "biased." An estimate should be made of the bias and, if the estimate is believable, a translation of the values of $\mathbf{d}$ should be made to remove the mean errors. Given that $\overline{\mathbf{d}}_e = 0$, we get

$$\overline{\mathbf{x}}_{el} = 0\text{, and } \mathbf{C}_{xl} = \mathbf{B}_o^{-1}\mathbf{A}_o^T\mathbf{W}\mathbf{G}_o\mathbf{C}_d\mathbf{G}_o^T\mathbf{W}\mathbf{A}_o\mathbf{B}_o^{-1}. \tag{3.5}$$

From Graupe [8]: If the data errors are (unbiased) Gaussian random variables, a common practical situation, then the maximum-likelihood, minimum-variance estimate of $\mathbf{C}_{xl}$ is obtained if $\mathbf{W}$ is chosen as $\mathbf{W} = (\mathbf{G}_o\mathbf{C}_d\mathbf{G}_o^T)^{-1}$. In this case,

$$\mathbf{C}_{xl} = \mathbf{B}_o^{-1}. \tag{3.6}$$

Although the maximum-likelihood, minimum-variance is exactly valid for Gaussian random variables, it is a practical approximation for almost any non pathological, single mode, symmetric distribution. In the following material, we shall refer to this weighting choice as "optimal" weighting.

If the weighting is normalized by using $trace(\mathbf{W}) = N$, then $\mathbf{C}_{xl}$ from equation (3.6) is an estimate of the conditioning of the event/sensor geometry, also referred to as Geometric Dilution of Precision (GDOP). For any weighting matrix, the condition of $\mathbf{C}_{xl}$ from equation (3.6) is the measure of GDOP if $\mathbf{C}_d$ is normalized by $trace(\mathbf{C}_d) = N.$ If the diagonal elements of $\mathbf{C}_{xl}$ are large compared to unity, the system is ill conditioned, and vice versa. Another, and essentially equivalent, estimate of GDOP is Error Probable (EP). For our location problem, EP has three components: Circular Error Probable (CEP), Vertical Error Probable (VEP), and Time Error Probable (TEP). These measures are directly computed from the location error covariance matrix (Aronson [1]).

At this juncture, we note that all location algorithms discussed in this report are nonlinear; therefore, if the data errors are "large" and/or the event/sensor geometry is ill conditioned, the actual location error statistics can differ substantially from their linear approximations.

4.0 TIME-OF-ARRIVAL LOCATION ALGORITHMS

For TOA location systems, there are four unknowns: the three-space location of the event and the time of the event. We assume that the pulse velocity v is constant and identical from event to all sensors. For simplicity, we take this velocity to be unity. Thus, the "time of event," $\tau = vt$, has the units of distance. The vector of unknowns has four dimensions:

$$\mathbf{x} = (x, y, z, \tau)^T. \tag{4.1}$$

For N sensors, the data vector is

$$\mathbf{d} = (d_1, ..., d_N)^T, (d_n = vt_n), \tag{4.2}$$

where t_n is the detected time of arrival at the n-th sensor. The location of the n-th sensor is

$$\mathbf{x}_n = (x_n, y_n, z_n)^T. \tag{4.3}$$

In the following, we assume that the sensor locations are known exactly. Analysis of event location errors when the sensor locations also contain errors (in addition to the data errors) is given in Aronson [3].

In all the following, we assume that the data errors are mutually independent random variables with zero mean. Thus, $E\{d_{en} d_{em}\} = 0$ if $n \neq m$, and the variance of the n-th data error is

$$\sigma_n^2 = E\{d_{en}^2\}. \tag{4.4}$$

The covariance matrix of data errors is therefore diagonal:

$$\mathbf{C}_d = diag(\sigma_1^2, ..., \sigma_N^2). \tag{4.5}$$

In Sections 4.1 through 4.8, we discuss four important algorithms for finding event location via least squares and the location errors associated with them.

4.1 TOA Method

Let the location of the n-th sensor be

$$\mathbf{x}_n = (x_n, y_n, z_n)^T. \tag{4.6}$$

The distance from the event to the n-th sensor is

$$r_n = \sqrt{(x - x_n)^2 + (y - y_n)^2 + (z - z_n)^2}. \tag{4.7}$$

For the TOA method the model equations are

$$f_n = r_n - (d_n - \tau), n = 1,..., N; \tag{4.8}$$

that is, the distance from the event to the n-th sensor should be equal to the difference between the arrival time at that sensor and the event time.

The matrix $\mathbf{A} = \partial \mathbf{f} / \partial \mathbf{x}$ has N rows and 4 columns, and its n-th row is

$$\mathbf{A}_n = [(x - x_n)/r_n, (y - y_n)/r_n, (z - z_n)/r_n, 1]. \tag{4.9}$$

The sensitivity of the model equations with respect to the data is the matrix $\mathbf{G} = \partial \mathbf{f} / \partial \mathbf{d}$, with N rows and N columns, and is

$$\mathbf{G} = -\mathbf{I}. \tag{4.10}$$

Thus, the "optimal" weighting matrix is diagonal; namely,

$$\mathbf{W} = [(-\mathbf{I})\mathbf{C}_d(-\mathbf{I})]^{-1} = diag(1/\sigma_1^2,...,1/\sigma_N^2). \tag{4.11}$$

The TOA equations are usually solved by Newton's iterative method. An initial location guess starts the iteration. Since there are four unknowns, each iteration requires the solution of a 4 by 4 set of linear equations; that is, the $\mathbf{B}$ matrix is 4 by 4.

Because the model equations are linear in τ, it is not necessary to make an initial guess for τ. Instead of using equation (2.6), we iterate on

$$\hat{\mathbf{x}}^{i+1} = \tilde{\mathbf{x}}^i - \mathbf{B}^{i-1}\mathbf{A}^{iT}\mathbf{f}(\tilde{\mathbf{x}}^i, \tilde{\mathbf{d}}),$$

where $\tilde{\mathbf{x}}^i = (x^i, y^i, z^i, 0)^T$, $\tilde{\mathbf{d}} = (0, d_2 - d_1, ..., d_N - d_1)^T$, and $\hat{\mathbf{x}}^{i+1} = (x^{i+1}, y^{i+1}, z^{i+1}, \tau^{i+1})^T$. The iteration terminates when $\left|\tilde{\mathbf{x}}^{i+1} - \tilde{\mathbf{x}}^i\right| \le \varepsilon$. Let the final iteration be at $i = I$, then the solution is

$$\mathbf{x} = (x^{I+1}, y^{I+1}, z^{I+1}, \tau^{I+1} + d_1)^T.$$

For TOA, it is prudent for numerical operation reasons to translate the data to the minimum datum value, for example. That is, if $\tilde{d}_n$ is the actual measured n-th datum value, $d_n = \tilde{d}_n - \min(\tilde{d}_n)$ should be used. The resultant value of τ should be negative, but can then be translated forward to the original time basis. Section 5.4 shows that although a data translation should be used, translation to the *minimum data value* should *not* be used for the Bancroft method.

4.2 Linear Location Errors in TOA

If the weighting is optimal, equation (3.6) is used to obtain the linear error covariance. We get

$$\mathbf{C}_{xl} = \mathbf{B}_o^{-1}, \tag{4.12}$$

where $\mathbf{B}_o = \mathbf{A}_o^T \mathbf{W} \mathbf{A}_o$,

$$\mathbf{A}_{on} = [(x_o - x_n)/r_{on}, (y_o - y_n)/r_{on}, (z_o - z_n)/r_{on}, 1],$$

$$r_{on} = \sqrt{(x_o - x_n)^2 + (y_o - y_n)^2 + (z_o - z_n)^2}.$$

If nonoptimal weighting is used, $\mathbf{C}_{xl}$ is computed from equation (3.5).

4.3 Time-Difference-of-Arrival Method

The Time-Difference-of-Arrival (TDOA) method is motivated by the fact that if the various TOA model equations are subtracted from each other in some pairwise fashion, the τ unknown drops out and only a 3 by 3 linear system need be solved in the Newton iteration. It is proven in Aronson [2] that "If the TOA data errors are independent random variables with zero means and the weighting matrix is chosen as the inverse of the data difference errors, then the (linear) covariance matrix of the position errors is independent of the particular choice of difference pairs—provided, of course, that a set of independent, non-redundant differences is used." As a TDOA example, we choose to subtract the first model equation from all the rest. We get

$$f_n = r_{n+1} - r_1 - d_{n+1} + d_1, n = 1, \ldots N-1, \tag{4.13}$$

$$\mathbf{A}_n = [(x - x_{n+1})/r_{n+1} - (x - x_1)/r_1, (y - y_{n+1})/r_{n+1} - (y - y_1)/r_1, (z - z_{n+1})/r_{n+1} - (z - z_1)/r_1], \tag{4.14}$$

and the n-th row of the $\mathbf{G}$ matrix is

$$\mathbf{G}_n = (0, \ldots, -1, \ldots, 1). \tag{4.15}$$

Note that $\mathbf{A}$ is (N-1) by 3, $\mathbf{G}$ is (N-1) by N, and the solution vector is $(x, y, z)^T$. The TDOA method does not directly yield τ, but it may be easily found by other means once the location is found.

The TDOA method was initially considered because –in the 1970s computer time and memory were very expensive, and the inversion of a 3 by 3 linear system was considered a significant advantage over inversion of a 4 by 4 system. Such consideration is no longer valid, and there is no need to use TDOA. One disadvantage of TDOA is that numerical roundoff problems may arise from taking TOA model equation differences.

4.4 Location Errors in TDOA

If the weighting is optimal, $\mathbf{W} = (\mathbf{G}\mathbf{C}_d\mathbf{G}^T)^{-1}$, then it is proved in Aronson [2] that the (3 by 3) linear location error matrix $\mathbf{C}_{xl}$ for TDOA is identical to the upper 3 by 3 submatrix of $\mathbf{C}_{xl}$ for TOA. Note that $\mathbf{W}$ is (N-1) by (N-1). The closed form for the inversion of $\mathbf{G}\mathbf{C}_d\mathbf{G}^T$ is given in Aronson [3] and Gregory and Karney [9]. For nonoptimal weighting, equation (3.5) is used; otherwise, equation (3.6) is used. We will not study TDOA further in this report.

4.5 Time-of-Arrival Squared Method

The model equations for the Time-of-Arrival Squared (TSQ) method are

$$f_n = [(x-x_n)^2 + (y-y_n)^2 + (z-z_n)^2 - (\tau - d_n)^2]/2 = [r_n^2 - (\tau - d_n)^2]/2, n = 1, \ldots, N; \quad (4.16)$$

that is, if the distance from the event is equal to the TOA minus the event time, then the squares of these quantities should also be equal. The division by 2 is for convenience. The TSQ least squares equations can be solved by Newton iteration with

$$\mathbf{A}_n^i = (x^i - x_n, y^i - y_n, z^i - z_n, d_n - \tau^i).$$

Note that, unlike TOA and TDOA, the model equations are nonlinear in τ as well in $x, y,$ and z, and that the data values themselves appear in the $\mathbf{B}^i$ matrix.

The associated $\mathbf{G}$ matrix is diagonal:

$$\mathbf{G} = diag(d_1 - \tau, \ldots, d_N - \tau) = diag(r_1, \ldots, r_N)$$

Thus the optimal weighting matrix is also diagonal;

$$\mathbf{W} = (\mathbf{G}\mathbf{C}_d\mathbf{G}^T)^{-1} = diag[1/\sigma_1^2(d_1-\tau)^2, \ldots, 1/\sigma_N^2(d_N-\tau)^2].$$

There are three apparent disadvantages in the iterated TSQ method. First, an initial estimate of all four unknowns (x, y, z, τ) must be made; second, a convergence criterion must be established, and third, one must know the "true" values of τ and $\mathbf{d}$ to achieve optimal weighting.

4.6 Linear Location Errors in TSQ

If optimal weighting is used in TSQ, the linear location error covariance matrix, $\mathbf{C}_{xl}$, is identical to that in the TOA method. Note that $d_{on} - \tau_o = r_{on}$, where r_{on} is the "true" distance from the event to the n-th sensor. Thus $\mathbf{B}_o = \mathbf{A}_o^T \mathbf{W} \mathbf{A}_o$ for TSQ is identical to $\mathbf{B}_o$ for TOA, and $\mathbf{C}_{xl} = \mathbf{B}_o^{-1}$ (QED).

4.7 TSQ with Bancroft's Method (TSQB)

Bancroft [10] has devised a scheme for solving the TSQ problem requiring neither iteration nor an initial guess of the unknowns. His method is a special case of a more general least squares method. Consider the following *parametric* least squares problem: Let the model equations have the form

$$f_n = f_n(\mathbf{x}, \mathbf{d}, p),$$

where p is a parameter and $\mathbf{d}$ is given.

Suppose that a solution can be found with $\mathbf{x}$ as a function of p, say, $\mathbf{x} = \mathbf{S}(p)$. Now, if we constrain p to be a function of $\mathbf{x}$, $p = f_p(\mathbf{x})$, we may substitute $\mathbf{S}$ in f_p and obtain $p = f_p[\mathbf{S}(p)]$. If we solve this last equation for p, we may then find $\mathbf{x}$ from $\mathbf{x} = \mathbf{S}(p)$.

If the model equations are linear in p and $\mathbf{x}$, $\mathbf{S}$ is easily found. Let

$$f_n = u_n p + \mathbf{A}_n \mathbf{x} + c_n,$$

where u_n, $\mathbf{A}_n$, and c_n are constants.

The solution to the weighted problem is

$$\mathbf{x}(p) = -\mathbf{B}^{-1}\mathbf{A}^T \mathbf{W} \mathbf{U} p - \mathbf{B}^{-1}\mathbf{A}^T \mathbf{W} \mathbf{C} = \mathbf{v}_p p + \mathbf{v}, \tag{4.17}$$

where $\mathbf{B} = \mathbf{A}^T \mathbf{W} \mathbf{A}$, $\mathbf{U} = (u_1, ..., u_N)^T$, and $\mathbf{C} = (c_1, ..., c_N)^T$. Solving $p = f_p[\mathbf{x}(p)]$ for p and substituting in equation (4.17) yields $\mathbf{x}$.

For the TSQB method, we write the TSQ model equations as

$$f_n = p - x x_n - y y_n - z z_n + \tau d_n + c_n, \tag{4.18}$$

$$c_n = (x_n^2 + y_n^2 + z_n^2 - d_n^2)/2. \tag{4.19}$$

$$p = (x^2 + y^2 + z^2 - \tau^2)/2. \tag{4.20}$$

Note that the model equations are linear in $\mathbf{x}$ and p. We have

$$\mathbf{A}_n = (-x_n, -y_n, -z_n, d_n), \tag{4.21}$$

$$\mathbf{U} = (1, ..., 1)^T. \tag{4.22}$$

Substituting equations (4.19), (4.21), and (4.22) in equation (4.17) yields values of $\mathbf{v}_p$ and $\mathbf{v}$.

From equation (4.20), we get the equation in p

$$(\mathbf{v}_p^T \mathbf{M} \mathbf{v}_p / 2) p^2 + (\mathbf{v}_p^T \mathbf{M} \mathbf{v} - 1) p + \mathbf{v}^T \mathbf{M} \mathbf{v} / 2 = ap^2 + bp + c = 0, \tag{4.23}$$

where $\mathbf{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

Equation (4.23) is solved for *p* by the quadratic formula $p = (-b \pm \sqrt{b^2 - 4ac})/2a$. The values of *p* are substituted in equation (4.17). The Bancroft method is computationally very fast because no iteration is required. Also, it requires neither an initial guess of the event location, nor a convergence criterion.

The two solutions represent two local minima. Numerical tests have shown that if the TOA method is initialized near each of these solutions, it converges to that solution. A major difficulty with TSQB is the resolution of the ambiguity in the quadratic solution. A possible method for resolving the two-solution ambiguity is to choose the value of p that offers the minimum scalar value of the TOA or TSQ problem, that is, minimizes either

$$\sum_{n=1}^{N} w_{nn}(r_n + \tau - d_n)^2, \text{ or } \sum_{n=1}^{N} w_{nn}[r_n^2 - (\tau - d_n)^2]^2,$$

where w_{nn} is the n-th diagonal element of **W**, for either the TOA or TSQ weighting matrix. Section 5.5 demonstrates that these are *not* good choices in general because even if they find the global minimum, there is no guarantee that the global minimum is the "correct" solution. Another possibility is that one of the resultant values of τ may be in "past" time and the other in "future" time. The "past" time solution is then chosen as the correct answer. However, if data errors are "large" or if the geometry is ill conditioned, this choice may also be poor. It appears that some criterion external to the Bancroft procedure itself must be employed to resolve the ambiguity.

For larger data errors and poorer geometries, the discriminant $b^2 - 4ac$ approaches zero. In this situation, the two minima approach each other, and it becomes impossible to choose the "correct" solution. Very large data errors and/or an ill-conditioned geometry are indicated if the discriminant becomes negative, so the quadratic has complex roots. In this situation no solution is possible, unless information in addition to the data themselves is employed (Section 7). In general, this situation is equivalent to nonconvergence of the TOA method.

Data translation is also recommended for TSQB, but *not* to the minimum data value as suggested for TOA. Note that the data values themselves appear in the $\mathbf{B}$ matrix and therefore can affect its conditioning. Data translation to the minimum data value appears to cause poor conditioning in some cases. Translation is still recommended, but to a different value. We suggest translation to a value where the minimum data value translates to a "typical" value of $\sqrt{x_n^2 + y_n^2 + z_n^2}$. This matter is discussed further in Section 5.4.

4.8 Linear Location Errors for TSQB

The TSQB method solves the identical problem as the TSQ method, assuming that TSQ converges properly and the correct value of p is chosen in TSQB. Therefore, $\mathbf{C}_{xl}$ for TSQB is identical to that of TSQ and TOA.

5.0 NUMERIC RESULTS FOR "REALISTIC" LOCATION SCENARIOS

A pair of FORTRAN codes, EPSCAN and MONTEC, was written to study location errors for various location algorithms and sensor groupings. A set of 24 sensors was used at fixed sites given by a "typical" ephemeris. They are placed very close to a sphere about 26,700 km (4.2 earth radii) from the center of the earth. A spherical earth of radius 6378 km was used. For light speed, this distance is equivalent to 21.26 ms. The distance unit of earth radius was used for all computations; that is, an object *on* the earth's surface is 1.0 unit from its center.

Each event was located by specifying its distance from earth center, longitude, and latitude (R,θ,ϕ). Although the codes can treat events at any location, all studies were conducted at $R = 1.00$ (i.e., on the earth's surface) and at $R = 1.10$, 637.8 km above the surface. The earth surface location offered maximum signal masking by the earth. No solution was discarded if it was at an impossible site, such as inside the earth. The higher altitude was used to simulate a nonatmospheric event. At each event location, the (Cartesian) coordinate system was rotated so that the first coordinate represented the east-west direction, the second the north-south, and the third the vertical direction.

The codes produce CEP, VEP, and TEP (Time Error Probable) results, which are deemed to be more informative than error covariances. The rotation choice translates easily into the metrics CEP, VEP, and TEP. The metric CEP is the radius of a circle in a plane tangent to the earth's surface within which the locations fall with probability 50%, with the assumption that the errors in the two orthogonal dimensions in this plane are jointly Gaussian random variables with zero means. VEP and TEP are one-dimensional 50% probabilities in the vertical and time dimensions, respectively, for zero-mean Gaussian random variables. The computation of these error probabilities is given in Appendix B of Aronson [1]. We use the notation EP to denote either CEP, VEP, or TEP.

Although the sensor positions were fixed, four different sensor scenarios were studied. A "candidate" sensor is one whose line of sight to the event is not masked by the earth. Suppose that at some specified event location there are M candidate sensors, indexed by increasing distance from the event. A code input option is $N \le M$, the number of sensors that "see" the event. There are four scenario options to choose which sensors are used in any computation. Let N_u be the set of used sensors. For example, let $M = 10$ and $N = 5$. The scenarios are

(1) NEAREST, $N_u = \{1,2,3,4,5\}$, (2) FARTHEST, $N_u = \{6,7,8,9,10\}$,
(3) NEAR ONE + FAR, $N_u = \{1,7,8,9,10\}$, (4) MIDDLE, $N_u = \{3,4,5,6,7\}$.

5.1 Code EPSCAN

This code was written to

1. Gain insight into location errors for "realistic" event and sensor locations and sensor sets.

2. Numerically verify that the linear error covariance estimates are identical for TOA and TSQB with optimum weighting.

3. Assess error covariance differences between optimum and non-optimum weighting for TSQB. Nonoptimum TSQB weighting uses the same weighting as TOA.

4. Find ill-conditioned situations for further study.

The code user chooses the number of sensors that "see" the event *N*, specifies the distance from earth center of the event *R*, the sensor scenario to be used, and the algorithm to be used. The code is currently implemented with TOA and TSQB. The event is positioned on its sphere of radius R in a set of locations that cover the sphere at (approximately) equal angular spacing DELP degrees apart. For example, with DELP = 6 degrees, 1134 event locations are used.

It is assumed that the data errors are independent random variables with mean zero and identical variance: $\sigma_n^2 = \sigma^2$ for all n. At each event location, *linear* CEP, VEP, and TEP (that is, from linear covariances) are computed. The results are normalized by assuming that $\sigma^2 = 1$. Normalized linear error probable (NLEP) statistics are presented as output. The magnitudes of the NLEP are measures of the geometric conditioning (GDOP) of the event/sensor system, as is $\mathbf{C}_{xl}$. If these elements are large compared to unity, the system is ill conditioned, and vice versa.

5.1.1 EPSCAN Input

N = Number of sensors to use. Error if $N < 4$ or $N >$ minimum number of unmasked sensors.
R = Distance of event from earth center. Error if $R < 1$.
DELP = Nominal angular (degree) spacing of events. Error if DELP < 3 or DELP > 30, or DELP is not a factor of 90.
Specification of sensor scenario; i.e., NEAREST, etc.
Type of algorithm: TOA or TSQB.

5.1.2 EPSCAN Output

The code computes certain statistics of Normalized Linear Error Probable (NLEP), with optimal weighting, over the set of event locations for the given input conditions. For TSQB, the statistical differences between optimal and nonoptimal weighting are also given. Output notation is

Lon/min Longitude of the minimum EP occurred.
Lat/min Latitude of the minimum EP.

EPmin	Minimum EP.
Lon/max	Longitude of maximum EP.
Lat/max	Latitude of maximum EP.
EPmax	Maximum EP.
Epavg	Average EP over events* [?].
Epstd	Standard deviation of EP*.
Epavgd	EPavg for nonoptimal weighting minus EPavg for optimal – TSQB only.
EPstdd	EPstd for nonoptimal weighting minus EPstd for optimal – TSQB only.

Some of the event/scenario situations exhibited very poor conditioning and yielded extreme NLEP values. In general, the largest NLEP values were the NLVEP values. These extreme values tended to dominate the NLEP statistics. Therefore, it was decided to eliminate all NLEP values that exceeded 75. A cutoff value of 75 was chosen because it was found that the TOA iteration would usually not converge if NLVEP exceeded that value, even if the data errors had the very small deviation of $\sigma = 0.000001$ (21.26 ns). The output listing notes how many NLEP data values were discarded in each situation. As an example of excessively poor conditioning, the value of NLVEP for $N = 4, Lon = 237.33, Lat = -42$ and the FARTHEST sensor set was 35893. This value was among the discards.

5.1.3 EPSCAN Results

The EPSCAN results are shown in Table 1a, Table 1b, and Table 1c for, $N = 4,5,6$ and $R = 1$. The results for $R = 1.10$ are in Table 2a through Table 2c. (Tables are located at the end of the report.) All results are for DELP = 6 degrees, which is likely a fine enough grid for my purposes. This angular spacing produced a scan of 1134 event locations. The results clearly indicate that the normalized linear EP, and therefore the linear error covariances, are identical for optimally weighted TOA and TSQB for the NEAREST scenario. The same equalities hold for all the other sensor scenarios and were not tabulated. The results also show that at least for all conditions studied, there is little differences between optimal and nonoptimal weighting in TSQB. However, all the work in this report assumes that the data error variances are identical. If the variances are not identical, it may be possible that optimal TSQB weighting may make a more significant difference in the NLEP values. Except for the possibility of a truly pathological case, we personally feel that there are very minor differences between optimal and nonoptimal TSQB.

5.2 Code MONTEC

This code was written to

1. Examine “true” error covariances and EP values for specified event locations, algorithms, data error variances, and sensor scenarios.

2. Assess differences among various data error probability distributions.

3. Ascertain the importance of optimal vs. nonoptimal weighting for TSQ.

4. Derive a choice method to resolve the ambiguity in the TSQB method.

MONTEC is a Monte Carlo analysis of EP values. In the absence of considerable, and perhaps extremely difficult (if at all possible) exact statistical analysis, we make the assumption that the statistics generated by the Monte Carlo procedure are close to the "true" values without speculating on how "close" the results are. An event/sensor geometry, data error variance, and location method are specified. With these specifications, location and time errors are computed using a set of (pseudo) random data errors with specified statistics. Each solution is a Monte Carlo trial. Numerous trials, each with a different set of random data errors, are processed. The statistics of the resultant location and time errors in the form of CEP, VEP, TEP, and the average position error are presented as output.

The user specifies the number of sensors that "see" the event *N*, the radial distance, longitude, and latitude of the event, the sensor scenario (NEAREST, etc.), the number of Monte Carlo trials, an initial seed for the pseudo-random number generator (input value zero generates a default), the probability distribution used to generate the data errors, and the standard deviation of the data errors. For fair comparisons, each algorithm is subject to the same data errors. The resultant EP statistics are computed from the Monte Carlo tries. It must be emphasized that *EP is defined only for distributions with mean zero*. If the means are not zero, as is the case with the results here, the EP values are to be taken as about the solution mean values.

In all cases, the data errors used are mutually independent (pseudo) random variables with zero mean and equal variance. Optimal weighting is used for all algorithms, except in Section 5.6. The code can use various algorithms as specified by the user. All units are in earth radii. Weighting is normalized such that $trace(\mathbf{W}) = N$. For all algorithms, all solutions are allowed.

Ten iterations are allowed for each *TOA* solution. If any trial exceeds 10 iterations, the code terminates with an error message. The initial TOA guess is randomly chosen by

$$x_g = x_o + 2\kappa(u - 0.5),$$

where u is a random variable uniform in [0,1] and independent of the data errors. The other guesses, y_g and z_g, are similarly chosen, with independent values of u for each dimension and trial. We use $\kappa = 0.3$ in the code. The TOA iteration is terminated when $\left|\tilde{\mathbf{x}}^{i+1} - \tilde{\mathbf{x}}^{i}\right| \le 0.001$, [$\tilde{\mathbf{x}}^{i} = (x^{i}, y^{i}, z^{i})^{T}$].

5.2.1 MONTEC Input

The algorithm to be used: TOA, TSQB, TSQBN, LPTOA

N = Number of sensors to use. Error if $N < 4$, if $N >$, QUIT.
R = Distance of event from earth center. Error if $R < 1$.
Lon = Longitude of event, degrees.
Lat = Latitude of event, degrees. Error if $|Lat| \geq 90$.
Sensor scenario; i.e., NEAREST, etc.
Number of Monte Carlo trials, must be positive. All results shown used 10,000 trials.
Pseudo-random generator seed. (Enter zero value for default.)
Data error probability distribution: Gaussian, uniform, or "tailed."
$\sigma =$ Standard deviation of data errors. Error if $\sigma \leq 0$.

The available data error probability distributions are

1. Gaussian – normal distribution, mean zero, variance σ^2, $Gau(0,\sigma^2)$.

2. Uniform – uniform distribution, mean zero, variance σ^2. $Unif(0,\sigma^2)$.

3. Tailed – This is a special distribution, concocted to generate more errors near the distribution tails. It chooses $Gau(0,\sigma^2)$ with probability $(1-p)$ and $Gau[0,(k\sigma)^2]$ with probability *p*. The resultant random variable is divided by $\sqrt{1+p(k^2-1)}$ so its variance will be σ^2. We use $p = 0.02$ and $k = 3$.

All studies were made using those event/sensor geometries that had the NLVEP max values, that is, the poorest conditioning. Since the NLVEP values generally exceeded the NLCEP and NLTEP values, the NLVEP were chosen as the criterion for poor conditioning. In general, large NLVEP implied large NLCEP and NLTEP values. Instead of doing the excessive computation for *all possible* combinations, it was decided that these cases represented the least favorable ones for accurate location results. The selected cases are printed in **bold** in Table 1a through Table 2c.

5.2.2 MONTEC Output

For the given event/sensor set, the linear EP values are estimated by Monte Carlo using the given σ. These values are *not* normalized; that is, the data error statistics are generated using the given value of σ. The Monte Carlo values of $\overline{\mathbf{x}}_e$ and EP values are computed. Note that $\overline{\mathbf{x}}_{el} = 0$. The LEP (Linear Error Probable) values are also shown. For comparison with the Monte Carlo results, these values are not normalized. For TOA, the average number of iterations and the value of $\kappa(0.3)$ are also listed. For TSQB, additional output information is given, as noted below. Each event/sensor combination is identified by a code of the form #n/r.dd/s, where n = (4, 5, 6) is the number of sensors, r.dd = (1.00, 1.10) is the distance of the event from the center of the earth, and s = (1, 2, 3, 4) is the sensor scenario index.

All studies were done with the data error standard deviation of $\sigma = 0.001$. This is a relatively large error, equivalent to 6.378 km or 21.26 μs at light speed.

5.3 Statistical Results for TOA

Table 3 gives the Monte Carlo TOA results for the selected cases using the “tailed” distribution. At a data error level of $\sigma = 0.001,$ the EP values were consistently in agreement with the LEP values; in fact, the EP values with TOA tended to be a bit lower than the LEP values. A possible rationale of this result may be that the Monte Carlo EP is taken about the MC mean, thereby tending to reduce the variance. The results suggest that the linear EP estimates are essentially valid in all cases. Certainly, they are valid for $\sigma \le 0.001$ and perhaps for larger σ values.

Table 4 shows that there is little change if the Gaussian or uniform distributions are used. To avoid excessive computation, we will use only the tailed distribution in all further study.

The mean values are also similar, although the four-sensor, uniform case shows a somewhat larger mean error in the z direction.

5.4 Statistical Results for TSQB – Optimal Weighting

The advantages of the TSQB method over TOA are that TSQB requires neither an initial location guess nor a convergence criterion. It is very fast and noniterative. It has two difficulties: the choice of a data translation value and the need to select one of two potentially ambiguous solutions.

Recall that unlike TOA, the data values themselves appear in the $\mathbf{A}$ matrix, equation (4.21), and therefore in the $\mathbf{B}$ matrix, which dominates the problem conditioning. Initially we used a data translation value of $\min(d_n),$ which placed a zero value in **A**. Even with very small data errors and correct solution choice, the procedure did not give the correct solution in some cases. When We used a translation value of $\min(d_n) - 3$ the solutions were correct in all cases. The translation bias value of 3 was chosen because it represented a nominal distance from the event to a nearby sensor. We do not know why the procedure is so sensitive to the translation value, or if the solutions are especially sensitive to the translation chosen. It appears from some preliminary study of this problem that the solutions are not particularly sensitive to the translation bias choice unless it was very near zero. We did not study this phenomenon further.

Before studying solution choices, we computed location error statistics for TSQB using “perfect” choice, that is, never choosing incorrectly. While this choice is not realistic in an actual location situation, it indicates what errors can ideally be expected with TSQB. The results are given in Table 5a and Table 5b. The TSQB results are virtually identical to the TOA results. However, the results presume that a method can be found to *always* choose the correct solution. In general, the errors can be enormous if the wrong solution is used. The number 1 after the TSQB characters indicates that “perfect” choice was made. The following two numbers indicate how many trials had complex roots (none) and how many did not choose the correct solution (obviously, none for this choice). If there were complex roots or if incorrect solutions were chosen, the output would indicate these facts.

Also, the results shown are for optimum TSQB weighting. This presents a dilemma, since optimal weighting requires a priori knowledge of the solution. One possibility around this difficulty is to solve TSQB using nonoptimal weighting (i.e., the same weighting as TOA), use that solution to estimate the optimal weighting, and do a TSQB solution again using weighting estimated from the first solution. However, such a procedure may not improve the accuracy enough to be worth the effort. In Section 5.6, we consider TSQB with nonoptimal weighting and conclude that optimal TSQB weighting makes very little difference in the results.

5.5 TSQB Solution Choice

Let the two solutions offered by TSQB be indexed by $\beta = (1,2)$. The solutions are $\mathbf{x}_\beta = (x_\beta, y_\beta, z_\beta, \tau_\beta)$. Associated with each solution is the parameter p_β. Five choice methods were tried. Define $r_{\beta n} \triangleq \sqrt{(x_\beta - x_n)^2 + (y_\beta - y_n)^2 + (z_\beta - z_n)^2}$. The five methods tested were to choose the value of β that satisfies the following:

$$\min(\mathbf{f}_{TOA}^T \mathbf{W} \mathbf{f}_{TOA}), \text{where } f_{TOA,n} = r_{\beta n} + \tau_{\beta n} - d_n, \tag{5.1a}$$

$$\min(\mathbf{f}_{TSQ}^T \mathbf{W} \mathbf{f}_{TSQ}) \text{, where } f_{TSQ,n} = (x_\beta - x_n)^2 + (y_\beta - y_n)^2 + (z_\beta - z_n)^2 - (\tau_\beta - d_n)^2, \tag{5.1b}$$

$$\min(x_\beta^2 + y_\beta^2 + z_\beta^2), \tag{5.1c}$$

$$\min\left|\sqrt{x_\beta^2 + y_\beta^2 + z_\beta^2} - 1\right|, \tag{5.1d}$$

$$\min\left|p_\beta\right|. \tag{5.1e}$$

The first two methods seek to minimize the TOA and TSQ residuals, respectively. Since both solutions represent local minima, these methods failed in many of the 24 test cases. Method (5.1c) chooses the solution nearest the center of the earth, and (5.1d) chooses the solution nearest the earth's surface. Both of these failed in some cases. However, method (5.1e) *always* chose the correct solution in all cases. Those results are not presented because they are identical to Table 5(a and b). Admittedly, this result is heuristic and is not a proof. However, the results offer strong evidence that this is the correct way to choose β. We shall refer to this method as "minimum p choice." The cases where the other choice methods produced failures in one or more trials are as follows:

(5.1a) 4/1.00/1, 4/1.00/2, 4/1.00/3, 4/1.00/4, 5/1.00/2
(5.1b) 4/1.00/2, 4/1.00/3, 4/1.00/4, 5/1.00/2
(5.1c) 6/1.00/4, 4/1.10/2, 4/1.10/3, 5/1.10/1, 6/1.10/4
(5.1d) 4/1.001/, 4/1.10/1, 4/1.10/2, 4/1.10/3, 5/1.10/1, 6/1.10/4

5.6 Statistical Results for TSQB with Nonoptimal Weighting (TSQBN)

For reasons discussed above, optimal weighting poses a difficulty for the Bancroft method. A natural question is whether the minimum p choice is valid for nonoptimal as well as optimal TSQB. Monte Carlo trials were made using TSQB with nonoptimal weighting: that is, weighting the same as TOA: $\mathbf{W} = (\mathbf{C}_d)^{-1}$. Table 6(a and b) indicates that the minimum p choice procedure is as valid for nonoptimal Bancroft, denoted TSQBN, as for optimal weighting TSQB for all cases tested. Note that the results for four sensors are identical in both situations. This result is expected, since the least squares minimum must solved exactly to satisfy four model equations in four unknowns.

The EP maximum, average, and standard deviations all decrease with data from an increasing number of sensors. This result is expected, since more information is always "better" than less information, and because with more sensors the event/sensor geometry is less coplanar and therefore better conditioned. In general, the MIDDLE scenario exhibited the smallest NLEP values, thereby showing the best overall conditioning.

6.0 AN ITERATIVE LINEAR PROGRAMMING LOCATION ALGORITHM (LPTOA)

As mentioned in Section 2.0, the L_2 norm in the least squares method has a tendency to exaggerate large data errors. It may be that some other norms may not exhibit this phenomenon and thereby reduce location errors. In particular, location solutions for two other interesting norms, L_1 and L_∞, can be found by Linear Programming (LP) (Gass [11] or Bradley, Hax, and Magnanti [12]). We suggest finding $\mathbf{x}$ to satisfy

$$L_1 : \min \sum_{n=1}^{N} \left| w_{nn}(r_n + \tau - d_n) \right|, \tag{6.1a}$$

$$L_\infty : \min[\max \left| w_{nn}(r_n + \tau - d_n) \right|], \tag{6.1b}$$

where w_{nn} is the n-th diagonal element of the (TOA) weighting matrix. Since equation (6.1a and b) is nonlinear, an iterative LP approach must be used.

An LP finds a vector of unknowns, $\mathbf{x}$, which minimizes (or maximizes) the linear combination

$$\mathbf{c}^T \mathbf{x} \tag{6.2a}$$

subject to the constraints

$$\mathbf{Ax}(\leq, =, \geq)\mathbf{b}\,,\ \mathbf{x} \geq 0, \tag{6.2b}$$

where $\mathbf{c}$, $\mathbf{A}$, and $\mathbf{b}$ are constants. Maximization is obtained by replacing $\mathbf{c}$ with $-\mathbf{c}$. The $\mathbf{x} \geq 0$ constraint is not very restrictive, since $x \leq 0$ may be effected by the transformation $x' = -x,$ an unrestricted value of x by $x = x' - x'',$ and a constraint of the form $-a \leq x$ (a positive) by $x' = x + a.$ LPs are solved by the "simplex" method described in [11] and [12]. For our nonlinear problem, each step in the iteration requires an LP solution of locally linearized TOA model equations.

Define $r_n^i \triangleq \sqrt{(x^i - x_n)^2 + (y^i - y_n)^2 + (z^i - z_n)^2}$. For the L_1 norm, we solve the following iterated LP. Find $(\delta x^i, \delta y^i, \delta z^i, \tau^i, e_1^i, \ldots, e_N^i)$ to

$$\min \sum_{n=1}^{N} e_n^i\,, \tag{6.3a}$$

subject to

$$-e_n^i \le w_{nn}[r_n^i + \delta x^i (x^i - x_n)/r_n^i + \delta y^i (y^i - y_n)/r_n^i + \delta z^i (z^i - z_n)/r_n^i + \tau^i - d_n] \le e_n^i, n = 1,...N \quad (6.3b)$$

$$0 \le e_n^i, \tau^i \le 0 \quad (6.3c)$$

$$-b \le \delta x^i, \delta y^i, \delta z^i \le b, \; 0 \le b. \quad (6.3d)$$

For L_∞, find $(\delta x^i, \delta y^i, \delta z^i, \tau^i, e^i)$ to

$$\min e^i, \quad (6.4a)$$

subject to

$$-e^i \le w_{nn}[r_n^i + \delta x^i (x^i - x_n)/r_n^i + \delta y^i (y^i - y_n)/r_n^i + \delta z^i (z^i - z_n)/r_n^i + \tau^i - d_n] \le e^i, n = 1,...N \quad (6.4b)$$

$$0 \le e^i, \tau^i \le 0 \quad (6.4c)$$

$$-b \le \delta x^i, \delta y^i, \delta z^i \le b, \; 0 \le b. \quad (6.4d)$$

The (6.3d) and (6.4d) constraints are used to control possible wild fluctuations in the iteration. For the initial guess $\mathbf{x}^0 = (x^0, y^0, z^0)^T$, the iteration is $x^{i+1} = x^i + \delta x^i$, etc. The iteration terminates when either $\left|e^{i+1} - e^i\right| \le \varepsilon$, or $\sqrt{\delta x^{i2} + \delta y^{i2} + \delta z^{i2}} \le \varepsilon.$

The simplex procedure is itself iterative and generally requires considerably more computation than TOA or TSQB. However, because of the special nature of our problem, a specialized and much faster simplex procedure could be devised if desired. The simplex process has two steps: it first funds a "solution" that is "feasible" and then finds the optimum (minimum or maximum) solution. An LP problem is "feasible" if there exists at least one set of unknowns that satisfies the constraints. The feasibility step can be skipped, since our LP is always feasible at any value of $(\delta x, \delta y, \delta z)$ that satisfies equations (6.3d) and (6.4d). Another concern in the general LP situation is that the optimum solution may be unbounded. All our potential "solutions" are bounded, so we need not be concerned with testing this condition.

As with TOA, an initial location guess must be made. Initially, we used the same random initial location for LPTOA as TOA (Section 5.2). However, the LP did not converge to near the "true" location in many cases. Therefore, we set the initial LPTOA location to the TSQB solution, and then proceeded with the LP iterations. The salient question to be answered is whether the L_1 or L_∞ norms yielded improved location errors statistics compared to the L_2 norms from TOA or TSQB. Table 7(a through d) shows the results of LPTOA for the 24 test cases. The same convergence criteria for TOA, as described in Section 5.2 were used for LPTOA: $\varepsilon = 0.001$ as the convergence criterion with up to 10 iterations allowed. The bounding constraint value was set to $b = \kappa = 0.3$. The tables indicate that there is very little, if any, improvement using norms other than L_2. These results suggest that the LPTOA method is not worth its additional complexity compared to TOA and TSQB.

7.0 LEAST SQUARES WITH CONDITIONING EQUATIONS

If some *a priori* information besides the TOA data themselves is available, it may be possible to reduce location errors by adding equations to the model equations. If properly applied, these added equations can make a considerable improvement in the condition of the event/sensor system. Conditioning equations are discussed in Aronson [3], where they are referred to as "augmenting" equations. We use the super tilde (~) to denote the augmenting equations and super caret (^) to denote the augmented, that is, conditioned, equations. The conditioned model equations become $\hat{\mathbf{f}} = (f_1, ..., f_N, \tilde{\mathbf{f}}^T)^T$, where the f_n are the original model equations, $\tilde{\mathbf{f}}$ are the conditioning equations, and $\hat{\mathbf{f}}$ is the conditioned (augmented) model equations. In general, $\tilde{\mathbf{f}}$ is a function of the location unknowns **x**, the data **d**, and a set of chosen parameters $\mathbf{p}$; thus. $\hat{\mathbf{f}} = [\mathbf{f}(\mathbf{x}, \mathbf{d})^T, \tilde{\mathbf{f}}(\mathbf{x}, \mathbf{d}, \mathbf{p})^T]^T$.

Linear estimations of the location errors with respect to the data errors and errors in the chosen conditioning parameters can be found as follows. Define the gradients in the usual way

$$\hat{\mathbf{A}} \triangleq \begin{pmatrix} \partial \mathbf{f} / \partial \mathbf{x} \\ \partial \tilde{\mathbf{f}} / \partial \mathbf{x} \end{pmatrix}, \; \hat{\mathbf{G}} \triangleq \begin{pmatrix} \partial \mathbf{f} / \partial \mathbf{d} \\ \partial \tilde{\mathbf{f}} / \partial \mathbf{d} \end{pmatrix}, \text{ and } \hat{\mathbf{U}} \triangleq \begin{pmatrix} \partial \mathbf{f} / \partial \mathbf{p} \\ \partial \tilde{\mathbf{f}} / \partial \mathbf{p} \end{pmatrix} = \begin{pmatrix} 0 \\ \partial \tilde{\mathbf{f}} / \partial \mathbf{p} \end{pmatrix}.$$

This last gradient expresses the sensitivity of the model equations to the conditioning parameters. The $\partial \mathbf{f} / \partial \mathbf{p} = 0$ relationship is valid if the conditioning parameters do not appear in the original model equations. In general, our external information is imperfect, and the conditioning parameters may be in error. Let $\mathbf{p} = \mathbf{p}_o + \mathbf{p}_e$, where $\mathbf{p}_o$ are the "true" parameter values and $\mathbf{p}_e$ are the errors in the chosen parameters. Expanding the model equations about the "true" values of the location, data, and parameters $(\mathbf{x}_o, \mathbf{d}_o, \mathbf{p}_o)$, we have the linearized least squares problem

$$\min(\hat{\mathbf{f}}_o + \hat{\mathbf{A}}_o \mathbf{x}_{el} + \hat{\mathbf{G}}_o \mathbf{d}_e + \hat{\mathbf{U}}_o \mathbf{p}_e)^T \hat{\mathbf{W}} (\hat{\mathbf{f}}_o + \hat{\mathbf{A}}_o \mathbf{x}_{el} + \hat{\mathbf{G}}_o \mathbf{d}_e + \hat{\mathbf{U}}_o \mathbf{p}_e)],$$

where $\hat{\mathbf{W}} = \begin{pmatrix} \mathbf{W} & \tilde{\mathbf{W}}_\xi^T \\ \tilde{\mathbf{W}}_\zeta & \tilde{\mathbf{W}} \end{pmatrix}$ is a symmetric augmented weighting matrix for most applications. Since $\hat{\mathbf{f}}_o = 0$, we get

$$\mathbf{x}_{el} = -\hat{\mathbf{B}}_o^{-1} \hat{\mathbf{A}}_o{}^T \hat{\mathbf{W}} (\hat{\mathbf{G}}_o \mathbf{d}_e + \hat{\mathbf{U}}_o \mathbf{p}_e), \hat{\mathbf{B}}_o = \hat{\mathbf{A}}_o{}^T \hat{\mathbf{W}} \hat{\mathbf{A}}_o. \tag{7.1}$$

With $E\{\mathbf{d}_e\} = 0$, the linearized mean location errors are

$$E\{\mathbf{x}_{el}\} = -\hat{\mathbf{B}}_o^{-1} \hat{\mathbf{A}}_o^T \hat{\mathbf{W}} \hat{\mathbf{U}}_o \mathbf{p}_e. \tag{7.2}$$

In general, $E\{\mathbf{x}_{el}\} \neq 0$. Thus, conditioning tends to bias the least squares location solutions.

The location error variance (about the mean) for the conditioned system is

$$\mathbf{C}_{xl} = \hat{\mathbf{B}}_o^{-1}\hat{\mathbf{A}}_o^T\hat{\mathbf{W}}\hat{\mathbf{G}}_o\mathbf{C}_d\hat{\mathbf{G}}_o^T\hat{\mathbf{W}}\hat{\mathbf{A}}_o\hat{\mathbf{B}}_o^{-1}, \qquad (7.3)$$

where $\mathbf{C}_d = E\{\mathbf{d}_e^T\mathbf{d}_e\}$. The analysis above applies to least squares in general and not just to location algorithms.

In general, conditioning, if properly applied, can reduce the solution error covariance, and therefore the EP values, but perhaps at the expense of increasing the mean errors.

7.1 Conditioning for TOA (CTOA)

We now apply the methods above to a specific conditioner for the TOA location method. One possible piece of external information may be that the event is "near" the surface of the earth: that is, $x^2 + y^2 + z^2 \approx 1$. We therefore use the (single) conditioning equation

$$\sqrt{x^2 + y^2 + z^2} - r_c = 0, \qquad (7.4)$$

with the (single) condition parameter $r_c \geq 1$, and conditioning weight $w_c > 0$. Defining $r \triangleq \sqrt{x^2 + y^2 + z^2}$, the various conditioned matrices become

$$\hat{\mathbf{A}} = \begin{pmatrix} \mathbf{A} \\ x/r, y/r, z/r, 0 \end{pmatrix}, \ \hat{\mathbf{G}} = \begin{pmatrix} \mathbf{G} \\ 0 \end{pmatrix}, \ \hat{\mathbf{U}} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \text{ and } \hat{\mathbf{W}} = \begin{pmatrix} \mathbf{W} & 0 \\ 0 & w_c \end{pmatrix}. \qquad (7.5)$$

For TOA, $\mathbf{W} = \mathbf{I}$. The parameter error is $\mathbf{p}_e = r_c - r_o$. The linear location error mean is computed from equation (7.2), and the linear location error covariance from equation (7.3). Note that since $\hat{\mathbf{A}}$, $\hat{\mathbf{W}}$, and $\hat{\mathbf{G}}$ are independent of r_c, the error covariance does not depend on the conditioning parameter choice.

A spherical earth was used in this study. However, a nonspherical earth can easily be implemented as a conditioning equation.

Although conditioning can reduce EP values, perhaps while increasing mean errors, it has another potential use. Suppose, for example, an event is located within the earth, which is clearly impossible for TOA location, or at some other unacceptable site. Instead of discarding the result, one may recompute the solution using conditioning that would result in an acceptable site.

7.1.1 TOA Conditioning with TSQB Initialization (CBTOA)

The CTOA method may be implemented by solving the conditioned equations by Newton's method. However, an initial location guess may be avoided by solving the nonconditioned system by the TSQB method (with or without optimal weighting) and using that solution as the initial condition for Newton's method with the conditioned TOA equations.

7.1.2 Statistical Results for CBTOA

Monte Carlo statistics were generated for conditioned TOA with the code CTOAMON, which has the same input as MONTEC, plus r_c and w_c. The results are shown in Table 8(a through d). The ill-conditioned event/sensor scenarios from Table 1(a through c) were used as conditioned examples. For all these cases, the "true" solutions are $\mathbf{x}_o = (0,0,1,0)^T$. We chose $r_c = 1.001,$ as the parameter. Thus, the parameter error was $p_e = 0.001$, equivalent to 6.378 km. The data error standard deviation used was $\sigma = 0.001,$ as before. A relatively small conditioning weight, $w_c = 0.01,$ was used.

It was found that the EP errors were dramatically reduced in cases where the nonconditioned EP values were substantial, and less dramatically where the nonconditioned EP values were less. The vertical mean errors were not much affected in the former cases, but degraded somewhat in the latter cases. The mean errors in the x and y directions were not very much affected by the conditioning in any cases. In all cases, there was substantial agreement between the linear estimations and Monte Carlo results for both mean errors and EP values.

To estimate the effect of the conditioned weight, w_c, Table 8d was generated for the scenarios with six sensors for $w_c = 0.1.$ As would be expected, these scenarios tended to have lower unconditioned EP values than those with four or five sensors. The mean errors were about doubled, but the EP values were decreased by one-half to one order of magnitude. In general, the conditioned mean does not appear to be especially sensitive to the conditioning weight, and the EP values are not very sensitive to the parameter choice. However, as w_c increases, the EP values decrease, but the mean errors increase, and vice versa.

Some of the scenarios computed with code EPSCAN had very large normalized linear EP values and thus were not included in Table 1. For error values of $\sigma = 0.001,$ these scenarios would not converge for TOA and would offer complex roots in Bancroft's method. Table 9 gives Monte Carlo and linear statistics for these scenarios with conditioning. The CTOA method had to be used, since an initial TSQB would not produce an initial location. The unconditioned Monte Carlo mean values are not presented because they cannot be computed for nonconvergent situations. Not only do these scenarios converge with conditioning, but the EP value improvement is very dramatic.

7.2 Time-Difference-Squared Conditioning

The TSQ model equations may also be used for conditioning. In this case, we use

$$(x^2 + y^2 + z^2 - r_c^{\,2})/2 \tag{7.6}$$

as the augmenting equation. Performing a Newton iteration on the TSQ system with equation (7.6) added is straightforward, and the linear mean and covariance estimates follow from equations (7.2) and (7.3), with $\mathbf{U} = (0,..,0,-1)^T$ and $p_e = (r_c^2 - r_o^2)/2$.

However, a given conditioning weight in TSQ is not equivalent to the same conditioning weight in TOA. We have used the convention that the trace of the (nonconditioned) weighting matrix equal N, the number of sensors, for both TSQ and TOA. Therefore, for independent data errors with equal variance, the weighting matrix is $\mathbf{W} = \mathbf{I}$ for both (nonoptimized) TSQ and TOA. Since $\mathbf{G}_{TOA} = -\mathbf{I}$ and $\mathbf{G}_{TSQ} = diag(r_1,..,r_N)$, a TSQ conditioning weight of

$$w_{cq} \approx (\Sigma r_n^2 / N) w_c$$

is equivalent to the TOA conditioning weight of w_c . For our sensor geometry and near-earth events, $w_{cq} = 11 w_c$ should be a reasonable approximation.

7.2.1 Conditioned Iterated Bancroft Method – CISQB

There does not seem to be a noniterative method for introducing conditioning of the form of equation (7.6) into the Bancroft method (Section 7.2.2). The difficulty is that the last row of the augmented $\hat{\mathbf{A}}$ matrix is zero, causing $\hat{\mathbf{B}} = \hat{\mathbf{A}}^T \hat{\mathbf{W}} \hat{\mathbf{A}}$ to be identical to the nonaugmented **B**, thereby effectively deleting the conditioning equation from the result.

An iterative Bancroft method is as follows: Solve the nonconditioned problem, yielding an initial solution, $\mathbf{x}^0 = (x^0, y^0, z^0, \tau^0)$. Then use this solution to create a datum for a pseudo sensor of index N+1 at the center of the earth that senses an event on the conditioning surface and uses the event time of τ^0; that is, $d_{N+1}^0 = r_c - \tau^0$. Now the augmented system may be solved iteratively by Bancroft's method using the conditioning equation $f^i = [x^2 + y^2 + z^2 - (d_{N+1}^{i-1} - \tau)^2]$, where $d_{N+1}^i = r_c - \tau^i$, and $\mathbf{x}^i = (x^i, y^i, z^i, \tau^i)$ is the solution at the *i-th* iteration. The iteration ends when $\left|\mathbf{x}^i - \mathbf{x}^{i-1}\right| \le \varepsilon$. A numerical example of the procedure is shown in Table 10 for one scenario and value of r_c and w_{cq}. In this case, the process converged in five iterations. The final Monte Carlo mean and EP values were in close agreement with their linear values. As far as computation speed is concerned, we see little preference for the CISQB method over the CBTOA method.

7.2.2 Conditioned Bancroft Method with Parameter Iteration

A possible scheme for using Bancroft's method to apply conditioning "directly" is by a translation of one coordinate. It makes no difference which coordinate is translated. We arbitrarily choose the x to translate. To avoid excessive subscripting, let the "new" coordinate be denoted as x, where $x = x_{old} + \xi, \xi \neq 0$. The original and augmented TSQ equations become

$$f_n = [(x - x_n - \xi)^2 + (y - y_n)^2 + (z - z_n)^2 - (d_n - \tau)^2]/2,$$

$$f_{N+1} = [(x - \xi)^2 + y^2 + z^2 - r_c^2]/2.$$

There equations may be written in the form

$$f_n = p + q - (x_n + \xi)x - y_n y - z_n z + d_n \tau + c_n,$$

$$f_{N+1} = p - \xi x + (\xi^2 - r_c^2)/2 = p - \xi x + c_{N+1},$$

where

$$p = (x^2 + y^2 + z^2)/2, \; q = -\tau^2/2, \text{ and} \tag{7.7}$$

$$c_n = [(x_n + \xi)^2 + y_n^2 + z_n^2 - d_n^2]/2.$$

The conditioned system is solved for $\mathbf{x}$ as a linear combination of the parameters p and q:

$$\begin{gathered} \mathbf{x} = \mathbf{u}p + \mathbf{v}q + \mathbf{w}, \text{ where} \\ \mathbf{u} = -\hat{\mathbf{B}}^{-1}\hat{\mathbf{A}}^T\hat{\mathbf{W}}(1,..,1)^T = (u_x, u_y, u_z, u_\tau)^T, \\ \mathbf{v} = -\hat{\mathbf{B}}^{-1}\hat{\mathbf{A}}^T\hat{\mathbf{W}}(1,...,1,0)^T = (v_x, v_y, v_z, v_\tau)^T, \\ \mathbf{w} = -\hat{\mathbf{B}}^{-1}\hat{\mathbf{A}}^T\hat{\mathbf{W}}(c_1,...,c_{N+1})^T = (w_x, w_y, w_z, w_\tau)^T. \end{gathered} \tag{7.8}$$

Substituting equation (7.8) in equation (7.7) yields a pair of quadratics in p and q:

$$\alpha_1 p^2 + \beta_1 q^2 + \gamma_1 pq + \delta_1 p + \varepsilon_1 q + \varphi_1 = 0 \tag{7.9a}$$

$$\alpha_2 p^2 + \beta_2 q^2 + \gamma_2 pq + \delta_2 p + \varepsilon_2 q + \varphi_2 = 0. \tag{7.9b}$$

$$\begin{gathered} \alpha_1 = (u_x^2 + u_y^2 + u_z^2)/2, \; \alpha_2 = u_\tau^2/2, \\ \beta_1 = (v_x^2 + v_y^2 + v_z^2)/2, \; \beta_2 = v_\tau^2/2, \\ \gamma_1 = u_x v_x + u_y v_y + u_z v_z, \; \gamma_2 = u_\tau v_\tau, \\ \delta_1 = u_x w_x + u_y w_y + u_z w_z - 1, \; \delta_2 = v_\tau w_\tau, \\ \varepsilon_1 = v_x w_x + v_y w_y + v_z w_z, \; \varepsilon_2 = v_\tau w_\tau + 1, \end{gathered}$$

$$\varphi_1 = (w_x^2 + w_y^2 + w_z^2)/2, \ \varphi_2 = w_\tau^2/2.$$

Since $\alpha_1, \beta_1, \alpha_2, \beta_2 > 0$, equation (7.9) represents a pair of simultaneous ellipses, which, if they intersect, may be solved by Newton's method for p and q. Equation (7.9) is the "model" equations, and the gradient matrix is

$$\mathbf{A} = \begin{pmatrix} 2\alpha_1 p + \gamma_1 q + \delta_1 & 2\beta_1 q + \gamma_1 p + \varepsilon_1 \\ 2\alpha_2 p + \gamma_2 q + \delta_2 & 2\beta_2 q + \gamma_2 p + \varepsilon_2 \end{pmatrix}.$$

This technique was tested with the usual scenarios. Two values of ξ were used: $\xi = 0.1$ and $\xi = 1.5$ It appeared to work well for very small data errors, but with $\sigma = 0.001$, it was found that the ellipses did not intersect in many cases for both ξ values. We do not recommend the method.

8.0 SUMMARY

Section 2. In this section, we formulated the least squares method and described solution procedures for linear problems and Newton's iterative method for nonlinear problems.

Section 3. Formulae for linear approximations of the statistics of errors in least squares solutions because of errors in the data were presented.

Section 4. Specific relationships for various TOA location methods were shown. In particular, the TOA, TDOA, TSQ, and TSQB methods and their linear location error estimates were discussed. The TSQB method is a special, noniterative scheme to solve the TSQ problem. In the rest of the report, only the TOA and TSQB algorithms were studied.

Section 5. In this section, we applied the location algorithms to sets of "realistic" event/sensor scenarios. Statistics of NLEP (Normalized Linear Error Probable) were generated using a fixed set of sensors in various combinations and numbers and sampling NLEP on a longitude and latitude grid on the surface of a spherical earth and also on a sphere of radius 1.10 earth radii. The results, Table 1a through Table 2c, indicated which event/sensor combinations were the most ill conditioned, that is, had the greatest GDOP or, equivalently, the greatest values of NLEP. As had been proved in Section 4, optimally weighted TOA and TSQB (TSQ) had identical NLEP values. There were very little differences between optimal and nonoptimal TSQB weighting.

The event locations in each of the 24 sets (4, 5, and 6 sensors with four sensor set scenarios each and events at R = 1 and R = 1.10), which had the largest NLEP values, were studied further by Monte Carlo trials. These event locations would be expected to yield the largest location errors compared to any other locations for the given number of data-producing sensors and sensor set scenarios at the given radial distances and thereby offer "worst case" conditions for location algorithms.

In all cases, the data errors were taken as mutually independent random variables with mean zero and equal variance. A relatively large data error standard deviation of $\sigma = 0.001$, equivalent to 6.378 km or 21.26 μs, was used in the studies. Table 3 indicated that for the TOA method, the EP values for the Monte Carlo results were in close agreement with their linear estimates, LEP, and that the mean errors tended to be near zero, as predicted by the linear approximation. This result implies that the linear estimators for TOA are valid for all event/sensor scenarios, even with the relatively large data error variances.

Three data error probability distributions were considered: Gaussian, uniform, and a specialized "tailed" distribution. Table 4 shows that the distribution used makes little difference in the results if all distributions have zero mean and the same variance. For the remaining studies, the tailed distribution was used.

The TSQB algorithm, with optimal weighting, was tested under the same conditions as TOA. As Table 5 shows, the location mean and EP values also agree with their linear approximations. In general, the TSQB method is preferable to TOA, since TSQB requires neither an initial guess of the event location nor convergence criteria. Also, since TSQB is not iterative, it is computationally very fast. We point out two important considerations in the application of the TSQB method: Since the sensor data are "clock" time, it is judicious to bias the data in any algorithmic application. For TOA, subtracting the minimum data value from all the data works very well. However, this biasing level was found *not* to be good procedure for TSQB. We suggest biasing so that the smallest data measurement is biased to a number that represents a nominal distance from event to the sensor set, say, 3 earth radii for our system. The other consideration is the choice of the parameter derived from Bancroft's quadratic equation. It was found that choosing the parameter value of least *absolute* value gave the best results in every case. However, this result is heuristic and not proven.

Table 6 indicates that there is little difference in TSQB if nonoptimal weighting is used in lieu of optimal weighting. Therefore, at least for the conditions studied, one should not be concerned with optimizing the TSQB weighting.

Section 6. Linear programming location methods using L_1 and L_∞ norms were described in this section. Note that least squares is an L_2 norm method. The LP methods are quite slow and offered no significant improvements over TOA or TSQB. We do not recommend them as location algorithms.

Section 7. In cases where TOA or TSQB generate poor location results, it may be helpful to add weighted "conditioning" relationships to the location model equations. In general, conditioning affects mean as well as EP location errors. For our location problem, a potentially effective single conditioning relationship is to specify that the event be located "near" a sphere with specified radius from the center of the earth. However, conditioning can be applied to a nonspherical earth. Three conditioning algorithms were presented, CTOA, CBTOA, and CISQB. The CBTOA scheme is identical to CTOA, except that a TSQB step initializes the CTOA procedure, thereby avoiding an initial location guess. For this reason, CBTOA is preferred to CTOA, if the problem initially converges (does not create complex roots in the TSQB quadratic equation). If the initial problem does not converge, CTOA must be used. Formulas are presented for linear estimators of conditioned mean and EP errors for TOA and TSQ equations.

Table 8 gives location error statistics for the CBTOA method for a specific choice of the conditioning parameter and weight. The improvement in EP values is significant, without appreciable deterioration of mean errors. The improvements are especially significant in large NLEP cases, and not as significant in cases with smaller location errors, which implies that proper conditioning can mitigate cases that have large GDOP but should not adversely affect small GDOP situations. There is close agreement between the Monte Carlo results and the linearized estimates.

Some cases were found in the EPSCAN runs that had extreme NLEP values, in particular 35,893 in one case. These cases would not converge with TOA or TSQB for very small data errors. The CTOA method was applied to a selected group of three of these cases (CBTOA could not be used here). The results in Table 9 show dramatic improvement in all cases, even with a relatively small conditioning weight.

The CISQB method is an iterative TSQB type of scheme. One CISQB result is shown in Table 10. The EP reduction from nonconditioning is significant, and the results agree with their conditioned linear approximations.

In addition to reducing location EP values, and thereby improving the GDOP of a an event/sensor scenario, conditioning can generate a solution that could not be found without conditioning.

We could not find a conditioning method that uses Bancroft's algorithm in a direct, noniterative way.

The following conclusions were reached:

1. Because it is computationally very fast and does not require an initial location guess or convergence criteria, we recommend that the TSQB method be used for event location. Care must be exercised in data biasing and in the choice of the TSQB parameter. We suggest the heuristic that the minimum absolute value of the parameter be chosen. Also, nonoptimal TSQB weighting has little effect on the solutions.

2. For all algorithms, conditioned or not, the linear approximation of the mean and EP errors is valid for substantial data error variances, at least up to about 21 μs of standard deviation for "realistic" event/sensor scenarios. Thus, the LEP formulas can be used to obtain useful estimates of the system's accuracy under most conditions.

3. If properly chosen, conditioning can dramatically reduce EP values, especially in excessive GDOP cases and/or large data errors. In addition to reducing EP, conditioning can generate a reasonable solution where no convergence is possible without conditioning. Also, if a nonconditioned solution seems impossible (for example, an optical event located inside the earth) conditioning can generate a solution that is physically feasible.

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ACRONYMS

CBTOA	**C**onditioned **TOA** method with Bancroft initialization – Section 7.1.1.
CEP	**C**ircular **E**rror **P**robable, EP in plane tangent to the earth's surface.
CISQB	**C**onditioned **I**terative T**SQB** – Section 7.2.1.
CTOA	**C**onditioned **TOA** method – Section 7.1.
EP	**E**rror **P**robable. Refers to CEP, VEP, and TEP.
GDOP	**G**eometric **D**ilution **O**f **P**recision.
LEP	**L**inear **E**rror **P**robable. The least squares linear estimate of EP, derived from the linear location error covariance matrix equations (3.5) or (3.6).
LP	Linear Programming
LPTOA	**L**inear **P**rogramming solution of **TOA** – **S**ection 6.0.
NLEP	**N**ormalized **L**inear **E**rror **P**robable. LEP normalized for unity data error variance.
TDOA	**T**ime-**D**ifference-**O**f-**A**rrival. Location model equations (4.13).
TEP	**T**ime **E**rror **P**robable, EP for the time-of-event.
TOA	**T**ime-**O**f-**A**rrival. TOA refers to a set of location model equations (4.8) and also to Newton's method for location using these equations.
TSQ	**T**ime-of-arrival-**SQ**uared. A set of model equations (4.16) and Newton's method using these equations.
TSQB	Solution of the **TSQ** equations using **B**ancroft's method - Section 4.7.
TSQBN	Solution of **TSQB** with **N**onoptimal weighting.
VEP	**V**ertical **E**rror **P**robable.

Table 1a. NLEP for Four Sensors and R = 1.00

```
EPSCAN,  N= 4, R= 1.000000, DELP=  6.0,  1134 Samples
 ----- TOA   -----      --- NEAREST SET ---
 * Statistics adjusted by discarding    0 samples with VEP >   75.
 NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight
                 CEP            VEP            TEP
 Lon/min     212.680000      61.000000       6.100000
 Lat/min     -42.000000      -6.000000      -6.000000
 EPmin          .976235       1.465563       1.001415
 Lon/max     229.400000        .000000     229.400000
 Lat/max     -12.000000     -84.000000     -12.000000
 EPmax*       43.206744      74.635565      53.920697
 EPavg*        3.531862       8.787046       7.171941
 EPstd*        3.980962      10.765508       9.210869

 EPSCAN,  N= 4, R= 1.000000, DELP=  6.0,  1134 Samples
 ----- TSQB  -----      --- NEAREST SET ---
 * Statistics adjusted by discarding    0 samples with VEP >   75.
 NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight
                 CEP            VEP            TEP
 Lon/min     212.680000      61.000000       6.100000
 Lat/min     -42.000000      -6.000000      -6.000000
 EPmin          .976235       1.465563       1.001415
 Lon/max     229.400000        .000000     229.400000
 Lat/max     -12.000000     -84.000000     -12.000000
 EPmax*       43.206744      74.635565      53.920697
 EPavg*        3.531862       8.787046       7.171941
 EPstd*        3.980962      10.765508       9.210869
 NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight
 EPavgd*        .000000        .000000        .000000
 EPstdd*        .000000        .000000        .000000
```

Table 1a. NLEP for Four Sensors and R = 1.00 (continued)

```
EPSCAN,  N= 4, R= 1.000000, DELP=  6.0,  1134 Samples
----- TSQB  -----      --- FARTHEST SET ---
* Statistics adjusted by discarding   69 samples with VEP >   75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight
                 CEP             VEP             TEP
Lon/min     120.000000     333.000000      75.720000
Lat/min      36.000000     -48.000000      18.000000
EPmin          .793287       1.140995        .419307
Lon/max     206.460000        .000000     352.920000
Lat/max      24.000000     -24.000000     -30.000000
EPmax*       21.927524      74.175651      19.915536
EPavg*        2.187061      11.376386       2.359001
EPstd*        1.925266      11.285158       2.394701
NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight
EPavgd*        .000000        .000000        .000000
EPstdd*        .000000        .000000        .000000
```

```
 EPSCAN,  N= 4, R= 1.000000, DELP=  6.0,  1134 Samples
----- TSQB  -----      --- NEAR ONE + FAR SET ---
* Statistics adjusted by discarding    4 samples with VEP >   75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight
                 CEP             VEP             TEP
Lon/min     103.700000     162.000000      90.000000
Lat/min      -6.000000     -48.000000     -48.000000
EPmin          .853610        .802395        .405617
Lon/max      91.500000      37.860000      91.500000
Lat/max       6.000000     -18.000000       6.000000
EPmax*      113.169941      53.017047      84.579999
EPavg*        2.907456       2.561263       1.939925
EPstd*        6.499907       4.369759       4.480409
NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight
EPavgd*        .000000        .000000        .000000
EPstdd*        .000000        .000000        .000000
```

```
EPSCAN,  N= 4, R= 1.000000, DELP=  6.0,  1134 Samples
----- TSQB  -----      --- MIDDLE SET ---
* Statistics adjusted by discarding   46 samples with VEP >   75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight
                 CEP             VEP             TEP
Lon/min     162.000000     135.000000     352.980000
Lat/min     -48.000000     -48.000000      24.000000
EPmin          .825535       1.214084        .646262
Lon/max      30.000000     309.190000      30.000000
Lat/max      36.000000      18.000000      36.000000
EPmax*       28.832014      68.387525      38.985024
EPavg*        2.624439       9.440561       4.526676
EPstd*        2.750373      10.497048       4.691551
NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight
EPavgd*        .000000        .000000        .000000
EPstdd*        .000000        .000000        .000000
```

Table 1b. NLEP for Five Sensors and R = 1.00

EPSCAN, **N= 5, R= 1.000000**, DELP= 6.0, 1134 Samples
----- **TOA** ----- --- **NEAREST SET** ---
* Statistics adjusted by discarding 0 samples with VEP > 75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight

	CEP	VEP	TEP
Lon/min	229.040000	297.000000	315.500000
Lat/min	-42.000000	-48.000000	18.000000
EPmin	.833109	1.044303	.710642
Lon/max	113.580000	**.000000**	220.000000
Lat/max	18.000000	**-84.000000**	-72.000000
EPmax*	5.957163	**14.144089**	9.826735
EPavg*	1.466580	3.097139	2.300699
EPstd*	.661211	2.052586	1.666587

EPSCAN, **N= 5, R= 1.000000**, DELP= 6.0, 1134 Samples
----- **TSQB** ----- --- **NEAREST SET** ---
* Statistics adjusted by discarding 0 samples with VEP > 75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight

	CEP	VEP	TEP
Lon/min	229.040000	297.000000	315.500000
Lat/min	-42.000000	-48.000000	18.000000
EPmin	.833109	1.044303	.710642
Lon/max	113.580000	**.000000**	220.000000
Lat/max	18.000000	**-84.000000**	-72.000000
EPmax*	5.957163	**14.144089**	9.826735
EPavg*	1.466580	3.097139	2.300699
EPstd*	.661211	2.052586	1.666587
NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight			
EPavgd*	.001069	.002024	.001429
EPstdd*	.000788	.000780	.000948

Table 1b. NLEP for Five Sensors and R = 1.00 (continued)

EPSCAN, **N= 5, R= 1.000000**, DELP= 6.0, 1134 Samples
----- **TSQB** ----- --- **FARTHEST SET** ---
* Statistics adjusted by discarding 0 samples with VEP > 75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight

	CEP	VEP	TEP
Lon/min	173.160000	343.560000	241.800000
Lat/min	24.000000	-42.000000	12.000000
EPmin	.700651	1.001697	.385132
Lon/max	6.310000	**.000000**	.000000
Lat/max	-18.000000	**-24.000000**	-12.000000
EPmax*	8.807447	**58.478647**	17.196016
EPavg*	1.180918	5.055940	1.260743
EPstd*	.783493	5.265241	1.253424
NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight			
EPavgd*	.000536	.001510	.000432
EPstdd*	.000612	.001374	.000706

EPSCAN, **N= 5, R= 1.000000**, DELP= 6.0, 1134 Samples
----- **TSQB** ----- --- **NEAR ONE + FAR SET** ---
* Statistics adjusted by discarding 0 samples with VEP > 75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight

	CEP	VEP	TEP
Lon/min	317.200000	162.000000	112.500000
Lat/min	-6.000000	-48.000000	36.000000
EPmin	.750661	.780055	.354764
Lon/max	56.790000	**318.680000**	56.790000
Lat/max	-18.000000	**54.000000**	-18.000000
EPmax*	6.139107	**10.251104**	5.880698
EPavg*	1.084627	1.336901	.740144
EPstd*	.489518	.735161	.550384
NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight			
EPavgd*	.000510	.001964	.000536
EPstdd*	.000207	.000554	.000359

EPSCAN, **N= 5, R= 1.000000**, DELP= 6.0, 1134 Samples
----- **TSQB** ----- --- **MIDDLE SET** ---
* Statistics adjusted by discarding 0 samples with VEP > 75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight

	CEP	VEP	TEP
Lon/min	75.720000	294.480000	298.120000
Lat/min	-18.000000	-42.000000	-54.000000
EPmin	.734655	1.022985	.515243
Lon/max	353.800000	**296.570000**	353.800000
Lat/max	6.000000	**18.000000**	6.000000
EPmax*	11.914987	**43.520621**	15.787218
EPavg*	1.305782	3.853867	1.927413
EPstd*	.775559	3.180443	1.435798
NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight			
EPavgd*	.000911	.001757	.000888
EPstdd*	.000323	.000451	.000317

Table 1c. NLEP for Six Sensors and R = 1.00

```
EPSCAN,  N= 6, R= 1.000000, DELP=  6.0,  1134 Samples
----- TOA   -----      --- NEAREST SET ---
* Statistics adjusted by discarding    0 samples with VEP >   75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight
                 CEP            VEP            TEP
Lon/min     249.120000     299.700000     299.700000
Lat/min     -30.000000      24.000000      24.000000
EPmin          .740346        .808640        .478802
Lon/max     304.480000     300.000000     300.000000
Lat/max      30.000000      84.000000      84.000000
EPmax*        2.350413       5.957451       3.237534
EPavg*        1.010571       1.785338       1.187961
EPstd*         .192602        .636111        .392723

EPSCAN,  N= 6, R= 1.000000, DELP=  6.0,  1134 Samples
----- TSQB  -----      --- NEAREST SET ---
* Statistics adjusted by discarding    0 samples with VEP >   75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight
                 CEP            VEP            TEP
Lon/min     249.120000     299.700000     299.700000
Lat/min     -30.000000      24.000000      24.000000
EPmin          .740346        .808640        .478802
Lon/max     304.480000     300.000000     300.000000
Lat/max      30.000000      84.000000      84.000000
EPmax*        2.350413       5.957451       3.237534
EPavg*        1.010571       1.785338       1.187961
EPstd*         .192602        .636111        .392723
NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight
EPavgd*        .001561        .002609        .001530
EPstdd*        .000832        .000440        .000382
```

Table 1c. NLEP for Six Sensors and R = 1.00 (continued)

EPSCAN, **N= 6, R= 1.000000**, DELP= 6.0, 1134 Samples
----- **TSQB** ----- --- **FARTHEST SET** ---
* Statistics adjusted by discarding 0 samples with VEP > 75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight

	CEP	VEP	TEP
Lon/min	148.800000	324.000000	62.280000
Lat/min	12.000000	-48.000000	-30.000000
EPmin	.636610	.793542	.363982
Lon/max	192.200000	**297.600000**	189.100000
Lat/max	-12.000000	**12.000000**	-6.000000
EPmax*	3.289812	**15.463500**	4.116478
EPavg*	.859233	2.668175	.848252
EPstd*	.255711	1.865299	.513992
NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight			
EPavgd*	.001004	.002177	.000715
EPstdd*	.000288	.000296	.000345

EPSCAN, **N= 6, R= 1.000000**, DELP= 6.0, 1134 Samples
----- **TSQB** ----- --- **NEAR ONE + FAR SET** ---
* Statistics adjusted by discarding 0 samples with VEP > 75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight

	CEP	VEP	TEP
Lon/min	210.000000	162.000000	162.000000
Lat/min	78.000000	-48.000000	-48.000000
EPmin	.671358	.769469	.315988
Lon/max	315.000000	**315.000000**	315.000000
Lat/max	48.000000	**48.000000**	48.000000
EPmax*	3.601296	**6.420097**	3.792572
EPavg*	.844805	1.192407	.596376
EPstd*	.188160	.402712	.269518
NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight			
EPavgd*	.001050	.004237	.001227
EPstdd*	.000528	.000702	.000703

EPSCAN, **N= 6, R= 1.000000**, DELP= 6.0, 1134 Samples
----- **TSQB** ----- --- **MIDDLE SET** ---
* Statistics adjusted by discarding 0 samples with VEP > 75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight

	CEP	VEP	TEP
Lon/min	257.000000	298.120000	304.480000
Lat/min	-54.000000	-54.000000	30.000000
EPmin	.664910	.817661	.423035
Lon/max	201.300000	**189.100000**	201.300000
Lat/max	-6.000000	**6.000000**	-6.000000
EPmax*	3.505335	**9.022502**	4.273669
EPavg*	.977853	2.334121	1.212889
EPstd*	.322479	1.389456	.691445
NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight			
EPavgd*	.001502	.002566	.001258
EPstdd*	.000568	.000769	.000553

Table 2a. NLEP for Four Sensors and R = 1.10

EPSCAN, **N= 4, R= 1.100000**, DELP= 6.0, 1134 Samples

----- **TOA** ----- --- **NEAREST SET** ---

* Statistics adjusted by discarding 0 samples with VEP > 75.

NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight

	CEP	VEP	TEP
Lon/min	212.680000	61.000000	6.100000
Lat/min	-42.000000	-6.000000	-6.000000
EPmin	.959897	1.414253	.943666
Lon/max	189.000000	**.000000**	.000000
Lat/max	48.000000	-**84.000000**	-84.000000
EPmax*	41.815490	**71.899152**	49.374987
EPavg*	3.443274	8.435259	6.782081
EPstd*	3.760671	10.293118	8.660070

EPSCAN, **N= 4, R= 1.100000**, DELP= 6.0, 1134 Samples

----- **TSQB** ----- --- **NEAREST SET** ---

* Statistics adjusted by discarding 0 samples with VEP > 75.

NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight

	CEP	VEP	TEP
Lon/min	212.680000	61.000000	6.100000
Lat/min	-42.000000	-6.000000	-6.000000
EPmin	.959897	1.414253	.943666
Lon/max	189.000000	.**000000**	.000000
Lat/max	48.000000	**-84.000000**	-84.000000
EPmax*	41.815490	**71.899152**	49.374987
EPavg*	3.443274	8.435259	6.782081
EPstd*	3.760671	10.293118	8.660070

NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight

EPavgd*	.000000	.000000	.000000
EPstdd*	.000000	.000000	.000000

Table 2a. NLEP for Four Sensors and R = 1.10 (continued)

EPSCAN, **N= 4, R= 1.100000**, DELP= 6.0, 1134 Samples

----- **TSQB** ----- --- **FARTHEST SET** ---

* Statistics adjusted by discarding 111 samples with VEP > 75.

NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight

	CEP	VEP	TEP
Lon/min	341.000000	60.000000	335.500000
Lat/min	12.000000	-72.000000	-6.000000
EPmin	.777959	1.577753	.344967
Lon/max	333.000000	**37.200000**	318.320000
Lat/max	24.000000	-**12.000000**	-30.000000
EPmax*	21.357906	**73.987646**	26.006126
EPavg*	2.366230	14.139197	3.894761
EPstd*	2.180054	13.828252	3.970494

NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight

EPavgd*	.000000	.000000	.000000
EPstdd*	.000000	.000000	.000000

EPSCAN, N= 4, R= 1.100000, DELP= 6.0, 1134 Samples

----- **TSQB** ----- --- **NEAR ONE + FAR SET** ---

* Statistics adjusted by discarding 2 samples with VEP > 75.

NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight

	CEP	VEP	TEP
Lon/min	352.920000	256.040000	.000000
Lat/min	-30.000000	30.000000	-36.000000
EPmin	.884115	.566432	.336498
Lon/max	120.000000	**120.000000**	120.000000
Lat/max	-72.000000	-**72.000000**	-72.000000
EPmax*	94.836900	**45.524977**	43.447406
EPavg*	3.049585	1.872826	1.411355
EPstd*	5.902331	3.650363	3.291706

NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight

EPavgd*	.000000	.000000	.000000
EPstdd*	.000000	.000000	.000000

EPSCAN, N**= 4, R= 1.100000**, DELP= 6.0, 1134 Samples

----- **TSQB** ----- --- **MIDDLE SET** ---

* Statistics adjusted by discarding 42 samples with VEP > 75.

NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight

	CEP	VEP	TEP
Lon/min	.000000	253.080000	269.880000
Lat/min	-30.000000	24.000000	30.000000
EPmin	.794432	1.257440	.343650
Lon/max	252.000000	**252.000000**	279.000000
Lat/max	48.000000	**48.000000**	-12.000000
EPmax*	31.455601	**69.250876**	18.094557
EPavg*	2.280933	10.258476	2.169835
EPstd*	2.152627	9.872375	2.459601

NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight

EPavgd*	.000000	.000000	.000000
EPstdd*	.000000	.000000	.000000

Table 2b. NLEP for Five Sensors and R = 1.10

```
EPSCAN,  N= 5, R= 1.100000, DELP=  6.0,  1134 Samples
----- TOA   -----      --- NEAREST SET ---
* Statistics adjusted by discarding    0 samples with VEP >   75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight
                 CEP            VEP            TEP
Lon/min     229.040000     288.000000     315.500000
Lat/min     -42.000000     -48.000000      18.000000
EPmin          .820978       1.010204        .672749
Lon/max     113.580000        .000000     220.000000
Lat/max      18.000000     -84.000000     -72.000000
EPmax*        5.812450      13.623901       9.253322
EPavg*        1.437730       2.987784       2.179298
EPstd*         .650206       1.998602       1.585718

EPSCAN,  N= 5, R= 1.100000, DELP=  6.0,  1134 Samples
----- TSQB  -----      --- NEAREST SET ---
* Statistics adjusted by discarding    0 samples with VEP >   75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight
                 CEP            VEP            TEP
Lon/min     229.040000     288.000000     315.500000
Lat/min     -42.000000     -48.000000      18.000000
EPmin          .820978       1.010204        .672749
Lon/max     113.580000        .000000     220.000000
Lat/max      18.000000     -84.000000     -72.000000
EPmax*        5.812450      13.623901       9.253322
EPavg*        1.437730       2.987784       2.179298
EPstd*         .650206       1.998602       1.585718
NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight
EPavgd*        .001375        .002558        .001771
EPstdd*        .001053        .001066        .001248
```

Table 2b. NLEP for Five Sensors and R = 1.10 (continued)

```
EPSCAN,  N= 5, R= 1.100000, DELP=  6.0,  1134 Samples
----- TSQB  -----       --- FARTHEST SET ---
* Statistics adjusted by discarding    2 samples with VEP >   75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight
                 CEP            VEP            TEP
Lon/min     161.200000      15.000000     216.000000
Lat/min      12.000000     -66.000000     -60.000000
EPmin          .688797       1.222677        .314325
Lon/max     140.300000     195.000000     195.000000
Lat/max       6.000000     -36.000000     -36.000000
EPmax*        7.714035      50.849013      14.949978
EPavg*        1.220277       6.250337       1.563049
EPstd*         .788584       5.668270       1.566608
NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight
EPavgd*        .000401        .001275        .000268
EPstdd*        .000299        .000520        .000137

EPSCAN,  N= 5, R= 1.100000, DELP=  6.0,  1134 Samples
----- TSQB  -----       --- NEAR ONE + FAR SET ---
* Statistics adjusted by discarding    0 samples with VEP >   75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight
                 CEP            VEP            TEP
Lon/min     347.200000     256.040000     126.540000
Lat/min      12.000000      30.000000     -24.000000
EPmin          .763774        .555285        .300975
Lon/max     302.880000     302.880000     302.880000
Lat/max      18.000000      18.000000      18.000000
EPmax*       11.136744       5.612180       7.134511
EPavg*        1.152361        .881691        .493340
EPstd*         .697970        .532173        .441578
NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight
EPavgd*        .000406        .001616        .000145
EPstdd*        .000440        .000122        .000314

EPSCAN,  N= 5, R= 1.100000, DELP=  6.0,  1134 Samples
----- TSQB  -----       --- MIDDLE SET ---
* Statistics adjusted by discarding    0 samples with VEP >   75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight
                 CEP            VEP            TEP
Lon/min     332.160000     254.200000     273.060000
Lat/min      30.000000      12.000000      24.000000
EPmin          .697513        .940891        .303013
Lon/max     322.500000      55.360000     277.640000
Lat/max     -36.000000     -30.000000     -18.000000
EPmax*        5.578667      33.521382       8.613396
EPavg*        1.122675       4.249704        .876287
EPstd*         .528513       3.449485        .683318
NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight
EPavgd*        .000718        .001853        .000339
EPstdd*        .000508        .000845        .000252
```

Table 2c. NLEP for Six Sensors and R = 1.10

```
EPSCAN,  N= 6, R= 1.100000, DELP=  6.0,  1134 Samples
----- TOA   -----       --- NEAREST SET ---
* Statistics adjusted by discarding    0 samples with VEP >   75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight
                 CEP            VEP            TEP
Lon/min     249.120000     299.700000     299.700000
Lat/min     -30.000000      24.000000      24.000000
EPmin          .730951        .788717        .459847
Lon/max     304.480000     300.000000     300.000000
Lat/max      30.000000      84.000000      84.000000
EPmax*        2.308296       5.829072       3.050950
EPavg*         .991996       1.723551       1.121449
EPstd*         .186974        .620153        .369062

EPSCAN,  N= 6, R= 1.100000, DELP=  6.0,  1134 Samples
----- TSQB  -----       --- NEAREST SET ---
* Statistics adjusted by discarding    0 samples with VEP >   75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight
                 CEP            VEP            TEP
Lon/min     249.120000     299.700000     299.700000
Lat/min     -30.000000      24.000000      24.000000
EPmin          .730951        .788717        .459847
Lon/max     304.480000     300.000000     300.000000
Lat/max      30.000000      84.000000      84.000000
EPmax*        2.308296       5.829072       3.050950
EPavg*         .991996       1.723551       1.121449
EPstd*         .186974        .620153        .369062
NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight
EPavgd*        .001978        .003261        .001847
EPstdd*        .001025        .000543        .000470
```

Table 2c. NLEP for Six Sensors and R = 1.10 (continued)

EPSCAN, **N= 6, R= 1.100000**, DELP= 6.0, 1134 Samples
----- **TSQB** ----- --- **FARTHEST SET** ---
* Statistics adjusted by discarding 0 samples with VEP > 75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight

	CEP	VEP	TEP
Lon/min	179.800000	60.000000	60.000000
Lat/min	-12.000000	-78.000000	-78.000000
EPmin	.634389	1.020526	.274243
Lon/max	138.000000	**16.360000**	16.360000
Lat/max	.000000	**42.000000**	42.000000
EPmax*	3.561343	**20.319520**	6.349617
EPavg*	.861498	3.438580	.809109
EPstd*	.297488	2.241193	.643157
NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight			
EPavgd*	.000785	.001947	.000337
EPstdd*	.000177	.000356	.000167

EPSCAN, **N= 6, R= 1.100000**, DELP= 6.0, 1134 Samples
----- **TSQB** ----- --- **NEAR ONE + FAR SET** ---
* Statistics adjusted by discarding 0 samples with VEP > 75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight

	CEP	VEP	TEP
Lon/min	279.720000	256.040000	201.920000
Lat/min	-24.000000	30.000000	18.000000
EPmin	.672817	.549722	.274464
Lon/max	243.000000	**310.000000**	243.000000
Lat/max	48.000000	**-12.000000**	48.000000
EPmax*	3.965208	**3.921964**	2.207646
EPavg*	.837653	.753566	.351299
EPstd*	.246380	.253499	.154661
NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight			
EPavgd*	.000847	.004536	.000232
EPstdd*	.000214	.000149	.000132

EPSCAN, **N= 6, R= 1.100000**, DELP= 6.0, 1134 Samples
----- **TSQB** ----- --- **MIDDLE SET** ---
* Statistics adjusted by discarding 0 samples with VEP > 75.
NLEP-Normalized Linear Error Probable - "OPTIMAL" Weight

	CEP	VEP	TEP
Lon/min	352.920000	249.120000	319.680000
Lat/min	30.000000	30.000000	24.000000
EPmin	.635979	.804587	.275354
Lon/max	84.000000	**302.880000**	51.400000
Lat/max	60.000000	**18.000000**	-54.000000
EPmax*	3.449705	**14.920299**	2.947787
EPavg*	.833379	2.469557	.572369
EPstd*	.224294	1.430775	.264936
NLEP-Normalized Linear Error Probable - NON-OPTIMAL Weight			
EPavgd*	.001257	.002648	.000445
EPstdd*	.000587	.001122	.000256

Table 3a. EP for the TOA Method, R = 1.00

```
 MONTE CARLO  N= 4  R= 1.000000 Lon=    .000 Lat= -84.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  1 *** NEAREST      *** Tailed                    #4/1.00/1
 ----TOA  ----      4.8232       .3000
Linear   CEP=     .009648 VEP=     .074636 TEP=     .052399
  MC       CEP=     .009684 VEP=     .075019 TEP=     .052709
  MC Mean      -.000037     .000151    -.000787    -.001514

 MONTE CARLO  N= 4  R= 1.000000 Lon=    .000 Lat= -24.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  2 *** FARTHEST     *** Tailed                    #4/1.00/2
 ----TOA  ----      4.6208       .3000
  Linear   CEP=     .008602 VEP=     .074176 TEP=     .010797
  MC       CEP=     .008610 VEP=     .074262 TEP=     .011125
  MC Mean      -.000156     .000062    -.001076    -.001713

 MONTE CARLO  N= 4  R= 1.000000 Lon=  37.860 Lat= -18.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  3 *** NEAR ONE+FAR *** Tailed                    #4/1.00/3
 ----TOA  ----      4.3968       .3000
Linear   CEP=     .030489 VEP=     .053017 TEP=     .032360
  MC       CEP=     .030354 VEP=     .052790 TEP=     .032200
  MC Mean       .000087     .000583    -.000814    -.001356

 MONTE CARLO  N= 4  R= 1.000000 Lon= 309.190 Lat=  18.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  4 *** MIDDLE       *** Tailed                    #4/1.00/4
 ----TOA  ----      4.6534       .3000
Linear   CEP=     .010310 VEP=     .068388 TEP=     .024297
  MC       CEP=     .010287 VEP=     .068229 TEP=     .024262
  MC Mean       .000102    -.000005     .001078    -.000873

 MONTE CARLO  N= 5  R= 1.000000 Lon=    .000 Lat= -84.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  1 *** NEAREST      *** Tailed                    #5/1.00/1
 ----TOA  ----      4.3352       .3000
Linear   CEP=     .002685 VEP=     .014144 TEP=     .009723
  MC       CEP=     .002663 VEP=     .014060 TEP=     .009667
  MC Mean       .000003     .000011     .000079     .000010

 MONTE CARLO  N= 5  R= 1.000000 Lon=    .000 Lat= -24.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  2 *** FARTHEST     *** Tailed                    #5/1.00/2
 ----TOA  ----      5.2191       .3000
Linear   CEP=     .006497 VEP=     .058479 TEP=     .008403
  MC       CEP=     .006497 VEP=     .058383 TEP=     .008505
  MC Mean      -.000165     .000069    -.001207    -.001140
```

Table 3a. EP for the TOA Method, R = 1.00 (continued)

```
 MONTE CARLO  N= 5  R= 1.000000 Lon= 318.680 Lat=  54.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  3 *** NEAR ONE+FAR *** Tailed                    #5/1.00/3
 ----TOA  ----      4.3601       .3000
Linear   CEP=     .004179 VEP=     .010251 TEP=     .005483
  MC       CEP=     .004141 VEP=     .010195 TEP=     .005448
  MC Mean       .000001     .000037     .000139     .000034

 MONTE CARLO  N= 5  R= 1.000000 Lon= 296.570 Lat=  18.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  4 *** MIDDLE       *** Tailed                    #5/1.00/4
 ----TOA  ----      5.5298       .3000
Linear   CEP=     .003326 VEP=     .043521 TEP=     .006887
  MC       CEP=     .003321 VEP=     .043493 TEP=     .006935
  MC Mean      -.000022    -.000111    -.001135    -.000715

 MONTE CARLO  N= 6  R= 1.000000 Lon= 300.000 Lat=  84.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  1 *** NEAREST      *** Tailed                    #6/1.00/1
 ----TOA  ----      4.0798       .3000
Linear   CEP=     .001077 VEP=     .005957 TEP=     .003238
  MC       CEP=     .001083 VEP=     .005987 TEP=     .003254
  MC Mean       .000008    -.000026     .000171     .000075

 MONTE CARLO  N= 6  R= 1.000000 Lon= 297.600 Lat=  12.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  2 *** FARTHEST     *** Tailed                    #6/1.00/2
 ----TOA  ----      4.3111       .3000
Linear   CEP=     .000957 VEP=     .015464 TEP=     .001296
  MC       CEP=     .000953 VEP=     .015412 TEP=     .001292
  MC Mean       .000003     .000003     .000189    -.000058

 MONTE CARLO  N= 6  R= 1.000000 Lon= 315.000 Lat=  48.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  3 *** NEAR ONE+FAR *** Tailed                    #6/1.00/3
 ----TOA  ----      4.0322       .3000
Linear   CEP=     .003601 VEP=     .006420 TEP=     .003793
  MC       CEP=     .003579 VEP=     .006386 TEP=     .003769
  MC Mean       .000008    -.000003     .000000    -.000020

 MONTE CARLO  N= 6  R= 1.000000 Lon= 189.100 Lat=   6.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  4 *** MIDDLE       *** Tailed                    #6/1.00/4
 ----TOA  ----      4.2064       .3000
Linear   CEP=     .001180 VEP=     .009023 TEP=     .004070
  MC       CEP=     .001174 VEP=     .008947 TEP=     .004032
  MC Mean      -.000015     .000006     .000102     .000019
```

Table 3b. EP for the TOA Method, R = 1.10

```
 MONTE CARLO  N= 4  R= 1.100000 Lon=    .000 Lat= -84.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  1 *** NEAREST      *** Tailed                  #4/1.10/1
----TOA  ----      4.8335      .3000
  Linear   CEP=     .009433 VEP=     .071899 TEP=     .049375
  MC       CEP=     .009467 VEP=     .072260 TEP=     .049658
  MC Mean      -.000033     .000137    -.000638    -.001387

 MONTE CARLO  N= 4  R= 1.100000 Lon=  37.200 Lat= -12.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  2 *** FARTHEST     *** Tailed                  #4/1.10/2
 ----TOA  ----      4.7397      .3000
Linear   CEP=     .010335 VEP=     .073988 TEP=     .021731
  MC       CEP=     .010308 VEP=     .073837 TEP=     .021832
  MC Mean      -.000019     .000023     .000219    -.001370

 MONTE CARLO  N= 4  R= 1.100000 Lon= 120.000 Lat= -72.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  3 *** NEAR ONE+FAR *** Tailed                  #4/1.10/3
----TOA  ----      4.3879      .3000
  Linear   CEP=     .094837 VEP=     .045525 TEP=     .043447
  MC       CEP=     .094554 VEP=     .045383 TEP=     .043465
  MC Mean      -.002644    -.001334    -.000696    -.003898

 MONTE CARLO  N= 4  R= 1.100000 Lon= 252.000 Lat=  48.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  4 *** MIDDLE       *** Tailed                  #4/1.10/4
----TOA  ----      4.7282      .3000
  Linear   CEP=     .031456 VEP=     .069251 TEP=     .007511
  MC       CEP=     .031699 VEP=     .069750 TEP=     .007801
  MC Mean       .000304    -.000520    -.000859    -.001749

 MONTE CARLO  N= 5  R= 1.100000 Lon=    .000 Lat= -84.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  1 *** NEAREST      *** Tailed                  #5/1.10/1
 ----TOA  ----      4.3377      .3000
Linear   CEP=     .002625 VEP=     .013624 TEP=     .009149
  MC       CEP=     .002604 VEP=     .013543 TEP=     .009096
  MC Mean       .000003     .000010     .000082     .000012

 MONTE CARLO  N= 5  R= 1.100000 Lon= 195.000 Lat= -36.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  2 *** FARTHEST     *** Tailed                  #5/1.10/2
 ----TOA  ----      5.1223      .3000
Linear   CEP=     .003134 VEP=     .050849 TEP=     .014950
  MC       CEP=     .003101 VEP=     .050420 TEP=     .014923
  MC Mean       .000024     .000019     .000978    -.000895
```

Table 3b. EP for the TOA Method, R = 1.10 (continued)

```
 MONTE CARLO  N= 5  R= 1.100000 Lon= 302.880 Lat=  18.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  3 *** NEAR ONE+FAR *** Tailed                   #5/1.10/3
----TOA  ----      4.3539       .3000
  Linear   CEP=      .011137 VEP=      .005612 TEP=      .007135
  MC       CEP=      .010948 VEP=      .005513 TEP=      .007014
  MC Mean        .000006      .000056      .000055      .000002

 MONTE CARLO  N= 5  R= 1.100000 Lon=  55.360 Lat= -30.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  4 *** MIDDLE       *** Tailed                   #5/1.10/4
----TOA  ----      5.1297       .3000
  Linear   CEP=      .002311 VEP=      .033521 TEP=      .001193
  MC       CEP=      .002325 VEP=      .033841 TEP=      .001232
  MC Mean        .000032      .000003      .000753     -.000300

 MONTE CARLO  N= 6  R= 1.100000 Lon= 300.000 Lat=  84.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  1 *** NEAREST      *** Tailed                   #6/1.10/1
----TOA  ----      4.0814       .3000
  Linear   CEP=      .001063 VEP=      .005829 TEP=      .003051
  MC       CEP=      .001068 VEP=      .005858 TEP=      .003066
  MC Mean        .000008     -.000026      .000168      .000070

 MONTE CARLO  N= 6  R= 1.100000 Lon=  16.360 Lat=  42.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  2 *** FARTHEST     *** Tailed                   #6/1.10/2
----TOA  ----      4.6049       .3000
  Linear   CEP=      .001532 VEP=      .020320 TEP=      .006350
  MC       CEP=      .001544 VEP=      .020377 TEP=      .006370
  MC Mean        .000039     -.000014      .000551     -.000275

 MONTE CARLO  N= 6  R= 1.100000 Lon= 310.000 Lat= -12.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  3 *** NEAR ONE+FAR *** Tailed                   #6/1.10/3
----TOA  ----      3.9654       .3000
  Linear   CEP=      .003235 VEP=      .003922 TEP=      .000868
  MC       CEP=      .003252 VEP=      .003938 TEP=      .000877
  MC Mean       -.000004      .000018      .000022     -.000011

 MONTE CARLO  N= 6  R= 1.100000 Lon= 302.880 Lat=  18.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  4 *** MIDDLE       *** Tailed                   #6/1.10/4
----TOA  ----      4.2078       .3000
  Linear   CEP=      .001445 VEP=      .014920 TEP=      .001182
  MC       CEP=      .001450 VEP=      .014975 TEP=      .001187
  MC Mean       -.000013     -.000032      .000404     -.000041
```

Table 4. EP for TOA Using Various Data Error Distributions

```
MONTE CARLO  N= 4  R= 1.000000 Lon=    .000 Lat= -84.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  1 *** NEAREST      *** Gaussian                 #4/1.00/1
 ----TOA  ----      4.8294       .3000
Linear   CEP=     .009648 VEP=     .074636 TEP=     .052399
  MC       CEP=     .009705 VEP=     .075144 TEP=     .052815
  MC Mean      -.000042     .000169    -.000947    -.001629

 MONTE CARLO  N= 4  R= 1.000000 Lon=    .000 Lat= -84.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  1 *** NEAREST      *** Uniform                  #4/1.00/1
----TOA  ----      4.8321       .3000
  Linear   CEP=     .009648 VEP=     .074636 TEP=     .052399
  MC       CEP=     .009638 VEP=     .074556 TEP=     .052412
  MC Mean      -.000089     .000392    -.002863    -.002952

 MONTE CARLO  N= 4  R= 1.000000 Lon=    .000 Lat= -84.000
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  1 *** NEAREST      *** Tailed                   #4/1.00/1
----TOA  ----      4.8232       .3000
  Linear   CEP=     .009648 VEP=     .074636 TEP=     .052399
  MC       CEP=     .009684 VEP=     .075019 TEP=     .052709
  MC Mean      -.000037     .000151    -.000787    -.001514
```

Table 5a. EP for the TSQB Method, R = 1.00
(Perfect Solution Choice – Optimal Weighting)

```
MONTE CARLO  N= 4  R= 1.000000 Lon=    .000 Lat= -84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #4/1.00/1
----TSQB ----1     0.     0.
 Linear   CEP=     .009648 VEP=     .074636 TEP=     .052399
 MC       CEP=     .009684 VEP=     .075019 TEP=     .052709
 MC Mean      -.000037     .000151    -.000787    -.001514

MONTE CARLO  N= 4  R= 1.000000 Lon=    .000 Lat= -24.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                  #4/1.00/2
----TSQB ----1     0.     0.
 Linear   CEP=     .008602 VEP=     .074176 TEP=     .010797
 MC       CEP=     .008610 VEP=     .074262 TEP=     .011125
 MC Mean      -.000156     .000062    -.001076    -.001713

MONTE CARLO  N= 4  R= 1.000000 Lon=  37.860 Lat= -18.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #4/1.00/3
----TSQB ----1     0.     0.
 Linear   CEP=     .030489 VEP=     .053017 TEP=     .032360
 MC       CEP=     .030354 VEP=     .052790 TEP=     .032200
 MC Mean       .000087     .000583    -.000814    -.001356

MONTE CARLO  N= 4  R= 1.000000 Lon= 309.190 Lat=  18.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #4/1.00/4
----TSQB ----1     0.     0.
 Linear   CEP=     .010310 VEP=     .068388 TEP=     .024297
 MC       CEP=     .010287 VEP=     .068229 TEP=     .024262
 MC Mean       .000102    -.000005     .001078    -.000873

MONTE CARLO  N= 5  R= 1.000000 Lon=    .000 Lat= -84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #5/1.00/1
----TSQB ----1     0.     0.
 Linear   CEP=     .002685 VEP=     .014144 TEP=     .009723
 MC       CEP=     .002664 VEP=     .014063 TEP=     .009668
 MC Mean       .000007    -.000016     .000226     .000111

MONTE CARLO  N= 5  R= 1.000000 Lon=    .000 Lat= -24.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                  #5/1.00/2
----TSQB ----1     0.     0.
 Linear   CEP=     .006497 VEP=     .058479 TEP=     .008403
 MC       CEP=     .006489 VEP=     .058326 TEP=     .008458
 MC Mean       .000011    -.000011     .000541    -.000887
```

Table 5a. EP for the TSQB Method, R = 1.00
(Perfect Solution Choice – Optimal Weighting) (continued)

```
MONTE CARLO  N= 5  R= 1.000000 Lon= 318.680 Lat=  54.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #5/1.00/3
----TSQB ----1     0.     0.
 Linear   CEP=     .004179 VEP=     .010251 TEP=     .005483
 MC       CEP=     .004163 VEP=     .010253 TEP=     .005478
 MC Mean      -.000001     .000050     .000172     .000051

MONTE CARLO  N= 5  R= 1.000000 Lon= 296.570 Lat=  18.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #5/1.00/4
----TSQB ----1     0.     0.
 Linear   CEP=     .003326 VEP=     .043521 TEP=     .006887
 MC       CEP=     .003430 VEP=     .045001 TEP=     .007156
 MC Mean      -.000003    -.000047    -.000237    -.000610

MONTE CARLO  N= 6  R= 1.000000 Lon= 300.000 Lat=  84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #6/1.00/1
----TSQB ----1     0.     0.
 Linear   CEP=     .001077 VEP=     .005957 TEP=     .003238
 MC       CEP=     .001084 VEP=     .006004 TEP=     .003263
 MC Mean       .000010    -.000030     .000205     .000094

MONTE CARLO  N= 6  R= 1.000000 Lon= 297.600 Lat=  12.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                  #6/1.00/2
----TSQB ----1     0.     0.
 Linear   CEP=     .000957 VEP=     .015464 TEP=     .001296
 MC       CEP=     .000956 VEP=     .015487 TEP=     .001297
 MC Mean       .000006     .000008     .000303    -.000050

MONTE CARLO  N= 6  R= 1.000000 Lon= 315.000 Lat=  48.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #6/1.00/3
----TSQB ----1     0.     0.
 Linear   CEP=     .003601 VEP=     .006420 TEP=     .003793
 MC       CEP=     .003588 VEP=     .006402 TEP=     .003778
 MC Mean       .000007     .000014     .000030    -.000002

MONTE CARLO  N= 6  R= 1.000000 Lon= 189.100 Lat=   6.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #6/1.00/4
----TSQB ----1     0.     0.
 Linear   CEP=     .001180 VEP=     .009023 TEP=     .004070
 MC       CEP=     .001176 VEP=     .008965 TEP=     .004040
 MC Mean      -.000023     .000002     .000177     .000053
```

Table 5b. EP for the TSQB Method, R = 1.10
(Perfect Solution Choice – Optimal Weighting)

```
MONTE CARLO  N= 4  R= 1.100000 Lon=    .000 Lat= -84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #4/1.10/1
----TSQB ----1     0.     0.
  Linear   CEP=      .009433 VEP=     .071899 TEP=     .049375
 MC       CEP=      .009467 VEP=     .072260 TEP=     .049658
 MC Mean      -.000033     .000137    -.000638    -.001387

MONTE CARLO  N= 4  R= 1.100000 Lon=  37.200 Lat= -12.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                  #4/1.10/2
----TSQB ----1     0.     0.
 Linear   CEP=      .010335 VEP=     .073988 TEP=     .021731
 MC       CEP=      .010511 VEP=     .073796 TEP=     .061708
 MC Mean       .000001    -.000010     .000236    -.000522

MONTE CARLO  N= 4  R= 1.100000 Lon= 120.000 Lat= -72.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #4/1.10/3
----TSQB ----1     0.     0.
 Linear   CEP=      .094837 VEP=     .045525 TEP=     .043447
 MC       CEP=      .094554 VEP=     .045383 TEP=     .043465
 MC Mean      -.002644    -.001334    -.000696    -.003898

MONTE CARLO  N= 4  R= 1.100000 Lon= 252.000 Lat=  48.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #4/1.10/4
----TSQB ----1     0.     0.
 Linear   CEP=      .031456 VEP=     .069251 TEP=     .007511
 MC       CEP=      .031699 VEP=     .069750 TEP=     .007801
 MC Mean       .000304    -.000520    -.000859    -.001749

MONTE CARLO  N= 5  R= 1.100000 Lon=    .000 Lat= -84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #5/1.10/1
----TSQB ----1     0.     0.
 Linear   CEP=      .002625 VEP=     .013624 TEP=     .009149
 MC       CEP=      .002604 VEP=     .013546 TEP=     .009098
 MC Mean       .000007    -.000015     .000219     .000104

MONTE CARLO  N= 5  R= 1.100000 Lon= 195.000 Lat= -36.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                  #5/1.10/2
----TSQB ----1     0.     0.
 Linear   CEP=      .003134 VEP=     .050849 TEP=     .014950
 MC       CEP=      .003103 VEP=     .050460 TEP=     .014934
 MC Mean       .000019     .000016     .000882    -.000867
```

Table 5b. EP for the TSQB Method, R = 1.10
(Perfect Solution Choice – Optimal Weighting) (continued)

```
MONTE CARLO  N= 5  R= 1.100000 Lon= 302.880 Lat=  18.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #5/1.10/3
----TSQB ----1     0.     0.
 Linear   CEP=     .011137 VEP=     .005612 TEP=     .007135
 MC       CEP=     .010978 VEP=     .005528 TEP=     .007033
 MC Mean       .000007     .000086     .000069     .000021

MONTE CARLO  N= 5  R= 1.100000 Lon=  55.360 Lat= -30.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #5/1.10/4
----TSQB ----1     0.     0.
 Linear   CEP=     .002311 VEP=     .033521 TEP=     .001193
 MC       CEP=     .002378 VEP=     .034688 TEP=     .001260
 MC Mean       .000060     .000004     .001168    -.000302

MONTE CARLO  N= 6  R= 1.100000 Lon= 300.000 Lat=  84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #6/1.10/1
----TSQB ----1     0.     0.
 Linear   CEP=     .001063 VEP=     .005829 TEP=     .003051
 MC       CEP=     .001070 VEP=     .005875 TEP=     .003075
 MC Mean       .000010    -.000030     .000200     .000087

MONTE CARLO  N= 6  R= 1.100000 Lon=  16.360 Lat=  42.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                  #6/1.10/2
----TSQB ----1     0.     0.
 Linear   CEP=     .001532 VEP=     .020320 TEP=     .006350
 MC       CEP=     .001544 VEP=     .020376 TEP=     .006370
 MC Mean       .000036    -.000013     .000515    -.000263

MONTE CARLO  N= 6  R= 1.100000 Lon= 310.000 Lat= -12.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #6/1.10/3
----TSQB ----1     0.     0.
 Linear   CEP=     .003235 VEP=     .003922 TEP=     .000868
 MC       CEP=     .003252 VEP=     .003938 TEP=     .000877
 MC Mean      -.000005     .000024     .000030    -.000009

MONTE CARLO  N= 6  R= 1.100000 Lon= 302.880 Lat=  18.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #6/1.10/4
----TSQB ----1     0.     0.
 Linear   CEP=     .001445 VEP=     .014920 TEP=     .001182
 MC       CEP=     .001455 VEP=     .015045 TEP=     .001191
 MC Mean      -.000016    -.000039     .000484    -.000035
```

Table 6a. EP for the TSQBN Method, R = 1.00
(Minimum p Choice, Nonoptimal Weighting)

```
MONTE CARLO  N= 4  R= 1.000000 Lon=    .000 Lat= -84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #4/1.00/1
----TSQBN----2     0.     0.
 Linear   CEP=      .009648 VEP=     .074636 TEP=     .052399
 MC       CEP=      .009684 VEP=     .075019 TEP=     .052709
 MC Mean      -.000037     .000151    -.000787    -.001514

MONTE CARLO  N= 4  R= 1.000000 Lon=    .000 Lat= -24.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                  #4/1.00/2
----TSQBN----2     0.     0.
 Linear   CEP=      .008602 VEP=     .074176 TEP=     .010797
 MC       CEP=      .008610 VEP=     .074262 TEP=     .011125
 MC Mean      -.000156     .000062    -.001076    -.001713

MONTE CARLO  N= 4  R= 1.000000 Lon=  37.860 Lat= -18.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #4/1.00/3
----TSQBN----2     0.     0.
 Linear   CEP=      .030489 VEP=     .053017 TEP=     .032360
 MC       CEP=      .030354 VEP=     .052790 TEP=     .032200
 MC Mean       .000087     .000583    -.000814    -.001356

MONTE CARLO  N= 4  R= 1.000000 Lon= 309.190 Lat=  18.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #4/1.00/4
----TSQBN----2     0.     0.
 Linear   CEP=      .010310 VEP=     .068388 TEP=     .024297
 MC       CEP=      .010287 VEP=     .068229 TEP=     .024262
 MC Mean       .000102    -.000005     .001078    -.000873

MONTE CARLO  N= 5  R= 1.000000 Lon=    .000 Lat= -84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #5/1.00/1
----TSQBN----2     0.     0.
 Linear   CEP=      .002686 VEP=     .014146 TEP=     .009724
 MC       CEP=      .002664 VEP=     .014062 TEP=     .009667
 MC Mean       .000007    -.000015     .000218     .000106

MONTE CARLO  N= 5  R= 1.000000 Lon=    .000 Lat= -24.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                  #5/1.00/2
----TSQBN----2     0.     0.
 Linear   CEP=      .006498 VEP=     .058483 TEP=     .008403
 MC       CEP=      .006490 VEP=     .058332 TEP=     .008460
 MC Mean       .000006    -.000009     .000490    -.000894
```

Table 6a. EP for the TSQBN Method, R = 1.00
(Minimum p Choice, Nonoptimal Weighting) (continued)

```
MONTE CARLO  N= 5  R= 1.000000 Lon= 318.680 Lat=  54.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #5/1.00/3
----TSQBN----2     0.     0.
 Linear   CEP=     .004182 VEP=     .010264 TEP=     .005489
 MC       CEP=     .004150 VEP=     .010210 TEP=     .005457
 MC Mean      -.000002     .000053     .000180     .000055

MONTE CARLO  N= 5  R= 1.000000 Lon= 296.570 Lat=  18.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #5/1.00/4
----TSQBN----2     0.     0.
 Linear   CEP=     .003327 VEP=     .043526 TEP=     .006888
 MC       CEP=     .003417 VEP=     .044837 TEP=     .007130
 MC Mean      -.000003    -.000048    -.000248    -.000608

MONTE CARLO  N= 6  R= 1.000000 Lon= 300.000 Lat=  84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #6/1.00/1
----TSQBN----2     0.     0.
 Linear   CEP=     .001077 VEP=     .005960 TEP=     .003239
 MC       CEP=     .001085 VEP=     .006010 TEP=     .003267
 MC Mean       .000010    -.000030     .000202     .000092

MONTE CARLO  N= 6  R= 1.000000 Lon= 297.600 Lat=  12.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                  #6/1.00/2
----TSQBN----2     0.     0.
 Linear   CEP=     .000958 VEP=     .015466 TEP=     .001296
 MC       CEP=     .000957 VEP=     .015507 TEP=     .001299
 MC Mean       .000007     .000008     .000311    -.000049

MONTE CARLO  N= 6  R= 1.000000 Lon= 315.000 Lat=  48.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #6/1.00/3
----TSQBN----2     0.     0.
 Linear   CEP=     .003616 VEP=     .006445 TEP=     .003808
 MC       CEP=     .003602 VEP=     .006414 TEP=     .003788
 MC Mean       .000006     .000012     .000031    -.000002

MONTE CARLO  N= 6  R= 1.000000 Lon= 189.100 Lat=   6.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #6/1.00/4
----TSQBN----2     0.     0.
 Linear   CEP=     .001180 VEP=     .009023 TEP=     .004070
 MC       CEP=     .001175 VEP=     .008959 TEP=     .004038
 MC Mean      -.000023     .000002     .000176     .000052
```

Table 6b. EP for the TSQBN Method, R = 1.10
(Minimum p Choice – Nonoptimal Weighting)

```
MONTE CARLO  N= 4  R= 1.100000 Lon=    .000 Lat= -84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                    #4/1.10/1
----TSQBN----2     0.     0.
 Linear   CEP=     .009433 VEP=     .071899 TEP=     .049375
 MC       CEP=     .009467 VEP=     .072260 TEP=     .049658
 MC Mean      -.000033     .000137    -.000638    -.001387

MONTE CARLO  N= 4  R= 1.100000 Lon=  37.200 Lat= -12.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                    #4/1.10/2
----TSQBN----2     0.     1.
 Linear   CEP=     .010335 VEP=     .073988 TEP=     .021731
 MC       CEP=     .010308 VEP=     .073837 TEP=     .021832
 MC Mean      -.000019     .000023     .000219    -.001370

MONTE CARLO  N= 4  R= 1.100000 Lon= 120.000 Lat= -72.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                    #4/1.10/3
----TSQBN----2     0.     0.
 Linear   CEP=     .094837 VEP=     .045525 TEP=     .043447
 MC       CEP=     .094554 VEP=     .045383 TEP=     .043465
 MC Mean      -.002644    -.001334    -.000696    -.003898

MONTE CARLO  N= 4  R= 1.100000 Lon= 252.000 Lat=  48.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                    #4/1.10/4
----TSQBN----2     0.     0.
 Linear   CEP=     .031456 VEP=     .069251 TEP=     .007511
 MC       CEP=     .031699 VEP=     .069750 TEP=     .007801
 MC Mean       .000304    -.000520    -.000859    -.001749

MONTE CARLO  N= 5  R= 1.100000 Lon=    .000 Lat= -84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                    #5/1.10/1
----TSQBN----2     0.     0.
 Linear   CEP=     .002625 VEP=     .013626 TEP=     .009150
 MC       CEP=     .002604 VEP=     .013545 TEP=     .009097
 MC Mean       .000006    -.000014     .000211     .000098

MONTE CARLO  N= 5  R= 1.100000 Lon= 195.000 Lat= -36.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                    #5/1.10/2
----TSQBN----2     0.     0.
 Linear   CEP=     .003134 VEP=     .050851 TEP=     .014951
 MC       CEP=     .003102 VEP=     .050441 TEP=     .014928
 MC Mean       .000019     .000016     .000890    -.000869
```

Table 6b. EP for the TSQBN Method, R = 1.10
(Minimum p Choice – Nonoptimal Weighting) (continued)

```
MONTE CARLO  N= 5  R= 1.100000 Lon= 302.880 Lat=  18.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #5/1.10/3
----TSQBN----2     0.     0.
 Linear   CEP=     .011161 VEP=     .005635 TEP=     .007149
 MC       CEP=     .010950 VEP=     .005515 TEP=     .007014
 MC Mean       .000009     .000102     .000081     .000031

MONTE CARLO  N= 5  R= 1.100000 Lon=  55.360 Lat= -30.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #5/1.10/4
----TSQBN----2     0.     0.
 Linear   CEP=     .002311 VEP=     .033524 TEP=     .001193
 MC       CEP=     .002387 VEP=     .034831 TEP=     .001266
 MC Mean       .000061     .000004     .001193    -.000304

MONTE CARLO  N= 6  R= 1.100000 Lon= 300.000 Lat=  84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #6/1.10/1
----TSQBN----2     0.     0.
 Linear   CEP=     .001063 VEP=     .005832 TEP=     .003053
 MC       CEP=     .001071 VEP=     .005883 TEP=     .003080
 MC Mean       .000010    -.000030     .000198     .000086

MONTE CARLO  N= 6  R= 1.100000 Lon=  16.360 Lat=  42.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                  #6/1.10/2
----TSQBN----2     0.     0.
 Linear   CEP=     .001532 VEP=     .020320 TEP=     .006350
 MC       CEP=     .001544 VEP=     .020377 TEP=     .006370
 MC Mean       .000037    -.000013     .000518    -.000264

MONTE CARLO  N= 6  R= 1.100000 Lon= 310.000 Lat= -12.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #6/1.10/3
----TSQBN----2     0.     0.
 Linear   CEP=     .003237 VEP=     .003923 TEP=     .000869
 MC       CEP=     .003252 VEP=     .003940 TEP=     .000877
 MC Mean      -.000004     .000022     .000031    -.000010

MONTE CARLO  N= 6  R= 1.100000 Lon= 302.880 Lat=  18.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #6/1.10/4
----TSQBN----2     0.     0.
 Linear   CEP=     .001445 VEP=     .014921 TEP=     .001182
 MC       CEP=     .001456 VEP=     .015048 TEP=     .001192
 MC Mean      -.000016    -.000039     .000484    -.000035
```

Table 7a. EP for the LPTOA Method, R = 1.00, L_1 Norm

```
MONTE CARLO  N= 4  R= 1.000000 Lon=    .000 Lat= -84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                 #4/1.00/1
----LPTOA----      1.0000       .3000 LPNORM=1
 Linear   CEP=     .009648 VEP=     .074636 TEP=     .052399*
 MC       CEP=     .009684 VEP=     .075019 TEP=     .052709
 MC Mean      -.000037     .000151    -.000787    -.001514

MONTE CARLO  N= 4  R= 1.000000 Lon=    .000 Lat= -24.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                 #4/1.00/2
----LPTOA----      1.0000       .3000 LPNORM=1
 Linear   CEP=     .008602 VEP=     .074176 TEP=     .010797
 MC       CEP=     .008610 VEP=     .074262 TEP=     .011125
 MC Mean      -.000156     .000062    -.001076    -.001713

MONTE CARLO  N= 4  R= 1.000000 Lon=  37.860 Lat= -18.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                 #4/1.00/3
----LPTOA----      1.0000       .3000 LPNORM=1
 Linear   CEP=     .030489 VEP=     .053017 TEP=     .032360
 MC       CEP=     .030354 VEP=     .052790 TEP=     .032200
 MC Mean       .000087     .000583    -.000814    -.001356

MONTE CARLO  N= 4  R= 1.000000 Lon= 309.190 Lat=  18.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                 #4/1.00/4
----LPTOA----      1.0000       .3000 LPNORM=1
 Linear   CEP=     .010310 VEP=     .068388 TEP=     .024297
 MC       CEP=     .010287 VEP=     .068229 TEP=     .024262
 MC Mean       .000102    -.000005     .001078    -.000873

MONTE CARLO  N= 5  R= 1.000000 Lon=    .000 Lat= -84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                 #5/1.00/1
----LPTOA----      2.6964       .3000 LPNORM=1
 Linear   CEP=     .002685 VEP=     .014144 TEP=     .009723
 MC       CEP=     .002801 VEP=     .014491 TEP=     .010003
 MC Mean      -.000026     .001017    -.003801    -.002808

MONTE CARLO  N= 5  R= 1.000000 Lon=    .000 Lat= -24.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                 #5/1.00/2
----LPTOA----      2.5650       .3000 LPNORM=1
 Linear   CEP=     .006497 VEP=     .058479 TEP=     .008403
 MC       CEP=     .006758 VEP=     .060757 TEP=     .009161
 MC Mean      -.001634     .000793    -.014899    -.003279
```

*** Note: LEP for least squares.**

Table 7a. EP for the LPTOA Method, R = 1.00, L_1 Norm (continued)

```
MONTE CARLO  N= 5  R= 1.000000 Lon= 318.680 Lat=  54.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                    #5/1.00/3
----LPTOA----      2.2781       .3000 LPNORM=1
 Linear   CEP=     .004179 VEP=     .010251 TEP=     .005483
 MC       CEP=     .004233 VEP=     .010332 TEP=     .005485
 MC Mean       .000011     .000025     .000112     .000022

MONTE CARLO  N= 5  R= 1.000000 Lon= 296.570 Lat=  18.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                    #5/1.00/4
----LPTOA----      2.4937       .3000 LPNORM=1
 Linear   CEP=     .003326 VEP=     .043521 TEP=     .006887
 MC       CEP=     .003363 VEP=     .043887 TEP=     .007124
 MC Mean      -.000151    -.000897    -.010833    -.002217

MONTE CARLO  N= 6  R= 1.000000 Lon= 300.000 Lat=  84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                    #6/1.00/1
----LPTOA----      2.5250       .3000 LPNORM=1
 Linear   CEP=     .001077 VEP=     .005957 TEP=     .003238
 MC       CEP=     .001182 VEP=     .006069 TEP=     .003308
 MC Mean      -.000054     .000129    -.001079    -.000603

MONTE CARLO  N= 6  R= 1.000000 Lon= 297.600 Lat=  12.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                    #6/1.00/2
----LPTOA----      2.3663       .3000 LPNORM=1
 Linear   CEP=     .000957 VEP=     .015464 TEP=     .001296
 MC       CEP=     .001034 VEP=     .016061 TEP=     .001387
 MC Mean      -.000199    -.000219    -.004409    -.000429

MONTE CARLO  N= 6  R= 1.000000 Lon= 315.000 Lat=  48.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                    #6/1.00/3
----LPTOA----      2.5687       .3000 LPNORM=1
 Linear   CEP=     .003601 VEP=     .006420 TEP=     .003793
 MC       CEP=     .003724 VEP=     .006633 TEP=     .003912
 MC Mean       .000071    -.000885    -.001564    -.000909

MONTE CARLO  N= 6  R= 1.000000 Lon= 189.100 Lat=   6.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                    #6/1.00/4
----LPTOA----      2.5778       .3000 LPNORM=1
 Linear   CEP=     .001180 VEP=     .009023 TEP=     .004070
 MC       CEP=     .001295 VEP=     .009773 TEP=     .004401
 MC Mean       .000389     .000190    -.003370    -.001562
```

Table 7b. EP for the LPTOA Method, R = 1.10, L_1 Norm

```
MONTE CARLO  N= 4  R= 1.100000 Lon=    .000 Lat= -84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #4/1.10/1
----LPTOA----      1.0000       .3000 LPNORM=1
 Linear   CEP=     .009433 VEP=     .071899 TEP=     .049375
 MC       CEP=     .009467 VEP=     .072260 TEP=     .049658
 MC Mean      -.000033     .000137    -.000638    -.001387

MONTE CARLO  N= 4  R= 1.100000 Lon=  37.200 Lat= -12.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                  #4/1.10/2
----LPTOA----      1.0000       .3000 LPNORM=1
 Linear   CEP=     .010335 VEP=     .073988 TEP=     .021731
 MC       CEP=     .010308 VEP=     .073837 TEP=     .021832
 MC Mean      -.000019     .000023     .000219    -.001370

MONTE CARLO  N= 4  R= 1.100000 Lon= 120.000 Lat= -72.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #4/1.10/3
----LPTOA----      1.0000       .3000 LPNORM=1
 Linear   CEP=     .094837 VEP=     .045525 TEP=     .043447
 MC       CEP=     .094554 VEP=     .045383 TEP=     .043465
 MC Mean      -.002644    -.001334    -.000696    -.003898

MONTE CARLO  N= 4  R= 1.100000 Lon= 252.000 Lat=  48.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #4/1.10/4
----LPTOA----      1.0000       .3000 LPNORM=1
 Linear   CEP=     .031456 VEP=     .069251 TEP=     .007511
 MC       CEP=     .031699 VEP=     .069750 TEP=     .007801
 MC Mean       .000304    -.000520    -.000859    -.001749

MONTE CARLO  N= 5  R= 1.100000 Lon=    .000 Lat= -84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #5/1.10/1
----LPTOA----      2.6928       .3000 LPNORM=1
 Linear   CEP=     .002625 VEP=     .013624 TEP=     .009149
 MC       CEP=     .002737 VEP=     .013955 TEP=     .009412
 MC Mean      -.000025     .000997    -.003658    -.002652

MONTE CARLO  N= 5  R= 1.100000 Lon= 195.000 Lat= -36.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                  #5/1.10/2
----LPTOA----      2.5772       .3000 LPNORM=1
 Linear   CEP=     .003134 VEP=     .050849 TEP=     .014950
 MC       CEP=     .003100 VEP=     .050776 TEP=     .014965
 MC Mean      -.000147    -.000156    -.003723     .000462
```

Table 7b. EP for the LPTOA Method, R = 1.10, L_1 Norm (continued)

```
MONTE CARLO  N= 5  R= 1.100000 Lon= 302.880 Lat=  18.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #5/1.10/3
----LPTOA----      2.5253       .3000 LPNORM=1
 Linear   CEP=     .011137 VEP=     .005612 TEP=     .007135
 MC       CEP=     .011476 VEP=     .005857 TEP=     .007380
 MC Mean      -.000492    -.002404    -.001270    -.001807

MONTE CARLO  N= 5  R= 1.100000 Lon=  55.360 Lat= -30.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #5/1.10/4
----LPTOA----      2.4526       .3000 LPNORM=1
 Linear   CEP=     .002311 VEP=     .033521 TEP=     .001193
 MC       CEP=     .002389 VEP=     .034353 TEP=     .001364
 MC Mean      -.000875    -.000120    -.010767    -.000727

MONTE CARLO  N= 6  R= 1.100000 Lon= 300.000 Lat=  84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #6/1.10/1
----LPTOA----      2.5128       .3000 LPNORM=1
 Linear   CEP=     .001063 VEP=     .005829 TEP=     .003051
 MC       CEP=     .001166 VEP=     .005936 TEP=     .003118
 MC Mean      -.000053     .000127    -.001046    -.000564

MONTE CARLO  N= 6  R= 1.100000 Lon=  16.360 Lat=  42.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                  #6/1.10/2
----LPTOA----      2.4301       .3000 LPNORM=1
 Linear   CEP=     .001532 VEP=     .020320 TEP=     .006350
 MC       CEP=     .001603 VEP=     .020880 TEP=     .006522
 MC Mean      -.000347     .000095    -.004060     .001207

MONTE CARLO  N= 6  R= 1.100000 Lon= 310.000 Lat= -12.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #6/1.10/3
----LPTOA----      2.3811       .3000 LPNORM=1
 Linear   CEP=     .003235 VEP=     .003922 TEP=     .000868
 MC       CEP=     .003386 VEP=     .004037 TEP=     .000918
 MC Mean       .000111    -.000672    -.000757    -.000154

MONTE CARLO  N= 6  R= 1.100000 Lon= 302.880 Lat=  18.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #6/1.10/4
----LPTOA----      2.3950       .3000 LPNORM=1
 Linear   CEP=     .001445 VEP=     .014920 TEP=     .001182
 MC       CEP=     .001544 VEP=     .015561 TEP=     .001304
 MC Mean       .000100     .000286    -.003620    -.000367
```

Table 7c. EP for the LPTOA Method, R = 1.00, L_∞ Norm

```
MONTE CARLO  N= 4  R= 1.000000 Lon=    .000 Lat= -84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                    #4/1.00/1
----LPTOA----      1.0000       .3000 LPNORM=inf
 Linear   CEP=      .009648 VEP=     .074636 TEP=     .052399
 MC       CEP=      .009684 VEP=     .075019 TEP=     .052709
 MC Mean      -.000037     .000151    -.000787    -.001514

MONTE CARLO  N= 4  R= 1.000000 Lon=    .000 Lat= -24.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                    #4/1.00/2
----LPTOA----      1.0000       .3000 LPNORM=inf
 Linear   CEP=      .008602 VEP=     .074176 TEP=     .010797
 MC       CEP=      .008610 VEP=     .074262 TEP=     .011125
 MC Mean      -.000156     .000062    -.001076    -.001713

MONTE CARLO  N= 4  R= 1.000000 Lon=  37.860 Lat= -18.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                    #4/1.00/3
----LPTOA----      1.0000       .3000 LPNORM=inf
 Linear   CEP=      .030489 VEP=     .053017 TEP=     .032360
 MC       CEP=      .030354 VEP=     .052790 TEP=     .032200
 MC Mean       .000087     .000583    -.000814    -.001356

MONTE CARLO  N= 4  R= 1.000000 Lon= 309.190 Lat=  18.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                    #4/1.00/4
----LPTOA----      1.0000       .3000 LPNORM=inf
 Linear   CEP=      .010310 VEP=     .068388 TEP=     .024297
 MC       CEP=      .010287 VEP=     .068229 TEP=     .024262
 MC Mean       .000102    -.000005     .001078    -.000873

MONTE CARLO  N= 5  R= 1.000000 Lon=    .000 Lat= -84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                    #5/1.00/1
----LPTOA----      2.4544       .3000 LPNORM=inf
 Linear   CEP=      .002685 VEP=     .014144 TEP=     .009723
 MC       CEP=      .002692 VEP=     .014249 TEP=     .009808
 MC Mean      -.000059     .000267    -.002036    -.001477

MONTE CARLO  N= 5  R= 1.000000 Lon=    .000 Lat= -24.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                    #5/1.00/2
----LPTOA----      2.6515       .3000 LPNORM=inf
 Linear   CEP=      .006497 VEP=     .058479 TEP=     .008403
 MC       CEP=      .006496 VEP=     .058372 TEP=     .008504
 MC Mean      -.000165     .000070    -.001220    -.001141
```

Table 7c. EP for the LPTOA Method, R = 1.00, L_∞ Norm (continued)

```
MONTE CARLO  N= 5  R= 1.000000 Lon= 318.680 Lat=  54.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #5/1.00/3
----LPTOA----      2.4634       .3000 LPNORM=inf
 Linear   CEP=      .004179 VEP=     .010251 TEP=     .005483
 MC       CEP=      .004249 VEP=     .010462 TEP=     .005592
 MC Mean        .000052    -.001240    -.002829    -.001538

MONTE CARLO  N= 5  R= 1.000000 Lon= 296.570 Lat=  18.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #5/1.00/4
----LPTOA----      2.0808       .3000 LPNORM=inf
 Linear   CEP=      .003326 VEP=     .043521 TEP=     .006887
 MC       CEP=      .003364 VEP=     .045055 TEP=     .007160
 MC Mean      -.000019    -.000100    -.000868    -.000712

MONTE CARLO  N= 6  R= 1.000000 Lon= 300.000 Lat=  84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #6/1.00/1
----LPTOA----      2.3999       .3000 LPNORM=inf
 Linear   CEP=      .001077 VEP=     .005957 TEP=     .003238
 MC       CEP=      .001133 VEP=     .006153 TEP=     .003353
 MC Mean        .000002     .000050    -.000900    -.000516

MONTE CARLO  N= 6  R= 1.000000 Lon= 297.600 Lat=  12.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                  #6/1.00/2
----LPTOA----      2.2859       .3000 LPNORM=inf
 Linear   CEP=      .000957 VEP=     .015464 TEP=     .001296
 MC       CEP=      .001001 VEP=     .016000 TEP=     .001359
 MC Mean      -.000065    -.000142    -.003188    -.000338

MONTE CARLO  N= 6  R= 1.000000 Lon= 315.000 Lat=  48.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #6/1.00/3
----LPTOA----      2.4338       .3000 LPNORM=inf
 Linear   CEP=      .003601 VEP=     .006420 TEP=     .003793
 MC       CEP=      .003655 VEP=     .006549 TEP=     .003860
 MC Mean        .000085    -.000900    -.001547    -.000941

MONTE CARLO  N= 6  R= 1.000000 Lon= 189.100 Lat=   6.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #6/1.00/4
----LPTOA----      2.4231       .3000 LPNORM=inf
 Linear   CEP=      .001180 VEP=     .009023 TEP=     .004070
 MC       CEP=      .001224 VEP=     .009198 TEP=     .004148
 MC Mean        .000185     .000106    -.001780    -.000785
```

Table 7d. EP for the LPTOA Method, R = 1.10, L_∞ Norm

```
MONTE CARLO  N= 4  R= 1.100000 Lon=    .000 Lat= -84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                    #4/1.10/1
----LPTOA----      1.0000       .3000 LPNORM=inf
 Linear   CEP=      .009433 VEP=     .071899 TEP=     .049375
 MC       CEP=      .009467 VEP=     .072260 TEP=     .049658
 MC Mean      -.000033     .000137    -.000638    -.001387

MONTE CARLO  N= 4  R= 1.100000 Lon=  37.200 Lat= -12.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                    #4/1.10/2
----LPTOA----      1.0000       .3000 LPNORM=inf
 Linear   CEP=      .010335 VEP=     .073988 TEP=     .021731
 MC       CEP=      .010308 VEP=     .073837 TEP=     .021832
 MC Mean      -.000019     .000023     .000219    -.001370

MONTE CARLO  N= 4  R= 1.100000 Lon= 120.000 Lat= -72.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                    #4/1.10/3
----LPTOA----      1.0000       .3000 LPNORM=inf
 Linear   CEP=      .094837 VEP=     .045525 TEP=     .043447
 MC       CEP=      .094554 VEP=     .045383 TEP=     .043465
 MC Mean      -.002644    -.001334    -.000696    -.003898

MONTE CARLO  N= 4  R= 1.100000 Lon= 252.000 Lat=  48.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                    #4/1.10/4
----LPTOA----      1.0000       .3000 LPNORM=inf
 Linear   CEP=      .031456 VEP=     .069251 TEP=     .007511
 MC       CEP=      .031699 VEP=     .069750 TEP=     .007801
 MC Mean       .000304    -.000520    -.000859    -.001749

MONTE CARLO  N= 5  R= 1.100000 Lon=    .000 Lat= -84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                    #5/1.10/1
----LPTOA----      2.4514       .3000 LPNORM=inf
 Linear   CEP=      .002625 VEP=     .013624 TEP=     .009149
 MC       CEP=      .002631 VEP=     .013726 TEP=     .009229
 MC Mean      -.000058     .000261    -.001959    -.001392

MONTE CARLO  N= 5  R= 1.100000 Lon= 195.000 Lat= -36.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                    #5/1.10/2
----LPTOA----      2.2580       .3000 LPNORM=inf
 Linear   CEP=      .003134 VEP=     .050849 TEP=     .014950
 MC       CEP=      .003133 VEP=     .050468 TEP=     .014917
 MC Mean      -.000049     .000000     .000045    -.000637
```

Table 7d. EP for the LPTOA Method, R = 1.10, L_∞ Norm (continued)

```
MONTE CARLO  N= 5  R= 1.100000 Lon= 302.880 Lat=  18.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #5/1.10/3
----LPTOA----      2.4447       .3000 LPNORM=inf
 Linear   CEP=     .011137 VEP=     .005612 TEP=     .007135
 MC       CEP=     .011070 VEP=     .005536 TEP=     .007084
 MC Mean      -.000041    -.000600    -.000248    -.000408

MONTE CARLO  N= 5  R= 1.100000 Lon=  55.360 Lat= -30.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #5/1.10/4
----LPTOA----      2.2329       .3000 LPNORM=inf
 Linear   CEP=     .002311 VEP=     .033521 TEP=     .001193
 MC       CEP=     .002433 VEP=     .035278 TEP=     .001297
 MC Mean      -.000110     .000013    -.001603    -.000398

MONTE CARLO  N= 6  R= 1.100000 Lon= 300.000 Lat=  84.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #6/1.10/1
----LPTOA----      2.3920       .3000 LPNORM=inf
 Linear   CEP=     .001063 VEP=     .005829 TEP=     .003051
 MC       CEP=     .001118 VEP=     .006021 TEP=     .003161
 MC Mean       .000002     .000049    -.000881    -.000487

MONTE CARLO  N= 6  R= 1.100000 Lon=  16.360 Lat=  42.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                  #6/1.10/2
----LPTOA----      2.4818       .3000 LPNORM=inf
 Linear   CEP=     .001532 VEP=     .020320 TEP=     .006350
 MC       CEP=     .001686 VEP=     .021333 TEP=     .006637
 MC Mean      -.000283     .000150    -.005005     .001489

MONTE CARLO  N= 6  R= 1.100000 Lon= 310.000 Lat= -12.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #6/1.10/3
----LPTOA----      2.4889       .3000 LPNORM=inf
 Linear   CEP=     .003235 VEP=     .003922 TEP=     .000868
 MC       CEP=     .003513 VEP=     .004142 TEP=     .000965
 MC Mean       .000347    -.000967    -.001327    -.000333

MONTE CARLO  N= 6  R= 1.100000 Lon= 302.880 Lat=  18.000
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #6/1.10/4
----LPTOA----      2.1490       .3000 LPNORM=inf
 Linear   CEP=     .001445 VEP=     .014920 TEP=     .001182
 MC       CEP=     .001480 VEP=     .015115 TEP=     .001202
 MC Mean       .000061     .000089    -.001142    -.000155
```

Table 8a. EP and Mean Errors, Conditioned
TOA – CBTOA (N = 4, R = 1.00, Rc = 1.001, Wc = 0.01)

```
   U=Unconditioned, L=Linear, C=Conditioned,  M=Monte Carlo
Conditioned Monte Carlo  N=4 R=1.0000 Lon=    .00 Lat=-84.00
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #4/1.00/1
 ----CBTOA---- 3.17 Rc=1.0010 Wc= .0100
 U/M Mean     -.000037     .000151    -.000787    -.001514
 C/L Mean      .000031    -.000124     .000992     .000696
 C/M Mean      .000026    -.000116     .001003     .000701
 U/L  CEP=     .009648 VEP=     .074636 TEP=     .052399
 C/L  CEP=     .001189 VEP=     .000599 TEP=     .000561
 C/M  CEP=     .001189 VEP=     .000602 TEP=     .000563

Conditioned Monte Carlo  N=4 R=1.0000 Lon=    .00 Lat=-24.00
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                  #4/1.00/2
 ----CBTOA---- 3.16 Rc=1.0010 Wc= .0100
 U/M Mean     -.000156     .000062    -.001076    -.001713
 C/L Mean      .000104    -.000048     .000992     .000144
 C/M Mean      .000093    -.000045     .001002     .000138
 U/L  CEP=     .008602 VEP=     .074176 TEP=     .010797
 C/L  CEP=     .000817 VEP=     .000603 TEP=     .000376
 C/M  CEP=     .000819 VEP=     .000602 TEP=     .000378

Conditioned Monte Carlo  N=4 R=1.0000 Lon=  37.86 Lat=-18.00
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #4/1.00/3
 ----CBTOA---- 3.47 Rc=1.0010 Wc= .0100
 U/M Mean      .000087     .000583    -.000814    -.001356
 C/L Mean     -.000077    -.000560     .000984     .000601
 C/M Mean     -.000064    -.000552     .000976     .000588
 U/L  CEP=     .030489 VEP=     .053017 TEP=     .032360
 C/L  CEP=     .001112 VEP=     .000837 TEP=     .000691
 C/M  CEP=     .001113 VEP=     .000833 TEP=     .000692

Conditioned Monte Carlo  N=4 R=1.0000 Lon= 309.19 Lat= 18.00
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #4/1.00/4
 ----CBTOA---- 3.02 Rc=1.0010 Wc= .0100
 U/M Mean      .000102    -.000005     .001078    -.000873
 C/L Mean      .000147    -.000026     .000990     .000352
 C/M Mean      .000144    -.000015     .000997     .000347
 U/L  CEP=     .010310 VEP=     .068388 TEP=     .024297
 C/L  CEP=     .000857 VEP=     .000653 TEP=     .000441
 C/M  CEP=     .000864 VEP=     .000650 TEP=     .000442
```

Table 8b. EP and Mean Errors, Conditioned
TOA – CBTOA (N = 5, R = 1.00, Rc = 1.001, Wc = 0.01)

```
Conditioned Monte Carlo  N=5 R=1.0000 Lon=    .00 Lat=-84.00
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  1 *** NEAREST      *** Tailed                   #5/1.00/1
  ----CBTOA---- 3.13 Rc=1.0010 Wc= .0100
  U/M Mean      .000007    -.000015     .000218     .000106
  C/L Mean      .000021    -.000147     .000816     .000561
  C/M Mean      .000024    -.000130     .000850     .000575
  U/L  CEP=     .002685 VEP=     .014144 TEP=     .009723
  C/L  CEP=     .001065 VEP=     .002602 TEP=     .001813
  C/M  CEP=     .001053 VEP=     .002587 TEP=     .001804

 Conditioned Monte Carlo  N=5 R=1.0000 Lon=    .00 Lat=-24.00
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  2 *** FARTHEST     *** Tailed                   #5/1.00/2
  ----CBTOA---- 3.14 Rc=1.0010 Wc= .0100
  U/M Mean      .000006    -.000009     .000490    -.000894
  C/L Mean      .000099    -.000046     .000987     .000142
  C/M Mean      .000081    -.000039     .000991     .000136
  U/L  CEP=     .006497 VEP=     .058479 TEP=     .008403
  C/L  CEP=     .000726 VEP=     .000761 TEP=     .000337
  C/M  CEP=     .000723 VEP=     .000759 TEP=     .000335

 Conditioned Monte Carlo  N=5 R=1.0000 Lon= 318.68 Lat= 54.00
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  3 *** NEAR ONE+FAR *** Tailed                   #5/1.00/3
  ----CBTOA---- 2.96 Rc=1.0010 Wc= .0100
  U/M Mean     -.000002     .000053     .000180     .000055
  C/L Mean     -.000034     .000280     .000700     .000373
  C/M Mean     -.000030     .000283     .000748     .000385
  U/L  CEP=     .004179 VEP=     .010251 TEP=     .005483
  C/L  CEP=     .001429 VEP=     .003079 TEP=     .001684
  C/M  CEP=     .001409 VEP=     .003062 TEP=     .001669

 Conditioned Monte Carlo  N=5 R=1.0000 Lon= 296.57 Lat= 18.00
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  4 *** MIDDLE       *** Tailed                   #5/1.00/4
  ----CBTOA---- 3.01 Rc=1.0010 Wc= .0100
  U/M Mean     -.000003    -.000048    -.000248    -.000608
  C/L Mean      .000020     .000071     .000977     .000154
  C/M Mean      .000022     .000052     .000958     .000142
  U/L  CEP=     .003326 VEP=     .043521 TEP=     .006887
  C/L  CEP=     .000726 VEP=     .001012 TEP=     .000361
  C/M  CEP=     .000721 VEP=     .001011 TEP=     .000356
```

Table 8c. EP and Mean Errors, Conditioned
TOA – CBTOA (N = 6, R = 1.00, Rc = 1.001, Wc = 0.01)

```
Conditioned Monte Carlo  N=6 R=1.0000 Lon= 300.00 Lat= 84.00
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  1 *** NEAREST      *** Tailed                   #6/1.00/1
  ----CBTOA---- 2.85 Rc=1.0010 Wc= .0100
  U/M Mean      .000010    -.000030     .000202     .000092
  C/L Mean      .000025    -.000051     .000440     .000238
  C/M Mean      .000029    -.000069     .000541     .000281
  U/L  CEP=     .001077 VEP=     .005957 TEP=     .003238
  C/L  CEP=     .000884 VEP=     .003334 TEP=     .001827
  C/M  CEP=     .000885 VEP=     .003351 TEP=     .001836

 Conditioned Monte Carlo  N=6 R=1.0000 Lon= 297.60 Lat= 12.00
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  2 *** FARTHEST     *** Tailed                   #6/1.00/2
  ----CBTOA---- 2.96 Rc=1.0010 Wc= .0100
  U/M Mean      .000007     .000008     .000311    -.000049
  C/L Mean      .000026     .000035     .000841     .000069
  C/M Mean      .000024     .000032     .000866     .000061
  U/L  CEP=     .000957 VEP=     .015464 TEP=     .001296
  C/L  CEP=     .000646 VEP=     .002454 TEP=     .000344
  C/M  CEP=     .000640 VEP=     .002445 TEP=     .000343

 Conditioned Monte Carlo  N=6 R=1.0000 Lon= 315.00 Lat= 48.00
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  3 *** NEAR ONE+FAR *** Tailed                   #6/1.00/3
  ----CBTOA---- 2.88 Rc=1.0010 Wc= .0100
  U/M Mean      .000006     .000012     .000031    -.000002
  C/L Mean     -.000026     .000262     .000477     .000281
  C/M Mean     -.000017     .000261     .000479     .000270
  U/L  CEP=     .003601 VEP=     .006420 TEP=     .003793
  C/L  CEP=     .001972 VEP=     .003355 TEP=     .002008
  C/M  CEP=     .001958 VEP=     .003336 TEP=     .001993

 Conditioned Monte Carlo  N=6 R=1.0000 Lon= 189.10 Lat=  6.00
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  4 *** MIDDLE       *** Tailed                   #6/1.00/4
  ----CBTOA---- 2.92 Rc=1.0010 Wc= .0100
  U/M Mean     -.000023     .000002     .000176     .000052
  C/L Mean     -.000063    -.000035     .000643     .000290
  C/M Mean     -.000073    -.000026     .000688     .000300
  U/L  CEP=     .001180 VEP=     .009023 TEP=     .004070
  C/L  CEP=     .000796 VEP=     .003217 TEP=     .001475
  C/M  CEP=     .000798 VEP=     .003190 TEP=     .001458
```

Table 8d. EP and Mean Errors, Conditioned
TOA – CBTOA (N = 6, R = 1.00, Rc = 1.001, Wc = 0.10)

```
Conditioned Monte Carlo  N=6 R=1.0000 Lon= 300.00 Lat= 84.00
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  1 *** NEAREST      *** Tailed                   #6/1.00/1
  ----CBTOA---- 2.83 Rc=1.0010 Wc= .1000
  U/M Mean      .000010    -.000030     .000202     .000092
  C/L Mean      .000050    -.000103     .000887     .000480
  C/M Mean      .000050    -.000112     .000907     .000482
  U/L  CEP=     .001077 VEP=     .005957 TEP=     .003238
  C/L  CEP=     .000788 VEP=     .000672 TEP=     .000459
  C/M  CEP=     .000785 VEP=     .000675 TEP=     .000461

 Conditioned Monte Carlo  N=6 R=1.0000 Lon= 297.60 Lat= 12.00
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  2 *** FARTHEST     *** Tailed                   #6/1.00/2
  ----CBTOA---- 2.84 Rc=1.0010 Wc= .1000
  U/M Mean      .000007     .000008     .000311    -.000049
  C/L Mean      .000030     .000040     .000981     .000080
  C/M Mean      .000028     .000037     .000984     .000072
  U/L  CEP=     .000957 VEP=     .015464 TEP=     .001296
  C/L  CEP=     .000644 VEP=     .000286 TEP=     .000280
  C/M  CEP=     .000637 VEP=     .000285 TEP=     .000279

 Conditioned Monte Carlo  N=6 R=1.0000 Lon= 315.00 Lat= 48.00
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  3 *** NEAR ONE+FAR *** Tailed                   #6/1.00/3
  ----CBTOA---- 2.93 Rc=1.0010 Wc= .1000
  U/M Mean      .000006     .000012     .000031    -.000002
  C/L Mean     -.000048     .000494     .000901     .000530
  C/M Mean     -.000040     .000493     .000902     .000522
  U/L  CEP=     .003601 VEP=     .006420 TEP=     .003793
  C/L  CEP=     .000869 VEP=     .000633 TEP=     .000533
  C/M  CEP=     .000864 VEP=     .000630 TEP=     .000526

 Conditioned Monte Carlo  N=6 R=1.0000 Lon= 189.10 Lat=  6.00
  Data error sigma= .001000  No. MC trials=10000 SEED= 573821
  4 *** MIDDLE       *** Tailed                   #6/1.00/4
  ----CBTOA---- 2.86 Rc=1.0010 Wc= .1000
  U/M Mean     -.000023     .000002     .000176     .000052
  C/L Mean     -.000093    -.000052     .000948     .000426
  C/M Mean     -.000099    -.000041     .000953     .000422
  U/L  CEP=     .001180 VEP=     .009023 TEP=     .004070
  C/L  CEP=     .000717 VEP=     .000474 TEP=     .000353
  C/M  CEP=     .000720 VEP=     .000470 TEP=     .000349
```

Table 9. EP and Mean Errors for Conditioned TOA – CTOA (Configurations that Would Not Converge Unless Conditioned)

```
  U=Unconditioned, L=Linear, C=Conditioned,  M=Monte Carlo
Conditioned Monte Carlo  N=4 R=1.0000 Lon= 237.22 Lat=-42.00
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                  #4/1.00/2
 ----CTOA ---- 3.97 Rc=1.0010 Wc= .0100
 C/L Mean     -.000033     .000090     .001000     .000121
 C/M Mean     -.000023     .000089     .000999     .000115
 U/L  CEP=    3.449598 VEP=   35.892991 TEP=    4.326352
 C/L  CEP=     .000777 VEP=     .000001 TEP=     .000339
 C/M  CEP=     .000780 VEP=     .000001 TEP=     .000340
 Conditioned Monte Carlo  N=4 R=1.0000 Lon= 352.92 Lat=-30.00
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #4/1.00/3
 ----CTOA ---- 4.25 Rc=1.0010 Wc= .0100
 C/L Mean      .000331     .000499     .001000     .000604
 C/M Mean      .000320     .000500     .001001     .000596
 U/L  CEP=     .194157 VEP=     .324088 TEP=     .195855
 C/L  CEP=     .001113 VEP=     .000139 TEP=     .000473
 C/M  CEP=     .001115 VEP=     .000139 TEP=     .000477
 Conditioned Monte Carlo  N=4 R=1.0000 Lon=  96.00 Lat= 60.00
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #4/1.00/4
 ----CTOA ---- 4.05 Rc=1.0010 Wc= .0100
 C/L Mean     -.000151    -.000237     .001000     .000532
 C/M Mean     -.000140    -.000232     .000996     .000528
 U/L  CEP=     .123571 VEP=     .439490 TEP=     .234044
 C/L  CEP=     .001154 VEP=     .000103 TEP=     .000373
 C/M  CEP=     .001161 VEP=     .000103 TEP=     .000372
```

Wc = 0.001

```
 Conditioned Monte Carlo  N=4 R=1.0000 Lon= 237.22 Lat=-42.00
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 2 *** FARTHEST     *** Tailed                  #4/1.00/2
 ----CTOA ---- 4.13 Rc=1.0010 Wc= .0010
 C/L Mean     -.000033     .000090     .001000     .000121
 C/M Mean     -.000023     .000089     .000999     .000115
 U/L  CEP=    3.449598 VEP=   35.892991 TEP=    4.326352
 C/L  CEP=     .000777 VEP=     .000013 TEP=     .000339
 C/M  CEP=     .000780 VEP=     .000012 TEP=     .000340
 Conditioned Monte Carlo  N=4 R=1.0000 Lon= 352.92 Lat=-30.00
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 3 *** NEAR ONE+FAR *** Tailed                  #4/1.00/3
 ----CTOA ---- 4.65 Rc=1.0010 Wc= .0010
 C/L Mean      .000330     .000497     .000996     .000602
 C/M Mean      .000324     .000506     .001013     .000603
 U/L  CEP=     .194157 VEP=     .324088 TEP=     .195855
 C/L  CEP=     .001577 VEP=     .001386 TEP=     .000958
 C/M  CEP=     .001571 VEP=     .001381 TEP=     .000955
 Conditioned Monte Carlo  N=4 R=1.0000 Lon=  96.00 Lat= 60.00
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 4 *** MIDDLE       *** Tailed                  #4/1.00/4
```

Table 9. EP and Mean Errors for Conditioned TOA – CTOA (Configurations That Would Not Converge Unless Conditioned) (continued)

```
----CTOA ---- 4.25 Rc=1.0010 Wc= .0010
C/L Mean     -.000151    -.000237     .000998     .000531
C/M Mean     -.000137    -.000228     .000980     .000519
U/L  CEP=     .123571 VEP=     .439490 TEP=     .234044
C/L  CEP=     .001187 VEP=     .001024 TEP=     .000659
C/M  CEP=     .001196 VEP=     .001022 TEP=     .000655
```

Table 10. Five Iterations of the CISQB Conditioning Method

```
   U=Unconditioned, L=Linear, C=Conditioned,  M=MONTE CARLO
Conditioned Monte Carlo  N=4 R=1.0000 Lon=    .00 Lat=-84.00
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #4/1.00/1
 ----CISQB----1 Rc=1.0010 Wc= .0100 Wcq=   .1143
 U/M Mean     -.000036     .000147    -.000765    -.001478
 C/L Mean      .000031    -.000124     .000993     .000697
 C/M Mean      .000049    -.000203     .001759     .001078
 U/L  CEP=     .009648 VEP=     .074636 TEP=     .052399
 C/L  CEP=     .001188 VEP=     .000589 TEP=     .000556
 C/M  CEP=     .004009 VEP=     .029893 TEP=     .020946

Conditioned Monte Carlo  N=4 R=1.0000 Lon=    .00 Lat=-84.00
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #4/1.00/1
 ----CISQB----2 Rc=1.0010 Wc= .0100 Wcq=   .1143
 U/M Mean     -.000036     .000147    -.000765    -.001478
 C/L Mean      .000031    -.000124     .000993     .000697
 C/M Mean     -.000007     .000015    -.000043    -.000061
 U/L  CEP=     .009648 VEP=     .074636 TEP=     .052399
 C/L  CEP=     .001188 VEP=     .000589 TEP=     .000556
 C/M  CEP=     .001969 VEP=     .012737 TEP=     .008948

Conditioned Monte Carlo  N=4 R=1.0000 Lon=    .00 Lat=-84.00
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #4/1.00/1
 ----CISQB----3 Rc=1.0010 Wc= .0100 Wcq=   .1143
 U/M Mean     -.000036     .000147    -.000765    -.001478
 C/L Mean      .000031    -.000124     .000993     .000697
 C/M Mean      .000016    -.000076     .000683     .000472
 U/L  CEP=     .009648 VEP=     .074636 TEP=     .052399
 C/L  CEP=     .001188 VEP=     .000589 TEP=     .000556
 C/M  CEP=     .001334 VEP=     .004723 TEP=     .003308

Conditioned Monte Carlo  N=4 R=1.0000 Lon=    .00 Lat=-84.00
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #4/1.00/1
 ----CISQB----4 Rc=1.0010 Wc= .0100 Wcq=   .1143
 U/M Mean     -.000036     .000147    -.000765    -.001478
 C/L Mean      .000031    -.000124     .000993     .000697
 C/M Mean      .000005    -.000033     .000336     .000231
 U/L  CEP=     .009648 VEP=     .074636 TEP=     .052399
 C/L  CEP=     .001188 VEP=     .000589 TEP=     .000556
 C/M  CEP=     .001220 VEP=     .002455 TEP=     .001721

Conditioned Monte Carlo  N=4 R=1.0000 Lon=    .00 Lat=-84.00
 Data error sigma= .001000  No. MC trials=10000 SEED= 573821
 1 *** NEAREST      *** Tailed                  #4/1.00/1
 ----CISQB----5 Rc=1.0010 Wc= .0100 Wcq=   .1143
 U/M Mean     -.000036     .000147    -.000765    -.001478
 C/L Mean.000031    -.000124     .000993     .000697
 C/M Mean      .000010    -.000052     .000489     .000340
 U/L  CEP=     .009648 VEP=     .074636 TEP=     .052399
 C/L  CEP=     .001188 VEP=     .000589 TEP=     .000556
 C/M  CEP=     .001189 VEP=     .000580 TEP=     .000376
```

Distribution

1		Eugene Aronson 13 Tennis Court Ln. NW Albuquerque, NM 87120
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1	0670	Julie J. Gregory, 6524
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