Ideal magnetohydrodynamics stability spectrum with a resistive wall

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The eigenvalue equations describing a cylindrical ideal magnetohydrodynamics plasma interacting with a thin resistive wall are presented in the standard mathematical form, $\mathcal{A}\cdot\mathbf{x} = \lambda\mathcal{B}\cdot\mathbf{x}$, without discretizing the vacuum regions surrounding the plasma. This is accomplished by using a finite element basis for the plasma perturbations, and by coupling the plasma surface perturbations to the perturbed electrical current in the wall using a Green's function approach. The perturbed wall current introduces a single additional degree of freedom into the system, which, together with an auxiliary variable, $\mathbf{u} = \omega\boldsymbol{\xi}$, allows the system to take the standard linear form. The standard form allows the use of linear eigenvalue solvers, without additional iterations, to compute the complete spectrum of plasma modes in the presence of a surrounding resistive wall at arbitrary separation. Standard results are recovered in the limits of (1) an infinitely resistive wall (no wall), and (2) a zero resistance wall (ideal wall).

A common model used to investigate plasma stability is that of an ideal magnetohydrodynamics (MHD) plasma configuration surrounded by a nearby conducting wall. It is well known that the location and electrical conductivity of the wall can have a profound effect upon the eigenmodes and eigenvalues of the system. While the complete mode spectrum of a configuration having no wall or a perfectly conducting wall has been obtained in previous numerical studies[1–3], the calculation of the complete mode spectrum has not been extended to the case where the surrounding wall has non-zero but finite conductivity (a resistive wall).

Several approaches have been presented to calculate the most unstable eigenvalue in the presence of a resistive wall in which a Green's function method is used to represent the solution in the vacuum region. In these, the eigenvalue appears nonlinearly and the most unstable mode is found by iteration.[4–6] While this is a valid technique, it is inherently not as efficient as a linear eigenvalue problem and is not a practical approach for finding the complete spectrum of modes. Another approach, which maintains the linear nature of the eigenvalue problem, is to enlarge the computational domain by discretizing the vacuum regions surrounding the plasma.[7]

In the present paper, we show how the Galerkin method can be used to convert the stability problem of an ideal MHD plasma surrounded by a thin resistive wall into a linear eigenvalue problem of the standard mathematical form, without discretizing the vacuum regions. This is achieved by introducing an auxiliary variable, $\mathbf{u} = \omega\boldsymbol{\xi}$, and by adding an additional degree-of-freedom (DOF) to the finite-element expansion of the plasma. This additional DOF corresponds to the perturbed current in the thin resistive wall, which is coupled to the plasma surface perturbation using a Green's function approach. The additional row in the eigenvalue matrix equation is the equation for the jump condition across the resistive wall of the perturbed magnetic field. The present paper deals only with a cylindrical plasma configuration, but the approach could readily be extended to non-circular and toroidal geometries.

The MHD equilibrium equation for a circular cylindrical, (r, θ, z), plasma is

$$\mu_0 p' + \left(B^2\right)'/2 + B_\theta^2/r = 0, \tag{1}$$

where $\{\}' \equiv \frac{d}{dr}\{\}$ and all equilibrium quantities are only a function of r. The linearized stability equations can be expressed as

$$\rho\mathbf{u} = \omega\rho\boldsymbol{\xi}, \tag{2a}$$

$$\omega\rho\mathbf{u} = -\tilde{\mathbf{J}} \times \mathbf{B} - \mathbf{J} \times \tilde{\mathbf{B}} + \nabla\tilde{p}. \tag{2b}$$

Here $\tilde{\mathbf{B}} = \nabla \times (\boldsymbol{\xi} \times \mathbf{B})$, $\tilde{\mathbf{J}} = \nabla \times \tilde{\mathbf{B}}/\mu_0$, $\tilde{p} = -\boldsymbol{\xi} \cdot \nabla p - \gamma p \nabla \cdot \boldsymbol{\xi}$, $\boldsymbol{\xi}$ is the usual Lagrangian displacement, $\mathbf{u}$ has been introduced so that the eigenvalue, ω, only appears linearly and the components of $\boldsymbol{\xi}$ and $\mathbf{u}$ are allowed to vary as $e^{i(m\theta + kz - \omega t)}$. The dependence on ω is such that a mode with $\mathrm{Im}(\omega) > 0$ is an instability.

An appropriate set of projections must be chosen to obtain accurate numerical results. The formulation here follows from using the set of projections that Appert et al.[8] introduced, namely

$$\xi_1 = \xi_r, \qquad \xi_2 = (\xi_r + im\xi_\theta)/r, \qquad \xi_3 = ik\xi_z, \tag{3}$$

with a similar set of projections for $\mathbf{u}$.

Each of the unknown eigenfunctions, $\xi_1(r) \ldots u_3(r)$, is expanded as a sum of N expansion functions,

$$\begin{gathered}\xi_1(r) = \textstyle\sum \xi_{1j}\phi_j(r), \quad \xi_2(r) = \sum \xi_{2j}\chi_j(r), \\ \xi_3(r) = \textstyle\sum \xi_{3j}\chi_j(r), \text{ etc.}\end{gathered} \tag{4}$$

Using these expansions, the set of projections in Eq. (3) has the property that

$$\begin{gathered}\nabla \cdot \boldsymbol{\xi} = \xi_{1j}\phi_j' + \xi_{2j}\chi_j + \xi_{3j}\chi_j, \\ \nabla \cdot \mathbf{u} = u_{1j}\phi_j' + u_{2j}\chi_j + u_{3j}\chi_j\end{gathered} \tag{5}$$

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which suggests that, in order for the compression to be arbitrarily small over a given grid interval, $\phi_j(r)$ should be a polynomial of one order higher than $\chi_j(r)$. One choice is to use a linear (tent) element for $\phi_j(r)$ and a constant (hat) element for $\chi_j(r)$.[8] However, we have used a cubic B-spline for $\phi_j(r)$ and the derivative of a cubic B-spline for $\chi_j(r)$.

Making the substitutions of Eqs. (3) and (4) into Eq. (2) then taking the projections

$$\begin{array}{c}\int_0^a \mathrm{d}r\ r\phi_i(r)\left(\hat{\mathbf{r}} - \frac{i}{m}\hat{\boldsymbol{\theta}}\right), \\ \int_0^a \mathrm{d}r\ r\chi_i(r)\left(\frac{ir}{m}\hat{\boldsymbol{\theta}}\right), \quad \int_0^a \mathrm{d}r\ r\chi_i(r)\frac{i}{k}\hat{\mathbf{z}},\end{array} \tag{6}$$

yields a system of equations of the form

$$\omega\mathcal{A}\cdot\mathbf{x} = \mathcal{B}\cdot\mathbf{x}. \tag{7}$$

The matrices $\mathcal{A}$ and $\mathcal{B}$ are made up of $N\times N$ submatrices of size 6×6 such that the i,j^{th} submatrix involves the inner product of the i^{th} test function and the j^{th} expansion function, and $\mathbf{x}$ is the vector of $6N$ unknown coefficients, $\mathbf{x} = [\xi_{11}, \xi_{21}, \xi_{31}, u_{11}, u_{21}, u_{31}, \xi_{12}\ldots, u_{3N}]$.

The key to coupling the plasma displacement to the vacuum comes through the $\hat{\mathbf{r}}$ projection of Eq. (2b). One term that comes from applying the first projection of Eq. (6) to Eq. (2b) is

$$\int_0^a \phi_i(r)\frac{d}{dr}\left[\left(\tilde{p} + \mathbf{B}\cdot\tilde{\mathbf{B}}/\mu_0\right)r\right]\mathrm{d}r \tag{8}$$

which is integrated by parts to yield

$$\left[\left(\tilde{p} + \frac{\mathbf{B}\cdot\tilde{\mathbf{B}}}{\mu_0}\right)r\phi_i(r)\right]_{r=a} - \int_0^a\left(\tilde{p} + \frac{\mathbf{B}\cdot\tilde{\mathbf{B}}}{\mu_0}\right)r\frac{d}{dr}\phi_i(r)\mathrm{d}r. \tag{9}$$

The part at $r = a$ can be replaced by the vacuum field using the jump condition

$$\left[\tilde{p} + \mathbf{B}\cdot\tilde{\mathbf{B}}/\mu_0\right]_{plasma} = \left[\mathbf{B}\cdot\tilde{\mathbf{B}}/\mu_0\right]_{vac}. \tag{10}$$

where the vacuum side is calculated next.

Let a thin resistive wall be placed at $r = b$ where $b \geq a$. Assume there is a vacuum in the regions $a < r < b$, which will be the inner region, and $r > b$, which will be the outer region. The perturbed magnetic field, $\tilde{\mathbf{B}} = \nabla\Phi$, in these two regions will satisfy $\nabla\cdot\tilde{\mathbf{B}} = \nabla^2\Phi = 0$. The general solutions in the two regions are

$$\Phi_{in} = [C_1K_m(kr) + C_2I_m(kr)]e^{i(m\theta+kz-\omega t)} \tag{11a}$$

$$\Phi_{out} = [C_3K_m(kr)]e^{i(m\theta+kz-\omega t)}, \tag{11b}$$

where C_1, C_2, C_3 are constants, and K_m and I_m are modified Bessel functions. Note that the resistive wall is of conductivity, σ, and thickness, d, with $d/b \ll 1$, such that $\tau_w = \mu_0\sigma bd$ is the resistive wall diffusion time.

The boundary conditions at the plasma surface and resistive wall, to lowest order in d/b, are[9]

$$\mathbf{n}\cdot\tilde{\mathbf{B}}_{in}\Big|_{r=a} = \mathbf{n}\cdot\nabla\times(\boldsymbol{\xi}\times\mathbf{B})|_{r=a} \tag{12a}$$

$$\mathbf{n}\cdot\tilde{\mathbf{B}}_{in}\Big|_{r=b} = \mathbf{n}\cdot\tilde{\mathbf{B}}_{out}\Big|_{r=b} \tag{12b}$$

$$\Phi_{out}(r) - \Phi_{in}(r)|_{r=b} = -\frac{i\omega\tau_w b}{m^2+k^2b^2}\mathbf{n}\cdot\tilde{\mathbf{B}}_{in}\Big|_{r=b} \tag{12c}$$

After substituting Equations (11) into these boundary conditions we have a system of 3 equations that could be solved for the three unknown coefficients, C_i, in terms of the boundary displacement $\xi_1(a)$. However, solving for all three has the consequence that ω ends up in the denominator of $\left[\mathbf{B}\cdot\tilde{\mathbf{B}}\right]_{vac}$ so that the standard form, Eq. (7), cannot be obtained.

Instead of solving for all of the unknown C's, let us only use Equations (12a) and (12b) to solve for C_1 and C_3 in terms of C_2 and $\xi_1(a)$. C_2 can then be expressed in terms of the perturbed current in the wall, $\mathbf{j}_{rw} = \nabla(j_s e^{i(m\theta+kz-\omega t)})\times\hat{\mathbf{r}}$, through the relation

$$j_s = C_2\left(\dot{I}_bK_b - I_b\dot{K}_b\right)/\dot{K}_b/\mu_0. \tag{13}$$

If there is no perturbed current in the wall, then $C_2 = 0$, which is correct for having no wall. Using the C_i in terms of j_s and $\xi_1(a)$ yields

$$\frac{\omega\tau_w b\dot{K}_b}{(m^2+k^2b^2)\dot{K}_a}\left[\xi_1(a)\alpha - \mu_0 j_s ik\frac{I_a\dot{K}_b - \dot{I}_bK_a}{\dot{I}_bK_b - I_b\dot{K}_b}\right] = -\mu_0 j_s \tag{14}$$

for the jump condition of the perturbed magnetic field across the wall, Eq. (12c), and

$$\left[\mathbf{B}\cdot\tilde{\mathbf{B}}/\mu_0\right]_{vac} = j_s i\alpha\frac{I_a\dot{K}_a - \dot{I}_aK_a}{I_b\dot{K}_b - \dot{I}_bK_b}\frac{\dot{K}_b}{\dot{K}_a} - \xi_1(a)\alpha^2\frac{K_a}{\mu_0k\dot{K}_a} \tag{15}$$

for the vacuum perturbed pressure. The notation used is $\alpha \equiv mB_\theta(a)/a + kB_z(a)$, $K_a \equiv K_m(ka)$, $\dot{K}_a \equiv \left[\frac{d}{d(kr)}K_m(kr)\right]_{r=a}$, etc.

To include the interaction of the plasma with the resistive wall, the following modifications to Eq. (7) are made. j_s is an additional unknown in the DOF vector, $\mathbf{x}$, which adds a single column to $\mathcal{A}$ and $\mathcal{B}$. The additional equation (row) for these matrices is given by Eq. (14). (The rank of the matrices is only changed from $6N$ to $6N+1$.) Finally, the vacuum perturbed pressure given by Eq. (15), which includes j_s, is substituted into Eq. (9) using Eq. (10).

To recover the jump condition for an ideal wall [no wall], solve Eq. (14) explicitly for j_s in the limit $\tau_w \to \infty$ [$\tau_w = 0$] then substitute this j_s into Eq. (15) which goes into Eq. (9).

By including the resistive wall boundary condition as an additional equation, the eigensystem takes the standard form of a generalized eigenvalue equation which can

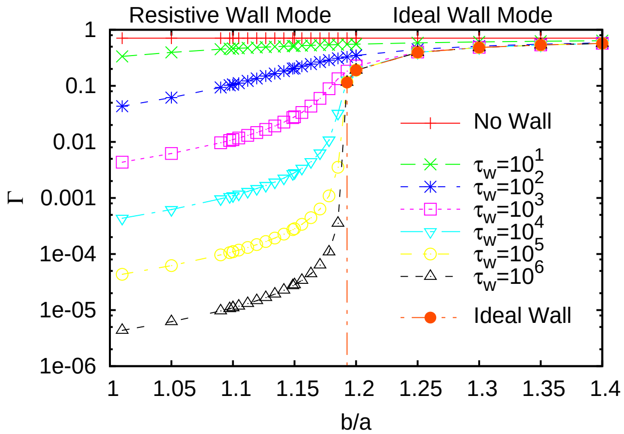

FIG. 1: The growth rate of the unstable free boundary kink mode depends on the location and time constant of the resistive wall. The plasma has a constant axial current density with $a = 1$, $B_z(a) = 15$, $B_\theta(a) = 1$, $\rho = 1$, $m = 2$, $k = -0.1$, and the discretization grid has 20 points.

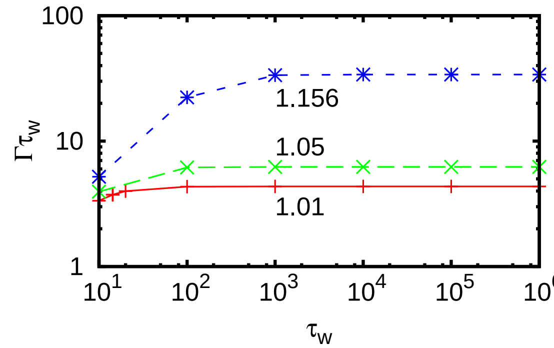

FIG. 2: The product of the wall time and the growth rate of the RWM (i.e. $\Gamma\tau_w$) as a function of τ_w showing that $\Gamma \propto \tau_w^{-1}$ for large enough wall times. The plasma parameters are the same as those of Fig. 1. The number labels indicate the wall location (b/a) for that curve.

be solved with any standard matrix eigensolver (we have used the LAPACK routine ZGGEV).

One profile of interest for examining the effects of wall resistivity is a constant current density, pressureless plasma. Let us choose $a = 1$, $B_z(a) = 15$, $B_\theta(a) = 1$, $\rho = 1$, $m = 2$, $k = -0.1$, $\mu_0 = 1$. This configuration has one unstable eigenvalue corresponding to the free boundary kink mode. The instability growth rate, $\Gamma[= Im(\omega)]$, as a function of wall location (b/a) and time constant (τ_w) is shown in Fig. 1 for the plasma discretized on 20 grid points. As predicted previously,[2] the ideal wall stabilizes the kink mode for a region of locations, $b < b_c$, where $b_c \approx 1.19$ for this case. Note that for $b < b_c$, a resistive wall reduces Γ but never completely stabilizes the kink mode, and thus the kink is called a resistive wall mode (RWM). The relationship between the RWM growth rate and τ_w is best seen in Fig. 2, which shows that for large enough values of τ_w, the growth rate varies inversely with the wall time, $\Gamma \propto 1/\tau_w$.

Because the resistive wall makes the problem non-self-adjoint, it is interesting to look at the spectrum of ω in the complex plane as shown in Fig. 3. The wall location is $b = 1.01$, which is within the range that the ideal wall stabilizes the kink mode. Thus as the wall time, τ_w, is varied from ∞ (ideal wall) to 0 (no wall), the spectrum varies from having a stabilized kink mode (label 1) to having an unstable and damped (4 and 7) no wall mode (NWM). In between ∞ and 0, the damped resistive wall mode (dRWM) shows up as the damped kink mode[9] (1 to 2) as long as $\tau_w \gtrsim \Gamma_{nw}^{-1}$, where Γ_{nw} is the NWM growth rate. For $\tau_w \lesssim \Gamma_{nw}^{-1}$ the dRWMs collapse onto the dNWM (2 to 3 to 4), and a super damped mode appears (2 to 3 to 5) which arises because of the extra degree of freedom introduced by including j_s as an additional unknown with Eq. (14) as an additional equation. At the same time that the damped modes are transitioning from the ideal wall regime to the no wall regime (1 to 3) the resistive wall mode growth rate increases (from 6 to 7) until it plateaus at Γ_{nw}. The modes at 8, while unexpected, are only damped for intermediate values of τ_w such that the correct results are approached for $\tau_w \to \infty$, $\tau_w \to 0$. Note that as the ideal wall is moved farther from the plasma (not shown explicitly), the stabilized kink modes decrease in magnitude along the real axis until they destabilize and move along the Γ axis toward the NWMs (1 to 6 to 4,7).

We have shown that it is possible to obtain a matrix eigenvalue equation in standard form to solve for the entire spectrum of an ideal MHD plasma, including the interaction with a resistive wall, without discretizing the vacuum regions. Adding the resistive wall only increases the rank of the matrices in Eq. (7) from $6N$ to $6N + 1$. The extra DOF is the current in the wall, j_s, and the extra equation is the wall jump condition, Eq. (14). As $\tau_w \to 0$ we recover the results of having no wall, and as $\tau_w \to \infty$ we recover the results of having an ideal wall. The method of including the jump condition at the resistive wall as an additional equation can be generalized to a non-circular, non-cylindrical geometry by having an additional equation for each poloidal harmonic, m, the details of which are shown elsewhere.[10] An equilibrium flow to stabilize the RWM[11,12] can be added to Eq. (2) very simply[10], and this stabilization will be shown later.

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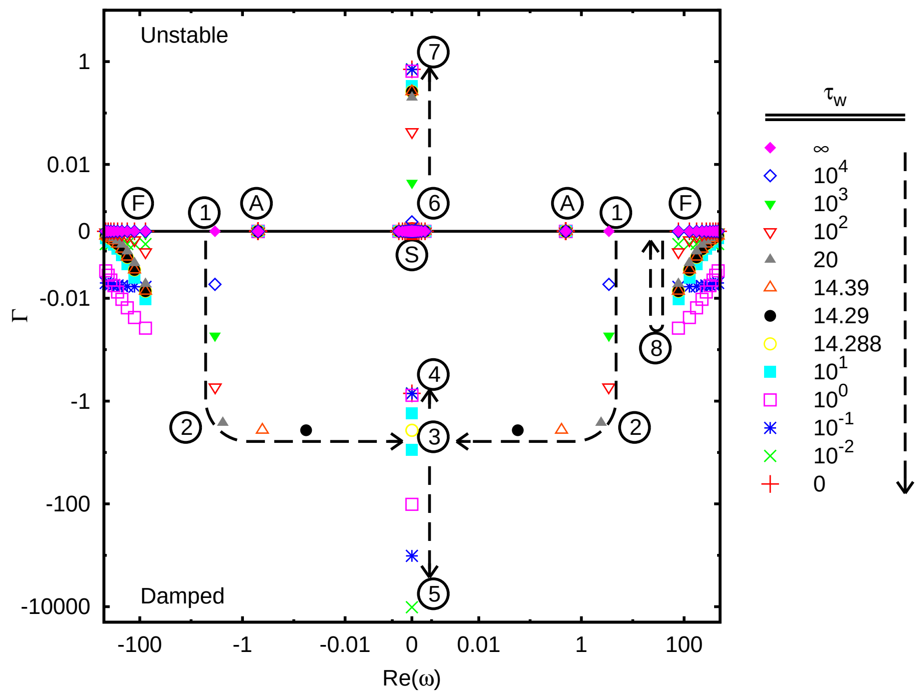

FIG. 3: Ideal MHD spectrum in the complex plane for various values of τ_w with the plasma parameters of Fig. 1 and $b/a = 1.01$. The letter-labelled modes are: F) Fast; A) Alfvén; S) Sound.

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