IDS 160B
Problem Set 3

1 Relativistic Kinematics & Doppler Effect

1.1 Speed - Relativistic Doppler Effect

The relativistic Doppler shift formula is $\nu'/\nu = \gamma(1 - \beta\cos\theta)$. It can be obtained by assuming the standard Doppler formula with a time dilation factor γ. (For a more difficult but rigorous derivation, consider pulses emitted from the origin of a coordinate system moving with velocity β at an angle θ, and use the Minkowski metric.) Write the formula from the point of view of both the emitter and the receiver coordinate systems, and find a relationship between $\cos\theta$ and $\cos\theta'$.
Hint: The formulae must look the same if time reversal invariance holds.

An expanding Robertson-Walker universe will produce a cosmological redshift, z, proportional to separation distance, d to first order. This can be seen by making a Taylor series expansion of the scale factor $a(t)$.

$$1 + z = \frac{\lambda_r}{\lambda_e} = \frac{\nu_e}{\nu_r} = \frac{a(t_r)}{a(t_e)} \cong 1 + \frac{1}{a}\frac{da}{dt}\bigg|_e \delta t = 1 + \frac{Hd}{c} \quad \text{or,} \quad cz \cong Hd,$$

where $H \equiv d\ln a/dt$ is the Hubble expansion rate. What relationship for recessional velocity versus distance would you determine if you did not know the Universe was expanding, but assumed the relation was due to a Doppler effect from recessional velocity? How fast (in km/sec) would a galaxy with cosmological redshift $z = 0.03$, be thought to be receding?

1.2 Greater Speed - Galaxy as a colorful Goldfish Bowl

In our effort to entertain you and train you for a potential new job, e.g. science advisor to *StarTrek Extragalactic*, which is the show that might be in the works when you complete graduate school, consider the following problem. A spacecraft is traveling outwards on a normal through the center of the Galaxy at $0.99c$. At a certain moment it is at an angle of $45°$ degrees to the plane of the Galaxy as viewed from the edge of the Galaxy. According to Hoyle, an observer in the spacecraft would then say that he appeared to be inside and heading toward the mouth of a goldfish bowl with a blue rim and a red body.

(a) Do you believe this to be true?

(b) Calculate the angle θ', the direction from which light from the edge of the Galaxy appears to come when viewed from the spacecraft.

(c) Determine the Doppler effect frequency ratio ν'/ν for light coming from the rim of the Galaxy, from $\theta' = 90°$, and from the center of the Galaxy.

(d) How far away is the Galactic Center from the spacecraft observer's point of view?

Frame of the Galaxy Frame of Spacecraft

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1.3 Travel Faster than Light

Suppose that there are tachyons (particles that travel faster than light) or that it is possible to travel through a "wormhole" in such a way that your space ship covers a greater distance than light could in the same time. (The same might be true of "Warp Drive".) Consider a tachyon traveling faster than light from the space-time point $x_1 = t_1 = 0$. It arrives at $x_2 = d$ at a later time where $t_2 = d/v$ and $v > c$. Is it possible to make a Lorentz transform to a coordinate system in which the arrival precedes the start of the tachyon trip? If so, for what range of β of the Lorentz transformation can this reversed time-ordering take place?

Using the Minkowski metric, $ds^2 = c^2 d\tau^2 = c^2 dt^2 - dx^2$ (suppressing y & z), what proper time has elapsed for the tachyon or on the space ship? What sign is $\Delta\tau^2$? Points in space-time that are outside each other's light-cones (future or past) are said to be separated in a space-like manner.

2 Cosmic Rays

Using the Lorentz force law, derive a formula for the radius of curvature of a particle with charge Ze and energy E. Next, express the formula in quantities useful for highly energetic cosmic rays. For units, use the radius R in parsecs, the energy E in 10^{15} eV, and the magnetic field in μGauss. Absorb the electron charge so that only R, B, E, and Z and a pure number appear in the formula.
Hint: One parsec is (1AU/1 arcsecond) $= 3.08568 \times 10^{18}$cm $\cong 3.262$ light years.

3 Schwarzschild Metric

$$ds^2 = \left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 - \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

The Schwarzschild Metric may be derived by assuming that the metric depends only on the radius for a static, spherically-symmetric mass. This means that when written in the form

$$ds^2 = e^{\lambda} dt^2 - e^{\nu} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

the metric can depend only upon the coordinate r. The Einstein equations imply that $\lambda + \nu = 0$ and that $e^{\lambda} = 1 - 2GM/r$ exterior to the spherical mass. (Note we are using $c = 1$ for these equations. Below, c is included when it is appropriate, i.e. when calculating numbers.) One could use the same Einstein equations to solve for the metric interior to the mass. However, the relativistic analog of the Newtonian theorem that says for the interior of a spherical shell, the potential is constant. This means that the metric at any location can be found by using the interior mass, i.e. $g_{00} = e^{\lambda} = 1 - 2GM(r)/r$ where the interior mass is the integral of the stress-energy tensor, T_0^0,

$$M(r) = \int_o^r T_0^0 4\pi R^2 dR.$$

This matches to the Schwarzschild metric if M is the total integrated $M(r)$ to the surface of the mass.

Make the same assumption as Einstein—that the energy density consists of only a uniform distribution of mass, ρ, and a cosmological constant energy density, $\Lambda/8\pi Gc^2$. Find the interior solution for the metric. What is the metric for the region outside of the mass, total value M, if the cosmological energy density continues? Note that this metric does not become asymptotically flat as $r \to \infty$. However, as we discussed in class, $\Lambda/c^2 \leq 10^{-58}$ cm^{-2}, is very small, so there is a range of distances

$$\frac{c}{\sqrt{\Lambda}} >> r >> \frac{GM}{c^2}$$

in which the metric is nearly flat.

In the weak field limit $g_{00} \cong 1 + 2\Phi/c^2$. What is the effective potential Φ for this case? What is the force as a function of distance? What sign does it have relative to gravity? This is why the cosmological constant is sometimes referred to as a force that increases with distance.

Consider a cluster of galaxies extending as far as 10 Mpc (10 million parsecs). How big could the cosmological constant correction be? If there are about 2000 galaxies in the cluster with typical masses on the scale of $10^{11}\mathrm{M}_\odot$, then what is its ratio relative to Newtonian gravity?

4 Maximum Field

Use the Uncertainty Principle to argue that there is a maximum electric field that can exist in space. Provide a formula for this electric field, and find its value in volts per cm.

5 Electrostatics: Reissner-Nördstrom Metric

An electrically charged mass will be surrounded by an electric field, which gives rise to a nonzero energy-momentum tensor through-out space. To find the metric requires solving the coupled Einstein and electromagnetic equations. For the stationary spherically symmetric solution, the metric will have the general form of the equations in Problem 3. However, the functions $\lambda(r)$ and $\nu(r)$ will have a different interpretation. The electric field will be in the radial direction and the electromagnetic field tensor, in spherical coordinates, is

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E(r) & 0 & 0 \\ E(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where T^{00} is the well-known energy density for an electromagnetic field, and T^{0k} is the Poynting vector:

$$T^{00} = \frac{1}{8\pi}(\mathbf{E}^2 + \mathbf{B}^2) \quad T^{0k} = \frac{1}{4\pi}(\mathbf{E} \times \mathbf{B})^k.$$

The trace is zero: $T^\mu_\mu = 0$. Setting $c = 1$ and using the same theorem for a spherically symmetric energy system, the results for the metric and the electric field are:

$$ds^2 = \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right) dt^2 - \left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right)^{-1} dr^2 + r^2\, d\Omega^2$$

and

$$E(r) = \frac{Q}{r^2}$$

Is it clear that Gauss's law is valid?

Clearly there is a radius of a charged particle at which the metric will go flat. What is that radius?

What is that radius for an electron? Express your answer in terms of the classical radius of the electron. What is the field strength E there in volts per cm? How does this compare to the maximum field strength computed in the previous problem? Can you conclude that the real electric charge on the electron is actually greater than e because of vacuum polarization effects? What if the electron was really a point particle? Does the metric have a singularity if $Q^2 > GM^2$?

6 White Dwarfs

In 1932 Landau gave a rough derivation of the Chandrasekhar (1931) mass limit for an electron-degenerate star. His argument applies to both white dwarf and neutron stars.

Suppose that there are N ferminons in a star of radius R, so that the number density of fermions is $n \sim N/R^3$. The volume per fermion is $\sim 1/n$ since the Pauli exclusion principle indicates that they cannot occupy the same phase space. Then, the Heisenberg uncertainty principle implies the momentum of the fermions is $\sim \hbar n^{1/3}$. Thus, the Fermi energy of a gas particle in the relativisitic regime is

$$E_{\mathrm{F-rel}} \sim \hbar n^{1/3} c \sim \frac{\hbar c N^{1/3}}{R}.$$

For a non-relativitistic particle $E \cong p^2/2m$, so that

$$E_{\mathrm{F-nonrel}} \sim \hbar n^{2/3} c \sim \frac{\hbar c N^{2/3}}{R^2}$$

The gravitational energy per fermion is

$$E_G \sim -\frac{GMm_b}{R}$$

where $M = Nm_b$ and we assume that most of the mass is in baryons (even if the pressure comes from electrons). Equilibrium is achieved at a minimum of the total energy, $E = E_{\mathrm{F}} + E_G$. In the relativistic case

$$E_{\mathrm{rel}} \sim \frac{\hbar c N^{1/3}}{R} - \frac{GMm_b}{R}$$

Both terms scale as $1/R$. When E is positive, decreasing R will increase both E and E_{F}; increasing r decreases both E and E_{F}. If E_{F} is decreased sufficiently the electrons (or neutrons) will become non-relativistic and $E_{\mathrm{F}} \sim \hbar c N^{2/3}/R^2$. Thus eventually E_G will become larger than E_{F}, and then E will be negative (eventually tending to zero as $R \to \infty$). There will therefore be a stable equilibrium at a finite value of R. (A stable equilibrium requires a minimum in the potential and the above argument is that the potential is positive at small R, zero at $R \to \infty$ and negative in between. This implies that there is a minimum in the potential at a finite value for R.)

If the sign of E is negative, E can be decreased without bound by decreasing R and no equilibrium exists so that gravitational collapse will occur.

The maximum baryon number for equilibrium is therefore determined by setting $E = 0$:

$$N_{\max} \sim \left(\frac{\hbar c}{Gm_b^2}\right)^{3/2} \approx 2 \times 10^{57}$$

$$M_{\max} \sim N_{\max} m_b \approx 1.5 M_\odot, \qquad M_\odot = 1.989 \times 10^{33} \text{ gm}$$

With the exception of composition-dependent numerical factors, the maximum mass of a degenerate star thus depends only on fundamental constants. What would you estimate for the ratio of the maximum masses of a star made of helium and one made of hydrogen?

The equilibrium radius associated with masses M approaching $M_{\max}$ is determined by the onset of relativistic degeneracy:

$$E_{\mathrm{F}} \geq mc^2$$

where m is the mass of the degenerate fermion (i.e. electrons or neutrons). The formula for $E_{\mathrm{F-rel}}$ gives an idea of the radius scale.

$$\begin{aligned} R &\sim \frac{\hbar}{mc}\left(\frac{\hbar c}{G m_b^2}\right)^{1/2} \\ &\sim 5 \times 10^8 cm, \qquad m = m_e \\ &\sim 3 \times 10^5 cm, \qquad m = m_b. \end{aligned}$$

There are thus two distinct regimes of collapse: one for densities above white dwarf values and another for densities above nuclear densities.

Though these arguments give the correct order of magnitude, note that the rough radius for collapse of a neutron star is approximately equal to the Schwarzschild radius. Thus we expect that the equilibrium value for the radius of such a neutron star is not much greater than the Schwarzschild radius and a more careful treatment will be necessary.

We expect a star to be in hydrostatic equilibrium; the inward pull of gravity should be matched by the outward push of the pressure. For a spherically symmetric distribution of matter, the mass interior to a radius r is

$$M(r) = \int_0^r \rho\, 4\pi R^2\, dR, \quad \text{or} \quad \frac{dM(r)}{dr} = 4\pi r^2 \rho$$

(compare with the expression in Problem 3). To find the equation of hydrostatic equilibrium, conder an infinitesimal fluid element lying between r and $r + dr$ and having area dA perpendicular to the radial direction. The gravitational attraction between $M(r)$ and the mass $dm = \rho\, dA\, dr$ is $GM(r) dm/r^2$.

The net outward pressure force on dm is $-[P(r+dr)-P(r)]dA$, so in equilibrium these are matched giving

$$-\frac{dP}{dr}\,dr\;dA=\frac{GM(r)}{r^2}\,dm$$

or

$$\frac{dP}{dr}=-\frac{GM(r)\rho}{r^2}.$$

These two equations can be combined to give the differential equation

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right)=-4\pi G\rho$$

This is useful, provided we know the equation of state (i.e., the relationship between the pressure P and density ρ) for matter. For many gases this is simply

$$P=\mathrm{K}\rho^\gamma \quad \text{where K and } \gamma \text{ are constants.}$$

This relation is true for a degenerate fermion gas. A volume V at pressure P has a pressure energy PV. A degenerate fermion gas with N particles in volume V has a number density $n=N/V$ and a Fermi energy E_{F} per particle given in the formulae above. Thus $P=NE_{\mathrm{F}}/V\sim nE_{\mathrm{F}}\sim n\hbar n^{1/3}=\hbar n^{4/3}$ for the relativistic case, and $\sim n\hbar n^{2/3}=\hbar n^{5/3}$ for the non-relativistic case. Since $n\propto\rho$, $\gamma=4/3$ for the relativistic case and $\gamma=5/3$ for the non-relativistic case. The last is familiar since

$$PV^\gamma=\text{constant}\quad\text{or}\quad P=K\rho^\gamma$$

with $\gamma=5/3$ is the adiabatic relation for an ideal monoatomic gas (having only translational degrees of freedom). Thus, this relation holds for the thermal phase of a star.

With the equation of state in hand, this differential equation can be solved (although usually numerically).

The central pressure required to support a self-gravitating sphere of radius R and mass M, where $P\propto\rho^{5/3}$ is

$$P_c=0.770\,\frac{GM^2}{R^4},$$

and the corresponding central density is given by

$$\rho_c = 5.99\langle\rho\rangle = 1.43\ \frac{M}{R^3}.$$

For a self-gravitating sphere with $P \propto \rho^{4/3}$, hydrostatic equilibrium requires that

$$P_c = 11.0\ \frac{GM^2}{R^4}, \quad \text{and} \quad \rho_c = 54.2\langle\rho\rangle = 12.9\ \frac{M}{R^3}$$

A careful calculation of the degenerate electron gas pressure gives

$$\begin{aligned} P_e &= \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2 n_e^{5/3}}{m_e} = 0.0485\ \frac{h^2 n_e^{5/3}}{m_e} \quad (\text{non}-\text{relativistic}) \\ &= \frac{1}{4}\left(\frac{3}{8\pi}\right)^{1/3}\ hc\ n_e^{4/3} = 0.123\ hc\ n_e^{4/3} \quad (\text{relativistic}). \end{aligned}$$

To a very good approximation, a white dwarf is charge neutral so that the number density of the electrons is equal to the sum over ion type i of the charge Z_i of the ion species, times the number density n_i of the ith ion species:

$$n_e = \sum_i Z n_i = \sum_i \frac{Z_i}{A_i} n_b = \left\langle\frac{Z}{A}\right\rangle n_b$$

Because electrons are so light

$$n_e = \left\langle\frac{Z}{A}\right\rangle \frac{\rho}{m_b}.$$

Show that

$$P_e = 0.0485\ \frac{h^2}{m_e}\ \left(\frac{Z}{A}\right)^{5/3}\ \frac{\rho^{5/3}}{m_b^{5/3}}$$

and that

$$R = 0.114\ \frac{h^2}{G m_e m_b^{5/3}}\ \left(\frac{Z}{A}\right)^{5/3} M^{-1/3}$$

where Z/A is now the average over all ion species. Notice that the volume is inversely proportional to the mass.

Compute R and ρ_c for $M = 0.5M_\odot$ and $M = 1.0M_\odot$. It is helpful to have the non-relativisitic ($\gamma = 5/3$) relations:

$$\frac{R}{10^4\ \text{km}} = 1.122 \times \left(\frac{\rho_c}{10^6\ \text{g cm}^{-3}}\right)^{-1/6} \left(\frac{A}{2Z}\right)^{-5/6}$$

and

$$\frac{M}{M_\odot} = 0.4964 \left(\frac{\rho_c}{10^6 \text{ g cm}^{-3}}\right)^{-1/2} \left(\frac{A}{2Z}\right)^{-5/2} = 0.7001 \left(\frac{R}{10^4 \text{ km}}\right)^{-3} \left(\frac{A}{2Z}\right)^{-5/2}.$$

For the relativistic electron gas the degenerate pressure is

$$P_e = 0.123 \; hc \; n_e^{4/3}.$$

It is easy to show that

$$P_e = 0.123 \; \frac{hc}{m_b^{4/3}} \; \left(\frac{Z}{A}\right)^{4/3} \rho^{4/3}.$$

Set the central pressure needed to that available from relativistic electron degeneracy. Show that R cancels out of each equation and solve for M to get Chandrasekhar's mass limit:

$$M_{\text{Ch}} = 0.2 \left(\frac{Z}{A}\right)^2 \left(\frac{hc}{Gm_b^2}\right)^{3/2} m_b.$$

Again, the maximum mass of a white dwarf is set by fundamental constants. Evaluate this as you did in the Landau case for a star made of helium or hydrogen to find its maximum mass and radius. Once again, if you do not want to put in the proton/neutron mass and G, h, and c, it will help to have the relations:

$$\frac{M}{M_\odot} = 1.4587 \left(\frac{2Z}{A}\right)^2$$

and

$$\frac{R}{10^4 \text{ km}} = 3.347 \times \left(\frac{\rho_c}{10^6 \text{ g cm}^{-3}}\right)^{-1/3} \left(\frac{A}{2Z}\right)^{2/3}$$

Note again that the mass M is independent of ρ_c and R in the extreme relativistic limit.

Concluding Remarks: After Chandrasekhar's orignal work a number of corrections were noticed. There is an electrostatic interaction between the electrons and ions which give smaller radii and higher central densities; though, in general these corrections are relatively small.

General Relativity induces a dynamical instability for white dwarfs when their radii become smaller than 1.1×10^3 km.

In thermonuclear reactions, the thermal energy of the reacting nuclei overcomes the Coulomb repulsion between them so that nuclear reactions can proceed. A white dwarf does not have sufficient mass and pressure to get past burning hydrogen to helium. This is why it can eventually cool and become a degenerate electron gas and begin its long slow collapse. If it reaches sufficiently high density, even at zero temperature, the high density and zero point energy of the nuclei in a lattice can lead to an appreciable rate of *pycnonuclear* nuclear reactions ("pyknos" is "dense" in Greek). Hamada and Salpeter estimated that in 10^5 years hydrogen would be converted to ^{4}He via pycnonuclear reactions above a density of 5×10^4 g cm^{-3}. Higher densities can burn the helium to Carbon and at even higher densities the Carbon may burn to Magnesium. These calculations of rate are highly uncertain, but it seems that the maximum white dwarf radius is $3.90 \times 10^{-2} R_\odot$ at $M = 2.2 \times 10^{-3} M_\odot$ for cold carbon stars.

Following the discovery of the neutron and its beta decay to a proton and electron

$$n \rightarrow p \; + \; e^- \; + \; \bar{\nu}_e$$

it was realized that at very high densities electrons would react with protons to form neutrons via inverse beta decay:

$$p \; + \; e^- \rightarrow n \; + \; \nu_e .$$

If all of a white dwarf's electrons and protons were converted to neutrons, what would be the equation for the radius of the neutron star? Hint: Set $Z/A = 1$ in the original equations and replace the electron mass with a neutron mass. What radius do you get for a $M = 1.4 M_\odot$ neutron star?

7 Neutron Stars

The first calculation of neutron star models was performed by UCB locals Oppenheimer and Volkoff (1939) who derived the relativistically correct hydrostatic equation in a Schwarzschild metric. Here one has to use the metric for the interior $M(r)$ that was used in Problem 3. Then Einstein's equations give

$$\frac{dM(r)}{dr} = 4\pi\rho$$

(nothing new there),

$$\frac{dP}{dr} = -\frac{\rho GM(r)}{r^2}\left(1+\frac{P}{\rho}\right)\left(1+\frac{4\pi P r^3}{GM(r)}\right)\left(1-\frac{GM(r)}{r}\right)$$

(note what it takes to get this back to the Newtonian case of the previous problem), and

$$\frac{d\Phi}{dr} = -\frac{1}{\rho}\frac{dP}{dr}\left(1+\frac{P}{\rho}\right)^{-1}$$

These are the Oppenheimer-Volkoff equations of hydrostatic equilibrium.

Low density neutron stars with the ideal neutron gas equation of state ($P = K\rho^{5/3}$) give

$$\frac{R}{1\ \mathrm{km}} = 14.64\left(\frac{\rho_c}{10^{15}\ \mathrm{g\ cm^{-3}}}\right)^{-1/6}$$

and

$$\frac{M}{M_\odot} = 1.102\left(\frac{\rho_c}{10^{15}\ \mathrm{g\ cm^{-3}}}\right)^{-1/2} = \left(\frac{15.12\ \mathrm{km}}{R}\right)^4.$$

Thus, in the Oppenheimer-Volkoff calculation, there is no minimum neutron star mass; in reality, neutrons become unstable to beta decay at sufficiently low density. They also found a maximum mass of $0.7M_\odot$. Calculate the radius and central density for this neutron star. How does this $\gamma = 5/3$ result compare with the relativistic limit of $R = 9.6$ km and $\rho_c = 5\times10^{15}$ g cm^{-3}? *Comment:* The real equation of state for neutrons as they approach nuclear densities is stiffer than the relativistic case, because repulsive nuclear forces become strong (as they must if nuclei are to be stable). It is more like the non-relativistic equiation of state or stiffer. Thus, it is likely that neutron stars of $1.4M_\odot$ are possible, and are perhaps favored by nature.

8 Black Holes

To explore the Schwarschild geometry further, consider the motion of a freely moving test particle. Such a particle must move on a geodesic in space-time.

This means that

$$\tau_{AB} = \int_A^B d\tau = \int_A^B \sqrt{g_{\alpha\beta} dx^\alpha dx^\beta}$$

is an extremum. In the case of particles with mass it is a maximum. The Euler-Lagrange equations that result from the calculus of variations give

$$\frac{d}{d\tau}\left(g_{\mu\nu} \frac{dx^\nu}{d\tau} \right) - \frac{1}{2} g_{\alpha\beta,\mu} \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0.$$

Using the simple relation from the metric,

$$\frac{dg_{\mu\nu}}{d\tau} = g_{\mu\nu,\alpha}\frac{dx^\alpha}{d\tau},$$

this becomes

$$\frac{d^2 x^\sigma}{d\tau^2} + \frac{1}{2} g^{\sigma\mu}\left(2 g_{\mu\beta,\alpha} - g_{\alpha\beta,\mu}\right) \frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

Since this looks so terrible, consider the connection to classical mechanics. The Lagrangian $\mathcal{L}$ is

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}\ m\ g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} \\ &= \frac{1}{2} m \Bigg[\left(1 - \frac{2GM(r)}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1 - \frac{2GM(r)}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 \\ &\qquad + r^2 \left(\frac{d\theta}{d\tau}\right)^2 + r^2 \sin^2\theta \left(\frac{d\phi}{d\tau}\right)^2 \Bigg]. \end{aligned}$$

A particle will follow a path that makes the action $A = \int \mathcal{L} d\tau$, an extremum. The Euler-Lagrange equations are:

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^\alpha} - \frac{\partial \mathcal{L}}{\partial x^\alpha} = 0.$$

For θ, ϕ, and t these equations are, respectively:

$$\begin{aligned} \frac{d}{d\tau}(r^2 \dot{\theta}) &= r^2 \sin\theta \cos\theta \dot{\phi} \\ \frac{d}{d\tau}(r^2 \sin^2\theta \dot{\phi}) &= 0 \\ \frac{d}{d\tau}\left[\left(1 - \frac{2GM}{r}\right)\dot{t} \right] &= 0. \end{aligned}$$

Instead of using the r equation it is easier to use the fact that

$$g_{\alpha\beta}u^\alpha u^\beta = 1 \quad \rightarrow \quad g_{\alpha\beta}p^\alpha p^\beta = m^2.$$

Now, orient the coordinate system so that initially the particle is moving in the equatorial plane ($\theta = \pi/2$, $\dot{\theta} = 0$). Then the particle will stay in the equatorial plane and

$$p_\phi \equiv r^2\dot{\phi} = \text{constant} = l,$$

$$p_t \equiv (1 - \frac{2GM}{r})\dot{t} = \text{constant} \equiv E,$$

where E is the energy at infinity. It is related to the local energy $E^2_{\rm local} = p^t p_t = g^{00}p_t p_t$ by the redshift factor $(1 - 2GM/r)^{-1/2}$. That is,

$$E = \left(1 - \frac{2GM}{r}\right)^{1/2} E_{\rm local} \approx \left(1 - \frac{GM}{r}\right) E_{\rm local}.$$

The final expression corresponds to the weak field limit.

Gravitational Redshift For an alternate derivation of the redshift formula, use the fact that E is constant along the photon path to show that

$$\frac{\nu_{\rm emitted}}{\nu_{\rm received}} = \frac{\left(1 - \frac{2GM}{r_{\rm received}}\right)^{1/2}}{\left(1 - \frac{2GM}{r_{\rm emitted}}\right)^{1/2}}.$$

What is it approximately for an observer at a very large distance receiving the photons? Explain why the Schwarzschild black hole radius (horizon) is sometimes called the "surface of infinite redshift"? How does this compare to the slowing of clocks, $dt/d\tau$?

The physical interpretation of l is that it is the angular momentum:

$$l = E_{\rm local} r v^\phi,$$

and it is conserved because the potential is radial. It is convenient to consider the cases of massive and massless particles separately. For a massive particle we can renormalize the energy E and the angular momentum l in to quantities per unit mass.

$$E' = E/m, \qquad l' = l/m$$

so that we can find a simple equation of motion with the form of an effective potential.

Then using the relation $m^2 = g_{\alpha\beta}p^\alpha p^\beta$ show that

$$\begin{aligned}
\left(\frac{dr}{d\tau}\right)^2 &= E'^2 - \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{l'^2}{r^2}\right), \\
\frac{d\phi}{d\tau} &= \frac{l}{r^2}, \quad \text{and} \\
\frac{dt}{d\tau} &= \frac{E'}{1-(2GM/r)}.
\end{aligned}$$

Show that a local observer at r finds that the velocity of a radially freely-falling particle released from rest at infinity is given by

$$v^r = \left(\frac{2GM}{r}\right)^{1/2},$$

which has precisely the same form as the Newtonian velocity. For a Schwarzschild mass equal to the Sun's, how long will it take to get to $r = 0$ from r_s?

The equation for the velocity of r is simply the equation expressing conservation of energy. Thus we can express the radial equation as

$$\left(\frac{dr}{d\tau}\right)^2 = E'^2 - V(r)$$

where the effective potential is

$$V(r) = \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{l'^2}{r^2}\right).$$

It is easy to show that this reduces to the Newtonian expression when $2GM/r << 1$.

The potential will be at a maximum or minimum when $\partial V/\partial r = 0$. Show that $GMr^2 - l'^2 r + 3GMl'^2 = 0$ for $\partial V/\partial r = 0$, and hence there are no maxima or minima of V for $l' < 2\sqrt{3}GM$.

Show that $V_{\max} = 1$ for $l' = 4GM$. Circular orbits ocur when $\partial V/\partial r = 0$ and $dr/d\tau = 0$. Using the equations above gives

$$\begin{aligned}
l'^2 &= \frac{GMr^2}{r - 3GM} \\
E'^2 &= \frac{(r-2GM)^2}{r(r-3GM)}.
\end{aligned}$$

Thus, circular orbits exist down to $r = 3GM/c^2 = 1.5r_s$. The limiting case corresponds to a photon (massless particle) orbit ($E' = E/m \to \infty$). Circular orbits will be stable if V is concave up; that is, $\partial^2V/\partial r^2 > 0$ and unstable if $\partial^2V/\partial r^2 < 0$. Why? Show that Schwarzschild orbits are stable if $r > 3r_s = 6GM/c^2$.

The fractional binding energy per unit mas of a particle in the last stable orbit at $r = 3r_s = 6GM/c^2$ is easily found from the last equation for the energy per unit mass:

$$\begin{aligned} E'^2 &= \frac{(r-r_s)^2}{r(r-1.5r_s)} = 8/9 \\ E'_{\text{binding}} &= 1 - E' = 1 - (8/9)^{1/2} = 5.72\%. \end{aligned}$$

This is the fraction of the rest-mass energy released when a particle, originally at rest at infinity, slowly spirals towards a black hole to the innermost stable orbit, and then plunges into the black hole. Thus, the conversion of the rest mass to other forms of energy is potentially more efficient for accretion onto a black hole than for nuclear burning, which releases a maximum of only 0.9% of the rest mass for burning H to Fe. This is the basis for invoking black holes as the energy source in many models seeking to explain astronomical observations of the huge energy output from compact regions (e.g. Cygnus X-1, AGNs, quasars, double-lobe radio galaxies).

To estimate accretion rates one needs to know the capture cross section for particles falling in from infinity. This is simply

$$\sigma_{\text{capture}} = \pi b_{\text{max}}^2$$

where b_{max} is the maximum inpact parameter of a particle that is captured. We can define the impact parameter b as

$$b = \lim_{r\to\infty} r \ \sin\phi.$$

The first two Euler-Lagrange (or geodesic) equations of motion combine together in the limit $r \to \infty$ to give

$$\frac{1}{r^4}\left(\frac{dr}{d\phi}\right)^2 \cong \frac{E'^2 - 1}{l'^2}.$$

Substituting $r \cong b/\phi$ gives,

$$\frac{1}{b^2} = \frac{E'^2 - 1}{l'^2},$$

or in terms of the velocity at infinity, $E' = (1 - v_\infty^2)^{-1/2}$,

$$l' = bv_\infty (1 - v_\infty)^{-1/2} \to bv_\infty \quad \text{for} \quad v_\infty << 1.$$

A non-relativistic particle moving towards the black hole ($E' \cong 1,\ v_\infty << 1$) is captured if $l' < 4GM$ (as shown earlier in the problem). Thus the maximum capture impact parameter is

$$b_{\max} = \frac{4GM}{v_\infty},$$

which gives a capture cross section

$$\sigma_{\text{capture}} = \frac{4\pi(2GM/c^2)^2}{v_\infty^2} = \frac{4\pi r_s^2}{v_\infty^2}.$$

Compare this value with the geometrical capture cross section of a particle by a gravitating sphere of radius R in Newtonian theory:

$$\sigma_{\text{Newtonian}} = \pi R^2 \left(1 + \frac{2GM}{v_\infty^2 R}\right).$$

Thus, a black hole captures nonrelativistic particles like a Newtonian sphere with radius $R = 8GM/c^2 = 4r_s$.

Now consider massless particles like photons. Since the mass is zero we cannot use $d\tau$ because it is zero also and zero over zero gives us an undefined answer. To get the action one uses an arbitrary variable, e.g. s, to parameterize the trajectory in 4-space. The equations of motion become

$$\begin{aligned}
\frac{dt}{ds} &= \frac{E}{1 - 2GM/r} \\
\frac{d\phi}{ds} &= \frac{l}{r^2} \\
\left(\frac{dr}{ds}\right)^2 &= E^2 - \frac{l^2}{r^2}\left(1 - \frac{2GM}{r}\right).
\end{aligned}$$

By the Equivalence Principle, the particle's world line should be independent of its energy. If we introduce a new parameter $s' = l\ s$ and $b \equiv l/E$ (this is equivalent to dividing the mass out to get an effective potential), the equations of motion become

$$\begin{aligned}
\frac{dt}{ds'} &= \frac{1}{b(1-2GM/r)} \\
\frac{d\phi}{ds'} &= \frac{1}{r^2} \\
\left(\frac{dr}{ds'}\right)^2 &= \frac{1}{b^2} - \frac{1}{r^2}\left(1-\frac{2GM}{r}\right).
\end{aligned}$$

The worldlines depend only on the parameter b, which is the particle's impact parameter, and not on E or l separately. Note that in the limit $m \to 0$ the two impact parameters are the same and thus the quantity defined previously is consistent.

The photons can be understood in terms of an effective potential in the same way the massive particles were earlier.

$$V_{\text{massless}} = -\frac{1}{r^2}\left(1-\frac{2GM}{r}\right)$$

and

$$\left(\frac{dr}{ds'}\right)^2 = \frac{1}{b^2} - V_{\text{massless}}(r)$$

Show that $V(r)$ has a maximum of $[1/27(GM)^2]$ at $r = 3GM$. Thus any massless particle heading towards a black hole with $b^2 < 27(GM/c^2)^2$ will be captured. The capture cross section for massless particles, including photons, from infinity is thus

$$\sigma_{\text{massless}} = \pi b_c^2 = 27\pi(GM/c^2)^2.$$

The Galaxy may have as many as 10^9 black holes with masses on the scale of $M_\odot$ confined in a volume of about 10^{15} cubic light years. Light coming from the center of the Galaxy must travel about 3×10^5 light years. What is its chance of getting absorbed by a black hole on the way to us?

IDS 160B
Solutions to Problem Set 2

1.1 Doppler Formulae

The Special Relativity Doppler shift formula in the two frames may be derived as follows. Assume the momentum of a photon in the unprimed frame has components $p_x = p\cos\theta, p_y = p\sin\theta$, and $p_z = 0$. In a frame that is moving with velocity v in the x-direction, the energy and momentum of this photon is described by the Lorentz transformations. Namely,

$$dt' = \gamma(dt - \beta\, dx) \quad \text{or} \quad m\frac{dt}{d\tau} = \gamma(\frac{dt}{d\tau} - v\,\frac{dx}{d\tau}),$$

in units where $\beta = v/c$ and $c = 1$. However, $m\, dt'/d\tau = E', m\, dt/d\tau = E$, and $m\, dx/d\tau = p_x = p\cos\theta$. Also, for a photon, $p = E/c$, so that

$$E' = \gamma(E - \beta p\cos\theta) = \gamma\, E\,(1 - \beta\cos\theta),$$

in units where $c = 1$. Since $E = h\nu$ and $E' = h\nu'$, the Lorentz transforms imply that

$$\frac{\nu'}{\nu} = \gamma(1 - \beta\cos\theta) = \frac{(1 - \beta\cos\theta)}{(1 - \beta^2)^{1/2}}.$$

The reverse transform is obtained by interchanging primes and signs:

$$\frac{\nu}{\nu'} = \gamma(1 + \beta\cos\theta') = \frac{(1 + \beta\cos\theta')}{(1 - \beta^2)^{1/2}}.$$

If we redefine the direction of θ and θ' so that they are 180° out of phase, then

$$\frac{\nu}{\nu'} = \gamma(1 - \beta\cos\theta'),$$

As before, β is the velocity of the emitter divided by the speed of light and θ is the angle between the velocity and the line of sight. The only difference is that now θ and θ' have opposite directions, i.e. they are 180° out of phase. Thus the relation between angles in the two coordinate systems is

$$\gamma(1 - \beta\cos\theta)\ \gamma(1 - \beta\cos\theta') = 1.$$

If we consider only recessional velocity $\beta_r = -\beta\cos\theta'$, then the Doppler shift formula becomes

$$\frac{\nu'}{\nu} = \frac{(1+\beta_r)}{(1-\beta_r^2)^{1/2}} = \sqrt{\frac{1+\beta_r}{1-\beta_r}}.$$

Since frequency is inversely proportional to wavelength ($\nu = c/\lambda$), the relationship between redshift and recessional velocity is

$$1+z = \frac{\lambda'}{\lambda} = \frac{\nu}{\nu'} = \sqrt{\frac{1+\beta_r}{1-\beta_r}}.$$

Squaring and transposing one finds

$$\beta_r = \frac{(1+z)^2-1}{(1+z)^2+1} = \frac{(\lambda'/\lambda)^2-1}{(\lambda'/\lambda)^2+1}$$

Plugging in $z = Hd/c$, means that

$$\beta_r = \frac{(1+z)^2-1}{(1+z)^2+1} = \frac{(1+Hd/c)^2-1}{(1+Hd/c)^2+1} \cong \frac{Hd}{c} - \frac{1}{2}\left(\frac{Hd}{c}\right)^2.$$

Thus, to first order, $v_r = Hd$. This is one form in which the Hubble law is quoted.

For a galaxy with cosmological redshift $z = 0.03$,

$$\beta = \frac{(1+z)^2-1}{(1+z)^2+1} = \frac{(1.03)^2-1}{(1.03)^2+1} = \frac{1.0609-1}{1.0609+1} = \frac{0.0609}{2.0609} \cong 0.29550.$$

This corresponds to a velocity of 8865 km/sec. Note that $v = cz = 3\times10^5 \times 0.03 = 9000$ km/sec. This is the recession velocity one would infer if the redshift was thought to be a Doppler effect. For small z, it is not so different from the exact answer.

1.2 Galaxy as goldfish bowl

1.2a The shape should be approximately like a goldfish bowl, although one would not expect the lip of the "bowl" to curl back. As shown below, this is because transverse distances are not changed by the Lorentz transformation.

1.2b Recall from the previous Problem that the Special Relativity Doppler shift formulae in the two frames are

$$\frac{\nu'}{\nu} = \gamma(1 - \beta\cos\theta) = \frac{(1 - \beta\cos\theta')}{(1 - \beta^2)^{1/2}}$$

and

$$\frac{\nu}{\nu'} = \gamma(1 - \beta\cos\theta') = \frac{(1 - \beta\cos\theta')}{(1 - \beta^2)^{1/2}},$$

where β is the velocity of the emitter divided by the speed of light, θ is the angle between the velocity and the line of sight in the frame of the Galaxy, and θ' is the angle in the frame of the spacecraft. The angles in the two coordinate systems are related by the expression

$$\gamma(1 - \beta\cos\theta)\,\gamma(1 - \beta\cos\theta') = 1.$$

This implies that

$$\beta\cos\theta' = 1 - \frac{1}{\gamma^2\,(1 - \beta\cos\theta)} = 1 - \frac{(1 - \beta^2)}{1 - \beta\cos\theta}.$$

So,

$$\cos\theta' = \frac{1}{0.99}\left(1 - \frac{(1 - 0.99^2)}{1 - 0.99\cos(45^\circ)}\right) = 1.01 - 0.0670.$$

This implies that $\theta' = 21^\circ$.

1.2c The Doppler shift formula of relevance is

$$\frac{\nu'}{\nu} = \frac{1}{\gamma(1 - \beta\cos\theta')} = \frac{(1 - \beta^2)^{1/2}}{(1 - \beta\cos\theta')},$$

and the edge of the galaxy is at $\theta' = 21^\circ$, which corresponds to $\theta = 45^\circ$. Since we know the angle in the unprimed coordinate system, we can use the original formula (or the angle transform) to obtain

$$\frac{\nu'}{\nu} = \gamma(1 - \beta\cos\theta) = \frac{(1 - 0.99/1.4142)}{(1 - 0.99^2)^{1/2}} = 0.29996/0.141 = 2.126.$$

Similarly, when $\theta' = 90°$:

$$\frac{\nu'}{\nu} = \frac{1}{\gamma} = \sqrt{1 - 0.99^2} = 0.141.$$

Finally, $\theta' = 180°$, is the center of the Galaxy so that

$$\frac{\nu'}{\nu} = \frac{1}{\gamma(1+\beta)} = 0.07.$$

1.2d The correct answer is $d = R_G/\gamma$. However, one must define what one means by distance carefully. In four dimensions we use the line element to calculate the distance. The Lorentz transformations are obtained by requiring the four distance,

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2,$$

to be the same for all observers (that is all coordinate systems). Now, we can define the spatial distance by taking the spatial part of the metric (line element),

$$dl^2 = dx^2 + dy^2 + dz^2,$$

but we must be careful when specifying the time aspects. The distance to a moving object, e.g. a truck hurdling towards you, clearly depends upon the time epoch. We have the unspoken convention that the distance between two objects is given by the spatial line element (metric) evaluated at the same time for each in our coordinate system. Thus, an observer on the Galaxy says that the space craft is at a distance r_g, equal to the radius of the Galaxy, away from the center of the Galaxy, on a normal to the Galaxy, in the direction the space craft is going.

The Lorentz transformation is

$$x = \gamma(x' - ct')$$

so that

$$\Delta x = \gamma(\Delta x' - c\Delta t')$$

Since we want to compare distances at the "same" times, $\Delta t' = 0$ and $\Delta x = \gamma \Delta x'$ or $\Delta x' = \Delta x/\gamma = r_g/\gamma = r_G/7.089$. Now this is not satisfying in that the same must be true in reverse. This is because we are taking different times of measurement for the two observers, and is the point of the next question.

Howver, if you are still not satisfied, consider doing the calculation with four-D vector algebra.

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 = -r_G^2 = c^2 \Delta t'^2 - \Delta x'^2$$

The shape of the Galactic "bowl"

Set $ds^2 = 0$ to get the null cone. Use this to consider the appearance of the Galaxy as seen (by light) by the observed on the spacecraft. From the point of view of the spacecraft, the Galaxy is receding at $0.99c$. Light from the center of the Galaxy comes from near by, since the Galaxy has passed by relatively recently (remember the time dilation for the spacecraft). Light from further out towards the edge of the galaxy must travel a longer distance. To arrive at the spacecraft at the same time as the light from the center of the Galaxy, therefore, it must start on its journey towards the spacecraft earlier. The further out from the center, the farther it must travel, so the earlier it must start. Since transverse dimensions are not affected by the Lorentz transformation, light coming to the observer on the spacecraft perpendicularly is that light which started towards the spacecraft when the Galaxy passed the space craft. Call this one unit of distance. This unit of distance has the correct travel time to get to the space craft at the appropriate time. Light from futher away on the Galaxy must have started toward the spacecraft earlier and thus be pointed that way. The further out the earlier and more tilted it must have started. Thus the Galaxy appears to be shaped as a giant open "U" or more like a rounded "V".

1.3 Faster than light?

Let $(x_1, t_1) = (0, 0)$ and $(x_2, t_2) = (d, d/v)$, where $v > c$ in the unprimed frame. The Lorentz transforms are

$$x' = \gamma(x - ut) \qquad \text{and} \quad ct' = \gamma(ct - ux/c),$$

where u is the relative velocity between the primed and unprimed frames, and $\gamma = c/\sqrt{c^2 - u^2}$. This means that $(x_1', t_1') = (0, 0)$, $x_2' = \gamma d(1 - u/v), t_2')$ and $t_2' = \gamma d(1/v - u/c^2)$. So, in the primed frame,

$$t_2' - t_1' = \gamma d \left(\frac{1}{v} - \frac{u}{c^2} \right),$$

which is negative when $1/v < u/c^2$. Of course, $t_2' - t_1' < 0$ means that the time ordering is reversed. This is allowed provided $u \leq c$ so that γ remains real. Thus, reversed time ordering is allowed provided

$$\frac{u}{c} > \frac{c}{v}.$$

The proper time that elapses is given by

$$(\Delta s)^2 = (c\Delta\tau)^2 = (c\Delta t)^2 - (\Delta x)^2 = (ct_2)^2 - d^2 = \left(\frac{c^2}{v^2} - 1 \right)$$

Since $c/v < 1$, $\Delta\tau^2 < 0$ so it is negative.

2. Cosmic Rays

Rearrange the expression from the previous Problem set to obtain

$$\frac{r}{1 \text{ pc}} = 1.08 \frac{(E/10^{15} \text{ eV})}{Z(B/1\mu G)}.$$

3. Schwarzschild Metric

The problem of determining the cosmological metric differs from that which Schwarzschild solved; in the Schwarzschild case, the metric was determined from the knowledge that the energy-momentum tensor was zero except at a single point. In the present case, both the metric and the energy density must be determined together.

The most general form of the Einstein equations, under certain reasonable mathmatical requirements, is

$$R^{\nu}{}_{\mu} - \frac{1}{2} R g^{\nu}{}_{\mu} + \Lambda g^{\nu}{}_{\mu} = \frac{-8\pi G}{c^2} T^{\nu}{}_{\mu},$$

where Λ is a constant. Though you have seen this equation in class before, you have not had to calculate it explicitly! For our purposes here, it is sufficient to know that each term in $R_{\mu\nu}$ is a special combination of certain second derivatives of the metric $g_{\mu\nu}$.

Now, consider the following dimensional argument. In any system of coordinates x^{μ} which have the dimensions of length, the terms in the metric $g_{\mu\nu}$ must be dimensionless. The elements of the contracted Riemann tensor (i.e., the $R_{\mu\nu}$ terms, must have dimensions of inverse length squared, since they are obtained by differentiating the terms in $g_{\mu\nu}$ twice. Therefore, to preserve homogeneity in the expression above, Λ must also have dimensions of inverse length squared. So, from dimensional considerations alone, one might expect that Λ should be related to the inverse square radius of the Universe. This means, of course, that Λ should play a negligible role on scales that are small compared to the scale of the Universe. We will show this more explicitly below.

Recall that the general static spherically symmetric metric is

$$ds^2 = e^{\lambda} dt^2 - e^{\nu} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2),$$

where λ and ν may be functions of r only. Analogously to the case where one is deriving the Schwarzschild metric, set $T_{\mu}{}^{\nu} = 0$ in Einstein's equations above. Contraction shows that $R = 4\Lambda$, and so $R_{\mu\nu} = \Lambda\, g_{\mu\nu}$. In particular, this means that $R_{00} = \Lambda\, g_{00}$ so the equation above requires that

$$\nu'' + \frac{\nu'^2}{2} - \frac{\nu'\lambda'}{2} + \frac{2\nu'}{r} = -2\Lambda e^{\lambda}.$$

Similarly, using the relation $R_{11} = \Lambda g_{11}$, and expressing R_{11} explicitly in terms of the relevant second derivatives of $g_{\mu\nu}$ yields

$$\nu'' + \frac{\nu'^2}{2} - \frac{\nu'\lambda'}{2} - \frac{2\lambda'}{r} = -2\Lambda e^{\lambda}.$$

Subtracting these two expressions yields

$$\nu' + \lambda' = 0, \qquad \nu + \lambda = \text{constant} = \log k,$$

or $\lambda = \log k - \nu$. Inserting this into the R_{00} relation above yields

$$(e^\nu)'' + \frac{2(e^\nu)'}{r} = \frac{(re^\nu)''}{r} = -2\Lambda k,$$

which can be integrated to give

$$re^\nu = \alpha + \beta r - \Lambda \frac{kr^3}{3},$$

where α and β are constants of integration. The analogous procedure for $R_{22} = \Lambda g_{22}$ implies that

$$(re^{-\lambda})' = 1 - \Lambda r^2.$$

Since $\nu' + \lambda' = 0$, this is the same as

$$(re^\nu)' = k(1 - \Lambda r^2),$$

which implies that $k = \beta$. Finally, all these relations imply that

$$e^\nu = k\left(1 + \frac{\alpha}{kr} - \frac{\Lambda r^2}{3}\right) = \frac{e^{-\lambda}}{k}.$$

It is straightforward to verify that $\alpha = 2Gm$ is sensible (in units where $c = 1$), and that setting $k = 1$ corresponds to specifying the units of time. Thus, the form of the metric has been fully specified.
In particular, inside the sphere (setting $c = 1$), analogously to the Schwarzschild case,

$$\begin{aligned} ds^2 &= e^\lambda dt^2 - e^\nu dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &= \left(1 - \frac{2GM(r)}{r} - \frac{\Lambda r^2}{3}\right) dt^2 - \left(1 - \frac{2GM(r)}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2\, d\Omega^2. \end{aligned}$$

For uniform mass and energy densities, ρ and $\rho_\Lambda = \Lambda/8\pi G$,

$$\begin{aligned} ds^2 &= \left(1 - \frac{8\pi G}{3}\big(\rho + \rho_\Lambda\big) r^2\right) dt^2 - \left(1 - \frac{8\pi G}{3}\big(\rho + \rho_\Lambda\big) r^2\right)^{-1} dr^2 + r^2\, d\Omega^2 \\ &= \left(1 - \left(\frac{8\pi G\rho + \Lambda}{3}\right) r^2\right) dt^2 - \left(1 - \left(\frac{8\pi G\rho + \Lambda}{3}\right) r^2\right)^{-1} dr^2 + r^2\, d\Omega^2 \\ &= \left(1 - 8\pi G\rho r^2 - \frac{\Lambda r^2}{3}\right) dt^2 - \left(1 - 8\pi G\rho r^2 - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2 d\Omega^2 \\ &= \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right) dt^2 - \left(1 - \frac{2GM}{r} - \frac{\Lambda r^2}{3}\right)^{-1} dr^2 + r^2\, d\Omega^2 \end{aligned}$$

In the weak field limit $g_{00} \cong 1 + 2\Phi/c^2$. Comparison with the expression above means that the potential is

$$\Phi = -\frac{GM}{r} - \frac{\Lambda r^2}{6}.$$

The force associated with this potential is $\boldsymbol{F} = m\boldsymbol{\nabla}\Phi$, so only the radial component is non-zero:

$$F_r = \frac{GMm}{r^2} - \frac{m\,\Lambda r}{3}.$$

This shows that whereas gravity is attractive, the Λ-force term is negative (repulsive) and proportional to the separation distance, r.

For r =10 Mpc, $\Lambda r^2/6c^2 \leq 10^{-58} \times (10^7 \times 3.08568 \times 10^{18})^2/6 = 1.587 \times 10^{-8}$. For a cluster of 2,000 galaxies, each having a mass of about $10^{11}\mathrm{M}_\odot$, extending over a scale of $r = 10$ Mpc, the ratio of the effect of the cosmological constant to that of Newtonian gravity is

$$\frac{-\Lambda r^3}{6GM} \leq \frac{10^{-58} \times (3.08568 \times 10^{25})^3}{(6 \times 2000 \times 10^{11} \times 3/2 \times 10^5)} = \frac{0.09994}{6}$$

or about 1.5%, where $2G\mathrm{M}_\odot/c^2 \cong 3$ km has been used.

4. Maximum Electric Field

The Uncertainty Principle states that canonical variables do not commute to the level of Planck's constant h. In other words,

$$g_{\alpha\beta}p^\alpha x^\beta = \delta p_\alpha \delta x^\beta \geq h\delta^\beta_\alpha,$$

where $\delta^\beta_\alpha = 0$ when $\alpha \neq \beta$ and $\delta^\beta_\alpha = 1$ when $\alpha = \beta$. In rectangular coordinates, this means that $\delta E\ \delta t \geq h$ and that $\delta p_x\ \delta x \geq h$, while in polar coordinates $\delta L_\phi\ \delta\phi = h$.

To make an electron-positron (e^-e^+) pair requires an energy of at least twice the rest mass of an electron: $E_{\rm min} = 2m_e c^2 = 2 \times 511$ keV. That energy can exist for a time $\delta t \leq h/E$ without violating the Uncertainty

Principle. During that time, the maximum separation that the electron-positron pair can have is $\sim 2c\delta t$. Therefore, the electric field must be such that $e2c\delta t\mathcal{E} \approx 2m_ec^2$. Thus

$$\mathcal{E} \approx \frac{m_ec}{eh/2m_ec^2} \approx \frac{2m_ec^2}{e\lambda_e} = \frac{1.022 \times 10^6 \text{ eV}}{e\ 2\pi \times 2.426 \times 10^{-10} \text{ cm}} = 4.26 \times 10^{15} V/\text{cm}.$$

Another way to derive this is by dimensional analysis. The only distance scale is the Compton wavelength of the electron (or positron): $\lambda_e = \hbar/m_ec = 2.462 \times 10^{-15}/2\pi$ cm. The energy available is $e\mathcal{E}\lambda_e$ and the energy needed is $2m_ec^2$. The maximum electric field allowed is when these are equal, because otherwise a current of counter-flowing electron-positron pairs will short out the field. Thus,

$$\mathcal{E} = \frac{2m_ec^2}{e\lambda_e} = 2\pi \times 4.26 \times 10^{15}\ V/\text{cm}.$$

A more precise treatment requires a quantum tunnelling calculation.

5. Electrostatics: Reissner-Nördstrom Metric

The Reissner-Nördstrom metric describes the field of a charged mass point, that is, a point singularity of the Einstein field equations with an energy-momentum tensor that is due to an electromagnetic field. Both the metric and the field are assumed to be static and spherically symmetric. The form of the Reissner-Nördstrom metric is similar to that of the Schwarzschild metric; it has an additional term that describes the effect of the charged mass point at the center. Since you are used to thinking of both the gravitational and the electromagnetic forces as being proportional to the inverse square of the distance, the electric and the gravitational terms look different in this metric. We will show, explicitly, that this form for the metric is correct.

From special relativity, the Maxwell tensor $F_{\mu\tau}$ for a static and spherically symmetric electric field $E(r)$ in the r direction is

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix},$$

in cartesian coordinates. In spherical coordinates, it is

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E(r) & 0 & 0 \\ E(r) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The antisymmetry of the electromagnetic field tensor implies that the electromagnetic energy-momentum tensor has zero trace. Hence, the Einstein field equations have the form

$$R_{\mu\nu} = \frac{8\pi G T_{\mu\nu}}{c^2}.$$

This system of equations must be solved together with the classical, free-space Maxwell equations. Thus, the system to be solved is

$$\begin{aligned} R_{\mu\tau} &= \frac{8\pi G T_{\mu\nu}}{c^2} = \frac{8\pi G T_{\mu\nu}}{c^4}\left(F_{\mu\alpha}F^{\alpha}_{\ \tau} + \frac{g_{\mu\tau}}{4}F_{\alpha\beta}F^{\alpha\beta}\right) \\ \left(\sqrt{-g}f^{\mu\tau}\right)_{|\tau} &= 0 \\ F_{\mu\tau|\lambda} &= 0. \end{aligned}$$

Since we are dealing with a Schwarzschild-like particle, the symmetry conditions which led to the Schwarzschild metric apply here also. Similarly, the restrictions on the form of the curvature tensor carry over as well. Since we expect a solution in which $F^{\mu\nu}$ has the form given above (i.e. static, spherical symmetry), this means that, in addition to the functions λ and μ introduced in the general form of the Schwarzschild metric we must also determine the form of $E(r)$. The general form of the Schwarzschild metric can be used to raise the indices of the Maxwell tensor. This provides several relations between λ, ν, and E, which may be solved to obtain the Reissner-Nördstron metric. The details of the calculation may be found, for example, in Section 13.1 of *Introduction to General Relativity* by Adler, Bazin & Schiffer.

The metric is flat when its coefficients are all ± 1. This happens when

$$\frac{2GM}{r} = \frac{GQ^2}{r^2} \quad \text{or} \quad r_{\text{flat}} = \frac{Q^2}{2Mc^2}.$$

An electron has charge e so that $r_{\text{flat}} = e^2/(2mc^2) = r_e$, the classical radius of the electron. At this distance, the field strength is

$$\mathcal{E} = \frac{e}{r_e^2} = \frac{e^2}{er_e^2} = \frac{2m_ec^2}{er_e} = \frac{1.022 \times 10^6 \text{ Volts}}{2.81794 \times 10^{-13} \text{ cm}} = 3.56 \times 10^{18} \text{ Volts/cm}.$$

Note that $\mathcal{E}$ is three orders of magnitude greater than the field strength computed in the previous problem. This suggests that, on these scales, vacuum polarization effects may be important. It may well be that the real electric charge on the electron is actually greater than e.

One way the metric can have a singularity is for the term

$$\left(1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}\right) = 0.$$

Multiplying through by r^2 gives the quadratic equation

$$r^2 - 2GMr + GQ^2 = 0.$$

Completing the square yields

$$(r - GM)^2 = (GM)^2 - GQ^2.$$

This has no real root if $Q^2 > GM^2$.

However, we must be careful because, in the above, we multiplied through by r^2. Inspection shows that there will be a coordinate singularity at $r = 0$. We can see if there will be a physical singularity by looking at the curvature as $r \to 0$. Even without calculating the full curvature tensor, it is clear that there will be a physical singularity if the electron is a point particle.

Gauss' law in the Euclidean three-space to which you are accustomed states that the integral of a divergence of a vector in a volume depends only on the values of the vector on the boundary of the volume. This law holds true in general relativity also, and is extremely useful when expressing conservation laws (see, e.g., Adler, Bazin & Schiffer, §3.2). However, you must be careful when calculating the integral and the divergence.

6. White Dwarfs

The difference between the Hydrogen and the Helium white dwarfs is that the number of electrons in the Helium star is not the same as the number of baryons (protons plus neutrons). One way to derive the ratio of the maximum masses in the two cases is as follows. The gravitational mass will equal the mass of a baryon times A/Z, the number of protons plus neutrons divided by the number of protons (and thus electrons), times the number of electrons:

$$M = \frac{A}{Z} N m_b$$

Using this relation we can go back to setting the total energy to zero and find that

$$E_{\mathrm{rel}} \sim \frac{\hbar c N^{1/3}}{R} - \frac{G M m_b}{R} = 0$$

$$\frac{\hbar c N^{1/3}}{R} = \frac{G(A/Z) N m_b^2}{R}$$

or, rearranging,

$$N^{2/3} = \frac{\hbar c}{G(A/Z) m_b^2}.$$

This implies that

$$M_{\mathrm{max}} \sim N_{\mathrm{max}} m_b = \left(\frac{\hbar c (Z/A)}{G m_b^2}\right)^{3/2} m_b \approx 1.5 M_\odot \left(\frac{Z}{A}\right)^{3/2}.$$

Thus, the ratio of maximum masses for a hydrogen to helium white dwarf is, approximately,

$$\frac{M_{\mathrm{max}_{\mathrm{H}}}}{M_{\mathrm{max}_{\mathrm{He}}}} = 2^{3/2} = 2.83,$$

since there are twice as many supporting electrons for every baryon in a hydrogen white dwarf than in a helium white dwarf.

Another way to do this problem is as follows. The maximum baryon number for equilibrium is $N_{\mathrm{max}} \sim (\hbar c / G m_b^2)^{3/2} \approx 2 \times 10^{57}$. The maximum mass scales as $M_{\mathrm{max}} \sim N_{\mathrm{max}} m_b$. For a Helium white dwarf, $m_b = 4 m_{\mathrm{H}}$ where m_{H} is the value of m_b for a Hydrogen dwarf. Thus,

$$\frac{M_{\mathrm{max}}^{\mathrm{He}}}{M_{\mathrm{max}}^{\mathrm{H}}} = \frac{1}{16}.$$

In fact, neither of these approaches gives the correct answer. The remainder of this problem is devoted to showing that the maximum mass scales as $(Z/A)^2$. These approaches get the wrong answer because they do not account for the fact that at the center of the star the pressure from the electron degeneracy must support the hydrostatic head of baryons above it. Both these answers should be compared with the correct answer, the Chandrasekhar mass limit which will be derived below.

We know that the non-relativistic degenerate electron gas pressure is

$$P_{\mathrm{e}} = 0.0485\,\frac{h^2 n_{\mathrm{e}}^{5/3}}{m_{\mathrm{e}}} = 0.0485\,\frac{h^2}{m_{\mathrm{e}}}\left(\frac{Z}{A}\right)^{5/3}\frac{\rho^{5/3}}{m_b^{5/3}},$$

where the final expression follows from the expressions for n_{e} that are given in the text.

The pressure at the center is

$$P_{\mathrm{c}} = 0.0485\,\frac{h^2}{m_{\mathrm{e}}}\left(\frac{Z}{A}\right)^{5/3}\frac{\rho_{\mathrm{c}}^{5/3}}{m_b^{5/3}} = 0.77\,\frac{GM^2}{R^4}.$$

However, we are told that $\rho_{\mathrm{c}} = 1.43\,M/R^3$. This, in the expression above, yields an expression for R. Namely,

$$\begin{aligned} R &= \frac{0.0485\,(1.43)^{5/3}}{0.77}\frac{h^2}{Gm_{\mathrm{e}}}\left(\frac{Z}{A}\right)^{5/3}\left(\frac{M}{m_b}\right)^{5/3}\frac{1}{M^2} \\ &= 0.114\frac{h^2}{Gm_{\mathrm{e}}m_b^{5/3}}\left(\frac{Z}{A}\right)^{5/3}M^{-1/3}, \end{aligned}$$

as required. Note that the radius decreases as the mass increases.

To illustrate this behaviour, calculate the radius or white dwarfs of mass $0.5M_\odot$, and $M_\odot$. This is most easily done using the expressions given in the text. Thus, $R = 1.12\times10^4(2Z/A)^{5/6}$ km and $R = 0.89\times10^4(2Z/A)^{5/6}$ km for $0.5M_\odot$ and $M_\odot$ white dwarfs, respectively. The corresponding central densities are $0.99\times10^6(2Z/A)^5$ g/cm^3 and $0.25\times10^6(2Z/A)^5$ g/cm^3, respectively.

To obtain Chandrasekhar's mass limit, first use the relations

$$P_{\rm c} = 11.0 \, \frac{GM^2}{R^4} = 0.123 \, hc \left(\frac{Z}{A}\right)^{4/3} \left(\frac{\rho_{\rm c}}{m_b}\right)^{4/3}.$$

However, $\rho_{\rm c} = 12.9 \, M/R^3$ (since in the relativistic case $P \propto \rho^{4/3}$). This, in the expression above, shows that there are R^4 terms in the denominators on both sides of the equation. Therefore, the equation can be rearranged to obtain an expression for M in terms of fundamental constants. This gives

$$M_{\rm Ch} = 0.2 \left(\frac{Z}{A}\right)^2 \left(\frac{hc}{Gm_b^2}\right)^{3/2},$$

as required.

7. Neutron Stars

To calculate the central density of a $0.7 M_\odot$ neutron star, use the relations that

$$\frac{M}{M_\odot} = 1.102 \left(\frac{\rho_{\rm c}}{10^{15} \ {\rm g \ cm^{-3}}}\right)^{-1/2} = \left(\frac{15.12 \ {\rm km}}{R}\right)^4.$$

Thus, when $M = 0.7 M_\odot$, then $R = 13.83$ km and $\rho_{\rm c} = 2.48 \times 10^{15}$ g/cm^3. In contrast, the relativistic limits are $R = 9.6$ km and $\rho_{\rm c} = 5 \times 10^{15}$ g/cm^3.

8. Black Holes

Gravitational Redshift   Recall that E is constant along the path of a photon. Since $E = h\nu$, the relation given in the problem can be rearranged to yield

$$E_{\rm emitted} \left(1 - \frac{2GM}{r_{\rm emitted}}\right)^{1/2} = E_{\rm received} \left(1 - \frac{2GM}{r_{\rm received}}\right)^{1/2}.$$

The left and right hand sides of this equation are both equal to E, and since E is constant (independent of r) along the path of the photon, they are equal to each other. This proves the consistency of the redshift formula.

The redshift formula has a singularity when

$$\frac{2GM}{c^2 r_{\text{emitted}}} = 1.$$

When $2GM/c^2 = r_{\text{emitted}}$, then $\nu_{\text{emitted}}/\nu_{\text{received}}$ diverges. In other words, photons emitted at the Schwarzschild radius are infinitely redshifted.

If we think of the redshift as describing the amount by which successive ticks on a clock at r_{emitted} are stretched when they reach r_{received}, then the result above means that when $r_{\text{emitted}} = r_{\text{s}}$, the ticks as viewed at r_{received} are infinitely far apart. In other words, $dt/d\tau \to \infty$ as $r \to$ the Schwarzschild radius.

Recall that the norm of the four velocity, $g_{\alpha\beta}x^\alpha x^\beta = 1$, implies that $g_{\alpha\beta}p^\alpha p^\beta = m^2$. Explicity, this relation is

$$\begin{aligned}
0 &= g_{00}p^0p^0 + g_{11}p^1p^1 + g_{22}p^2p^2 + g_{33}p^3p^3 - m^2 \\
&= E_{\text{local}}^2 - \frac{1}{1-(2GM/r)}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{d\phi}{d\lambda}\right)^2 - m^2 \\
&= \frac{E^2}{1-(2GM/r)} - \frac{1}{1-(2GM/r)}\left(\frac{dr}{d\lambda}\right)^2 - r^2\left(\frac{l}{r^2}\right)^2 - m^2 \\
&= \frac{E^2}{1-(2GM/r)} - \frac{1}{1-(2GM/r)}\left(m\frac{dr}{d\tau}\right)^2 - \frac{l^2}{r^2} - m^2 \\
&= \frac{E'^2}{1-(2GM/r)} - \frac{1}{1-(2GM/r)}\left(\frac{dr}{d\tau}\right)^2 - \frac{l'^2}{r^2} - 1.
\end{aligned}$$

The second line follows from the form of the metric, namely $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$, from the relation $g^{00}p_0p_0 = E_{\text{local}}^2$, and from the fact that we have chosen a coordinate system in which the particle is always in the equatorial plane, so that $d\theta/d\lambda = 0$, and $\sin\theta = 1$. In all the above, λ is called the affine parameter; it describes the trajectory of the particle. The third line expresses E_{local} in terms of the energy at infinity, E, and uses the definition

$$l = r^2\frac{d\phi}{d\lambda} = r^2 p^\phi = g_{\phi\phi}p^\phi = p_\phi$$

which was given in the problem. The fourth line uses the relation, $\lambda = \tau/m$, between the affine parameter and the proper time. The fifth line is obtained by dividing through by m^2; primes denote quantities per unit mass. This final expression can be rearranged to give

$$\left(\frac{dr}{d\tau}\right)^2 = E'^2 - \left(1 - \frac{2GM}{r}\right)\left(1 + \frac{l'^2}{r^2}\right).$$

These relations imply that

$$\frac{d\phi}{d\tau} = \frac{1}{m}\frac{d\phi}{d\lambda} = \frac{1}{m}\frac{l}{r^2} = \frac{l'}{r^2},$$

and that

$$\frac{dt}{d\tau} = \frac{1}{m}\frac{dt}{d\lambda} = \frac{p^0}{m} = \frac{g^{00}p_0}{m} = g^{00}E' = \frac{E'}{1-(2GM/r)}.$$

For a local observer, $dr/d\tau = v^r$, and a radially freely-falling particle released from rest at infinity has $d\phi/d\tau = 0$ and $E' = 1$. This, in the equation above, implies that

$$(v^r)^2 = E'^2 - 1 + \frac{2GM}{r} = \frac{2GM}{r},$$

which is identical to the Newtonian expression.

Now, since $v^2 = dr/d\tau = \sqrt{2GM/r}$,

$$\int_{r=0}^{r=r_s} d\tau = \int_0^{r_s} \sqrt{\frac{r}{2GM}}\, dr = \frac{2r_s}{3}\sqrt{\frac{r_s}{2GM}} = \frac{4GM}{3} = \tau.$$

Setting $M = M_\odot$ in the expression above, and using the fact that $2GM_\odot/c^2 \approx 3$ km, means that the time to get to $r = 0$ from $r = r_s$ is $2 \times 10^5/3 \times 10^{10} = 0.66 \times 10^{-5}$ seconds.

First, find the extrema of effective potential. These are at those r where

$$\begin{aligned}\frac{\partial V}{\partial r} = 0 \quad &= \quad \frac{2GM}{r^2}\left(1 + \frac{l'}{r^2}\right) - \frac{2l'^2}{r^3}\left(1 - \frac{2GM}{r}\right) \\ &= \quad \frac{GM}{r^2} - \frac{l'^2}{r^3} + \frac{3GMl'^2}{r^4},\end{aligned}$$

as required. This can be rearranged to provide a quadratic equation that expresses r in terms of G, M and l':

$$r = \frac{l'^2}{2GM} \pm \frac{l'\sqrt{l'^2 - 12G^2M^2}}{2GM}.$$

For r to be real valued, $l'^2 - 12G^2M^2$ must be positive. In other words, the requirement that r be real valued means that there are no maxima or minima of V for $l' < 2GM\sqrt{3}$. When $l' = 4GM$, then $r = 8GM \pm 4GM$. So,

$$V_{\max}(l' = 4GM) = \left(1 - \frac{2GM}{r_{\max}}\right)\left(1 + \frac{16G^2M^2}{r_{\max}^2}\right).$$

When $r = 12GM$ then $V(r = 12GM, l' = 4GM) = 5/6(160/144) = 0.93$. When $r = 4GM$, then $V(r = 12GM, l' = 4GM) = 1$. Thus, $V_{\max}(l' = 4GM) = 1$ when $r = 4GM$.

Circular orbits occur when $\partial V/\partial r = 0$ and $dr/d\tau = 0$. These orbits are stable if $\partial^2 V/\partial r^2 > 0$. This means that,

$$\begin{aligned} \frac{\partial^2 V}{\partial r^2} = \frac{-4GM}{r^3} + \frac{6l'^2}{r^4} - \frac{24GMl'^2}{r^5} &> 0 \qquad \text{or} \\ 2GMr^2 - 3l'^2 r + 12GMl'^2 &< 0. \end{aligned}$$

However, since $\partial V/\partial r = 0$ for a circular orbit, $l'^2 = GMr^2/(r - 3GM)$. This, in the equation above, implies that the stability condition takes the form

$$\begin{aligned} 0 &> 2GMr^2 - \frac{3GMr^3}{r - 3GM} + \frac{12(GMr)^2}{r - 3GM} \qquad \text{or} \\ 0 &< \frac{GMr^2}{r - 3GM}(r - 6GM). \end{aligned}$$

Thus, orbits are only stable if $r > 6GM$. Is this consistent with the earlier relation between l' and GM that was obtained by requiring that r be real? Substituting $l' < \sqrt{12}GM$ into the expression for the stability condition implies that

$$r^2 - 18GMr + 72G^2M^2 = (r - 6GM)(r - 12GM) < 0.$$

Therefore, for this value of l', stable orbits only exist for those values of r which satisfy $6GM/c^2 < r < 12GM/c^2$. This provides the same lower bound on r as before, and it also provides an upper bound, for this value of l'.

Massless particles can be understood in terms of an effective potential in the same way that massive particles were above. For massless particles the effective potential is

$$V_{\text{massless}} = -\frac{1}{r^2}\left(1 - \frac{2GM}{r}\right).$$

The extrema occur at

$$\frac{\partial V}{\partial r} = 0 = \frac{2}{r^3} - \frac{6GM}{r^4},$$

i.e., when $r = 3GM$. At this value of r, $\frac{\partial^2 V}{\partial r^2 = 2/(81G^4M^4) > 0}$, so that the extremum is a maximum. This implies that

$$V_{\max}^{\text{massless}} = \frac{-1}{27G^2M^2}.$$

The capture cross section for massless particles by a black hole is

$$\sigma_{\text{massless}} = \pi b_c^2 = 27\pi(GM/c^2)^2.$$

If the number density of black holes is $n = 10^9/10^{15}$ lyrs^{-3}, then the mean free path of a massless particle is

$$\ell = \frac{1}{n\sigma} \approx \frac{10^6}{2 \times 10^{-24}} \text{ light years},$$

where the numerical value is obtained by assuming that most black holes have masses M that are on the order of $M_\odot$. Light travelling from the center of the galaxy must travel about $L = 3 \times 10^5$ light years, so the probability it is absorbed by a black hole scales as $\frac{L}{\ell}$. If we assume that this distribution is exponential, then it is highly improbable that a given photon will get absorbed by this distribution of black holes.