IDS160B - Relativistic Astrophysics and Cosmology
Problem Set 4 1995 March

G. F. SMOOT
Department of Physics,
University of California, Berkeley, USA 94720

1 Relativistic Kinematics & Compton Scattering

1.1 Cross Section

For low energy photons ($E_\gamma = h\nu << mc^2$), the scattering of radiation from free charges reduces to the classical case of Thomson scattering covered in your EM course.

$$\frac{d\sigma_T}{d\Omega} = \frac{1}{2} r_o^2 (1 + cos^2\theta) \quad \text{and} \quad \sigma_T = \frac{8\pi}{3} r_o^2$$

In this case $E'_\gamma \cong E_\gamma$, that is the scattering does not change the energy E_γ of the incident photon. These relationships are derived in the classical low energy regime. They are modified when quantum (photons) relativistic effects are taken into account.

The differential cross section for unpolarized radiation in the theory of quantum electrodynamics (Heitler 1954) is given by the Klein-Nishina formula:

$$\frac{d\sigma_T}{d\Omega} = \frac{1}{2} r_o^2 (\frac{E'_\gamma}{E_\gamma})^2 (\frac{E'_\gamma}{E_\gamma} + \frac{E_\gamma}{E'_\gamma} - 1 + cos^2\theta)$$

Note that if $E'_\gamma \cong E_\gamma$, this reduces to the classical Thomson scattering case.

Derive the relationship between the incident photon wavelength λ and scattered photon wavelength λ' and the scattering angle θ for a photon hitting an electron at rest.

$$\lambda' - \lambda = \lambda_c (1 - cos\theta), \quad \text{where} \quad \lambda_c \equiv \frac{h}{mc}$$

For an electron $\lambda_c = 2.426 \times 10^{-10}$ cm.

Suggestion: Draw a diagram with the incoming photon, the scattered photon leaving at angle θ to the incident direction and the scatter electron having momentum p_e going off at angle ϕ from the incident photon direction. Write down the three equations of motion from the conservation of energy-momentum four vector. (You only have to deal with three since the scattering must take place in a plane.) Since you have three equations in four unknowns, you can eliminate the unknown electron recoil momentum and direction. It is easiest if you first find an expression for the scattered photon energy in terms of the incident energy and the angle θ.

Note that the scattered photon always ends up with less energy than the incident photon.

1.2 Scattering from Electons in Motion

Whenever a moving electon has sufficient kinetic energy compared to the photon, net energy may be transfered to the photon from the electron. Such a case is called *inverse Compton Scattering.*

Consider the case where $E_\gamma = h\nu << m_e c^2$ is valid in the reference frame where the electron is at rest, so that one can use the Thomson scattering limit. What is the maximum energy photon that satisfies this limit (i.e. $< 0.1 m_e c^2$) for an electron with Lorentz factor γ_e? What about for typical energetic electrons in a galaxy cluster where X-rays observations indicate temperatures of 10 to 30 KeV?

Show that for a typical photon hitting an electron with Lorentz factor γ_e that the energy of the photon transforms from the incident E_γ to roughly $\gamma_e E_\gamma$ in the electron rest frame to $\gamma_e^2 E_\gamma$ when the scattered photon is observed in the rest frame. The spread is very wide from losing energy to gaining more than this factor.

This process therefore can convert a low-enrgy photon to a high energy photon by a factor of γ_e^2. For an electron kinetic energy kT what is the fractional change in photon energy to first order in $kT/m_e c^2$ per scattering? How much is this for a galactic cluster with plasma temperature $kT = 10$ KeV?

2 Advance of Perihelion

One of the early successes of Einstein's General Relativity was the correct prediction of the advance of the perhelion of Mercury. In the last problem of the previous set we derived the equations of motion for a particle in the Schwarzschild metric (black hole).

$$\dot{r}^2 + (1 - \frac{2Gm}{r})(1 + \frac{\ell'^2}{r^2}) = E'^2$$

where the $'$ indicated per unit mass for the energy E and angular momentum ℓ. Change variables as is usually done in the Newtonian calculations of solar system orbits to

$$u = \frac{1}{r}$$

so that

$$\dot{r} = \frac{dr}{d\phi}\dot{\phi} = -\frac{1}{u^2}\frac{du}{d\phi}\frac{\ell'}{r^2} = -\frac{du}{d\phi}\ell'$$

to get

$$E'^2 - \ell'^2(\frac{du}{d\phi})^2 - \ell'^2 u^2(1 - 2GMu) = (1 - 2GMu)$$

Take the derivative with respect to ϕ to get

$$\frac{d^2u}{d\phi^2} + u - \frac{GM}{\ell^2} - 3GMu^2 = 0.$$

Compare this to the Newtonian equation

$$\frac{d^2u}{d\phi^2} + u - \frac{GM}{\ell^2} = 0.$$

Remember or show that the expression

$$u = \frac{GM}{\ell'^2}[1 + \epsilon cos(\phi - \phi_o)]$$

is a solution to the Newtonian equation, where ϵ is a constant which is the eccentricity of the orbit and the perhelion is located at the angle ϕ_o.

For the case we are considering, Mercury going around the Sun, $M_\odot \cong 2.0 \times 10^{33}$ gms and $r = 5.5. \times 10^{12}$ cm, the extra term

$$\frac{3GM}{rc^2} \approx 10^{-7}$$

is quite small so it is sufficient to solve the full equation by successive approximation. That is replace u^2 in the small term by the Newtonian solution of an elipse and try to solve the equation.

$$\frac{d^2u}{d\phi^2} + u - \frac{GM}{\ell^2} - \frac{6\epsilon(GM)^3}{\ell'^4}cos(\phi - \phi_o) - \frac{3\epsilon^2(GM)^3}{\ell'^4}cos^2(\phi - \phi_o) = 0$$

This is the eqaution for a harmonic oscillator, with a restoring force $-u$, and two oscillating driving forces. The first of the oscillating driving forces is in resonance with the natural oscillation and thus gradually builds up, while the second merely produces a periodic perturbation of small amplitude. Thus to first order we can ignore the second driving force to consider the equation

$$\frac{d^2u}{d\phi^2} + u - \frac{GM}{\ell^2} - \frac{6\epsilon(GM)^3}{\ell'^4}cos(\phi - \phi_o) = 0$$

Show that the solution to this differential equation is

$$u = \frac{GM}{\ell'^2}[1 + \epsilon cos(\phi - \phi_o)] + \frac{3\epsilon(GM)^3}{\ell'^4}\phi sin(\phi - \phi_o)$$

Use the trigonometric relation for the sines and cosines of sums of angles to show that for small times this is equivalent to the advance of the perhelion of the orbit.

Hint: sum the angles so that the approximate solution is of the form

$$u = \frac{GM}{\ell'^2}[1 + \epsilon cos(\phi - \phi_o - \frac{3(GM)^2}{\ell'^2}\phi)]$$

as long as $(GM)^2\phi/\ell'^2 << 1$.

Find an expression for the change in the location of the perhelion from one orbit to the next (i.e. the angle which perhelion exceeds 2π). Which direction does the perhelion move - in the same as the orbit or the reverse? For Mercury how much does this work out to be per orbit? If Mercury's orbital period is roughly 88 days how much does this work out to be in a century? Is the Newtonian approximation very good? By what factor should the precession of earth's perhelion advance differ if the earth is at $r = 1.5 \times 10^{13} cm$?

3 Radiation Dominated or Ulra-Relativistic Era

3.1 Number and Energy Density

Integrate the occupation of states formula

$$\frac{dn(p)}{dp}dp = \frac{8\pi}{h^3}\frac{p^2 dp}{exp[\frac{E-\mu}{kT}] \pm 1}$$

from 0 to ∞ in the case that the particles are ultra-relativisitc meaning that their mass and μ are negligble to find the number of particles of each species and mode. First express the answer in terms of a dimensionaless integral with a coefficient that is proportional to $(kT)^3$ indicating the density of particles is $\propto T^3$ and thus expansion conserves the number of particles but dilutes them. Compute the proportionality coefficient.

Now find the total energy of these relativistic particles making use of the relation needed above that $E \cong pc$ for ultrarelativisitc particles. Hint: multiply the differential density of particles at momentum p times the energy E and integrate. Again express the answer in terms of a dimensionaless integral and a coefficient out front showing that it is proportional to $(kT)^4$ or that the energy density is $\propto T^4$. You can then evaluate the integrals or look them up in a table of integrals. What is the difference between fermions and bosons in terms of the final answer? For photons what is an expression for the constant of proportionality a_r and the Stephan-Boltzmann constant $E/V = a_r T^4 = 4/c\, \sigma_{SB} T^4$ in terms of the fundamental constants? Note that the Stephan-Boltzmann constant is for energy flux. Know that the reason that there are two names attached to the constant is that the photon has spin 1 and thus has two degrees of freedom and thus the constant should be twice the original derivation.

So the total energy density in ultrarelativistic phase is

$$E/V = \rho_{rel} c^2 = \frac{g_{eff}}{2} a_r T^4 = \frac{g_{eff}}{2}\frac{4}{c}\sigma_{SB} T^4$$

where g_{eff} is the effective number of degrees of freedom for all the particle species present.

3.2 Expansion Rate and Temperature

In the radiation dominated era we know that $a \propto t^{1/2}$ So that

$$H = \frac{\dot{a}}{a} = \frac{1}{2}\frac{1}{t}$$

One Friedmann equation (energy conservation) gives us

$$(\frac{\dot{a}}{a})^2 = \frac{8\pi}{3}G\rho$$

for the early universe the curvature is negligible (see the notes). Taking the square root gives

$$\frac{\dot{a}}{a} = \sqrt{\frac{8\pi}{3}G\rho} = \sqrt{\frac{8\pi}{3}G\frac{g_{eff}}{2}\sigma_{SB}}\ T^2$$

Thus we have the relationship

$$2H = \frac{1}{t} = \sqrt{\frac{16\pi G\sigma_{SB}}{3c^2}g_{eff}}T^2 \cong 3.027 \times 10^{-21}[\frac{T}{1K}]^2 sec^{-1}$$

By dimensional analysis this should depend on the inverse square of the kinetic energy kT in terms of the the Planck energy (Planck mass, $M_{Pl} = \sqrt{\hbar c/G}$ times c^2), times the Planck time $t_{Pl} = \sqrt{\hbar G/c^5}$, and clearly from above the coefficient includes $1/\sqrt{g_{eff}}$. Find the proportionality factor for the dimensional analysis relation

$$t \propto \frac{1}{\sqrt{g_{eff}}}(\frac{kT}{M_{Pl}c^2})^{-2}t_{Pl}$$

3.3 Entropy Density

The entropy is defined by the relation $DS = dQ/T$. For our simple case

$$S = \frac{V}{kT}(E + P)$$

In the relativisitic phase the pressure $P = \frac{1}{3}E$ so that

$$S = \frac{4}{3}\frac{V}{kT}E$$

and to first order $\delta S = 0$ since the expansion of the Universe is nearly adiabatic in this phase.

We define the entropy per unit volume s as

$$s \equiv \frac{S}{V} = \frac{2}{3}g_{eff}\frac{\sigma T^4}{kT} \cong 1.80 g_{eff} n_\gamma$$

4 Big Bang Nucleosynthesis

4.1 Deuterium Production Rate

The mean time for neutron capture by a proton (or proton capture by a neutron) is

$$t = (n_p \sigma v)^{-1}$$

where $\sigma \approx 10^{-28}$ cm^2 and n_p is the density of protons and v is the relative velocity between the protons and the neutron. Compute this mean time (its inverse is the rate) for a neutron to be captured and make a deuteron at temperature T and baryon density ρ_B. For example, consider $T = 10^9$ K and $\rho_B = 2 \times 10^{-5} gcm^{-3}$. Since there is a relation between the temperature and the density, $\rho_B \propto T^3$, express your answer in a form that would let you figure how the time (rate) scales with temperature.

4.2 Deuteron Destruction Rate

Free nucleons are favored over deuterons down to a radiation temperature of $kT \cong 0.07 MeV$. If the ratio of baryons to photons is $\eta = 2.76 \times 10^{-8} \Omega_B h^2$, what is the ratio of energies photons with energies exceeding the deuteron binding energy, 2.22 MeV, to the number of baryons at $kT \sim 0.07$ MeV? How would this scale with energy?