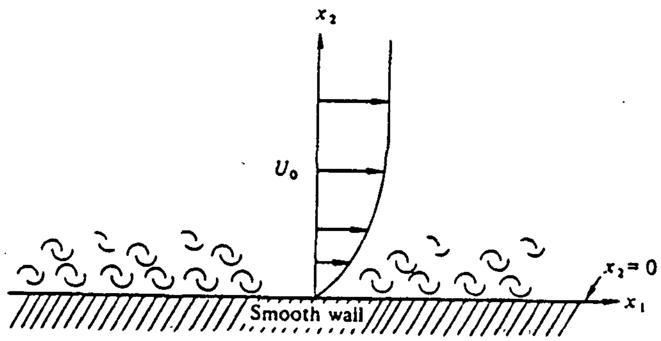
Acoustics of Boundary Layers

The role of the surface shear stress "dipole"
(JSV 65, 159 (1979))



At low mean flow Mach numbers we may use Lighthill's equation:

$$\left\{\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right\} p = \frac{\partial^2}{\partial x_i \partial x_j}(\rho_0 v_i v_j) - \mathrm{div}\left(\rho_0 \nu \nabla^2 \underline{v} + \frac{\rho_0 \nu}{3}\nabla(\mathrm{div}\,\underline{v})\right)$$

Take the Fourier transform w.r.t. (x_1, x_3, t), and let $\underline{k} = (k_1, 0, k_3)$:

$$\left\{\gamma(\underline{k})^2 + \frac{\partial^2}{\partial x_2^2}\right\}\hat{p} = \hat{S} + \hat{Z} \quad \text{(I)} \quad (\gamma = (\omega^2/c^2 - k^2)^{\frac{1}{2}})$$

$\hat{S}$: transform of Reynolds stress; $\hat{Z}$: viscous effects.

Use the Green's function $G(x_2, y_2)$ defined by:

$$\left(\gamma^2 + \frac{\partial^2}{\partial y_2^2}\right) G = \delta(x_2 - y_2) \qquad \text{(II)}$$

namely: $G(x_2, y_2) = \dfrac{-i}{\gamma(\underline{k})}\left\{e^{i\gamma x_2}\cos(\gamma y_2) H(x_2 - y_2) + e^{i\gamma y_2}\cos(\gamma x_2) H(y_2 - x_2)\right\}$

This satisfies the radiation condition and $\partial G/\partial x_2 = 0, x_2 = 0$;
$\partial G/\partial y_2 = 0,\ y_2 = 0$.

Apply Green's Theorem to equations (I), (II) (as on page 2.3) in the region $y_2 > 0$:

$$\hat{p}(x_2, k) - G(x_2, 0)\left(\frac{\partial \hat{p}}{\partial y_2}\right)_{y_2=0} = \int_0^\infty G(x_2, y_2)\left[\hat{S} + \hat{Z}\right] dy_2. \qquad \text{(III)}$$

On $x_2 = +0$ the x_2-component of the momentum equation (page 1.2) becomes:

$$\frac{\partial \hat{p}}{\partial x_2} - \frac{4\rho_0 \nu}{3}\frac{\partial}{\partial x_2}(\operatorname{div} \hat{v}) = -\rho_0 \nu k_i \frac{\partial \hat{v}_i}{\partial x_2}$$

Hence, when viscous stresses within the boundary layer are neglected in comparison with Reynolds stresses we obtain from (III):

$$\hat{p}(\underset{\sim}{k}, \omega) + \frac{\rho_0 \nu k_i}{\gamma(k)}\left(\frac{\partial \hat{v}_i}{\partial x_2}\right)_{x_2=0} = \int_0^\infty G(0, y_2)\hat{S}(\underset{\sim}{k}, \omega, y_2)\, dy_2$$

surface pressure

represents the modification of the turbulence induced surface pressure due to the surface shear stress $\partial v_i / \partial x_2$.

"known" from the distribution of $\rho_0 v_i v_j$ in boundary layer

The Lighthill-Curle view is to interpret this force as a surface "dipole" source of sound. It is actually an acoustic sink! To see this note that in the acoustic region $1/k \gg \delta$ = boundary layer width,

$$\therefore \quad \hat{v}_i(x_2) = \frac{k_i \hat{p}}{\rho_0 \omega}\left\{1 - \frac{H_0^{(1)}\left[\left(\left(\frac{\kappa v_* x_2}{\nu} + 1\right)\cdot \frac{4i\omega\nu}{\kappa^2 v_*^2}\right)^{\frac{1}{2}}\right]}{H_0^{(1)}\left[\left(\frac{4i\omega\nu}{\kappa^2 v_*^2}\right)^{\frac{1}{2}}\right]}\right\},$$

(see page 4.20). This may be used to calculate $\partial \hat{v}_i / \partial x_2$ in terms of $\hat{p}$, which is constant through the boundary layer at low wavenumbers.

Hence, the surface pressure in the acoustic region is given by:

$$\hat{p}(\underline{k},\omega) = \frac{\gamma(\underline{k}) \int_0^\infty G(0,y_2)\hat{S}(\underline{k},\omega,y_2)\,dy_2}{\left\{ \gamma(\underline{k}) + \frac{k^2 \varkappa \nu_*}{2\omega} F\left(\sqrt{\frac{4i\omega\nu}{\varkappa^2 \nu_*^2}}\right)\right\}} \qquad (*)$$

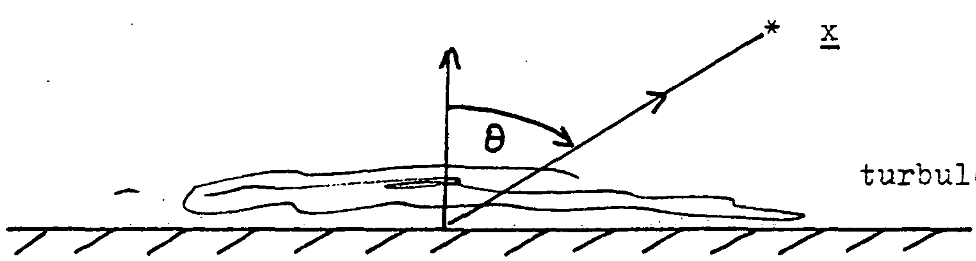
in which F(z) is defined on page 4.21. Re(F) > 0, ∴ shear stress reduces the surface pressure.

The acoustic radiation:

Since $\hat{p}$ may be regarded as constant across the boundary layer when $k\delta \ll 1$, it follows that the far field acoustic pressure satisfies the correspondence

$$k_1 = \frac{\omega}{c}\sin\theta\cos\varphi \;;\; \gamma(\underline{k}) = \frac{\omega}{c}\cos\theta \;;\; k_3 = \frac{\omega}{c}\sin\theta\sin\varphi$$

in equation (*), where θ, φ are spherical polar angles:



If $Q(\underline{k},\omega)$ $(\underline{k} = (k_1,0,k_3))$ denotes the power spectral density of

$$\gamma(\underline{k})\int_0^\infty G(0,y_2)\hat{S}\,dy_2 = i\rho_0\int_0^\infty e^{i\gamma y_2}\left[k_i - \gamma\delta_{i2}\right]\left[k_j - \gamma\delta_{j2}\right]\widehat{v_i v_j}\,dy_2$$

then:

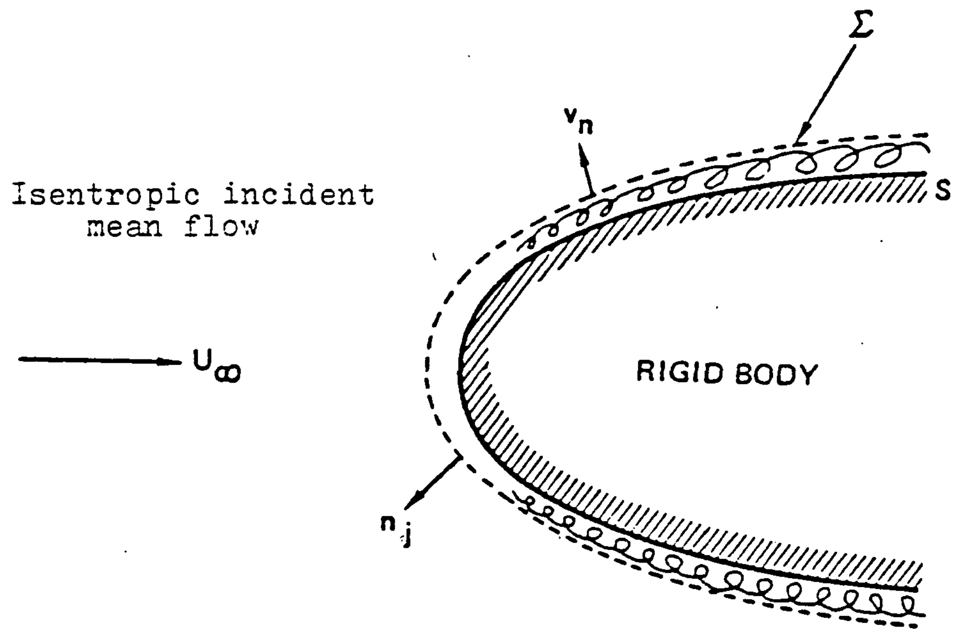
$$\Delta I = \frac{2\cos\theta\, Q(\underline{k},\omega)\,\Delta\sigma\,\Delta\omega}{\rho_0 c\left|\cos\theta + \frac{\varkappa}{2} m_* \sin^2\theta\, F\left(\sqrt{\frac{4i\omega\nu}{\varkappa^2 \nu_*^2}}\right)\right|^2}$$

where: ΔI = acoustic radiation intensity per unit area of wall into $\Delta\sigma$ = solid angle; $\Delta\omega$ = frequency interval.

Note: $Q \sim \rho_0^2 \delta^* M^4 U^3$ ∴ $\Delta I \sim O(U^8)$.

$\Delta I = \infty$ at $\theta = \pi/2$ if surface shear stress is neglected!

Liepmann's Theory - for low Mach number mean flows



From page 2.4:

$$B(\underline{x},t) = -\int \frac{\partial G}{\partial y_j}(\underline{x},\underline{y};t,\tau)(\underline{\omega}\wedge\underline{v})_j(\underline{y},\tau)\,d^3\underline{y}\,d\tau + \frac{\partial}{\partial t}\oint_\Sigma G(\underline{x},\underline{y};t,\tau)\,v_n(\underline{y},\tau)\,d\Sigma\,d\tau$$

This is valid for arbitrary control surface Σ. Let Σ be the smooth surface marking the outer edge of the boundary layer, then

$$B(\underline{x},t) = \frac{\partial}{\partial t}\oint G(\underline{x},\underline{y};t,\tau)\,\upsilon(\underline{y},\tau)\,d\Sigma\,d\tau$$

υ = boundary layer displacement velocity.

Note:

For 2-dimensional flow $\delta^* = \int_0^\delta \left\{1 - \frac{v_\alpha}{U_\infty}\right\} dx_n$
the direction α being parallel to the surface.

hence: $\frac{\partial \delta^*}{\partial x_\alpha} = -\frac{1}{U_\infty}\int_0^\delta \frac{\partial v_\alpha}{\partial x_\alpha}\,dx_n = \frac{1}{U_\infty}\int_0^\delta \frac{\partial v_n}{\partial x_n}\,dx_n = \upsilon/U_\infty$

i.e., $\upsilon = U_\infty \frac{\partial \delta^*}{\partial x_\alpha}$.

Liepmann's hypothesis: $\Sigma \longrightarrow S$:

$$B(\underline{x},t) \simeq \frac{\partial}{\partial t}\oint_S G(\underline{x},\underline{y};t,\tau)\,v(\underline{y},\tau)\,dS\,d\tau.$$

Justification: for low Mach number flows:

(i) The case of a curved surface for which

boundary layer width $<<$ surface radius of curvature $<<$ acoustic wavelength

i.e., $\delta << \mathcal{R} << \lambda \quad (\sim O(\mathcal{R}/M))$

The principal contributions to the integral are from components of $v(\underline{y},\tau)$ having length scales $\gtrsim O(\mathcal{R})$ (since when $\underline{x}$ is in the far field the smallest length scale of variation of $G \sim O(\mathcal{R})$); smaller scale variations integrate to zero over surface elements $\Delta S, \Delta\Sigma \sim O(\mathcal{R}^2)$ in which G may be regarded as constant. This implies that phase differences between the integrands on S and Σ are of no importance.

(ii) The plane boundary layer:

The length scale of $G \sim \delta/M$,; phase differences are again negligible.

Example: Curved surface:

$$\oint G v \, dS = 0$$ if G is assumed to be constant (i.e., if retarded position differences are ignored)

$$\therefore \oint G v \, dS \simeq \int y_\alpha \frac{\partial G}{\partial y_\alpha} v \, dS$$

$$\simeq \frac{1}{c}\frac{\partial}{\partial t}\int y_\alpha G v \, dS$$

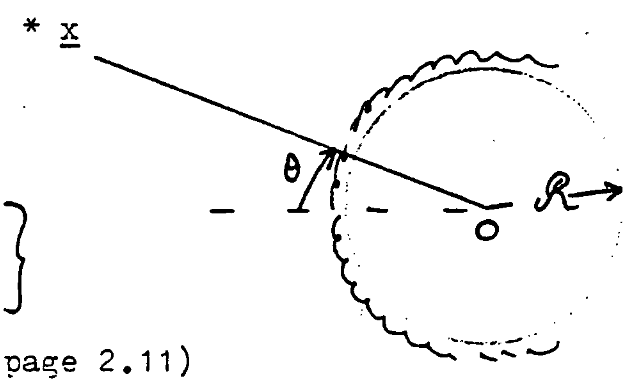
(since in order of magnitude $\partial G/\partial y_\alpha \sim (1/c)\partial G/\partial t$)

hence: $$\frac{p}{\rho_0} \simeq \frac{1}{c}\frac{\partial^2}{\partial t^2}\left(\frac{\mathcal{R}^3 v}{|\underline{x}|}\right) \sim \frac{\mathcal{R}}{|\underline{x}|} MU^2$$: Dipole radiation

Example: Axisymmetric boundary layer on a sphere.

$$v(t,\theta) = \sum_{n \geq 1} v_n P_n(\cos\theta)$$

$(\oint v \, dS = 0)$.



For a compact sphere

$$G = \frac{1}{4\pi|\underline{x}|}\delta\left\{t - \tau - \frac{|\underline{x} - \underline{y}|}{c}\right\}$$

$$Y_i = y_i\left(1 + \frac{\mathcal{R}^3}{2|\underline{y}|^3}\right)$$ (see page 2.11)

Hence: $$\frac{p}{\rho_0} \simeq \frac{\cos\theta(\underline{x})}{4\pi c|\underline{x}|}\frac{\partial^2}{\partial t^2}\sum_{n\geq 1}\int_0^\pi 2\pi\left[v_n(t)\right]\frac{3\mathcal{R}^3}{2}\cos\theta \, P_n(\cos\theta)\sin\theta \, d\theta$$

$$= \frac{\mathcal{R}^3\cos\theta}{2c|\underline{x}|}\left[\frac{\partial^2 v_1}{\partial t^2}\right] \sim \frac{\mathcal{R}}{|\underline{x}|} MU^2\cos\theta.$$

<u>Effective representation of displacement velocity in terms of boundary layer vorticity:</u>

$$B(\underline{x},t) = -\int \frac{\partial G}{\partial \underline{y}}(\underline{x},\underline{y};t,\tau)\cdot \underline{\omega}\wedge\underline{v}\, d^3\underline{y}\, d\tau$$

$$\simeq -\int \frac{\partial G}{\partial y_\alpha}(\underline{x},\underline{y};t,\tau)(\underline{\omega}\wedge\underline{v})_\alpha d^3\underline{y}\, d\tau$$

$$+ \quad \text{error} \sim O(\delta/\mathcal{R}),$$

where y_α = tangential coordinate. $\partial G/\partial y_\alpha$ varies by a negligible amount across the boundary layer, hence, integrating by parts:

$$B(\underline{x},t) = \oint_S \underbrace{G(\underline{x},\underline{y};t,\tau)}_{\text{evaluated on the surface S}} \cdot \left[\frac{\partial}{\partial y_\alpha}\underbrace{\int_0^\infty (\underline{\omega}\wedge\underline{v})_\alpha dy_n}_{\text{integral across the boundary layer}}\right] dS\, d\tau$$

i.e.,

$$\frac{\partial v}{\partial t} \underset{\text{acoustically}}{=\!=\!=} \frac{\partial}{\partial y_\alpha}\int_0^\infty (\underline{\omega}\wedge\underline{v})_\alpha dy_n$$

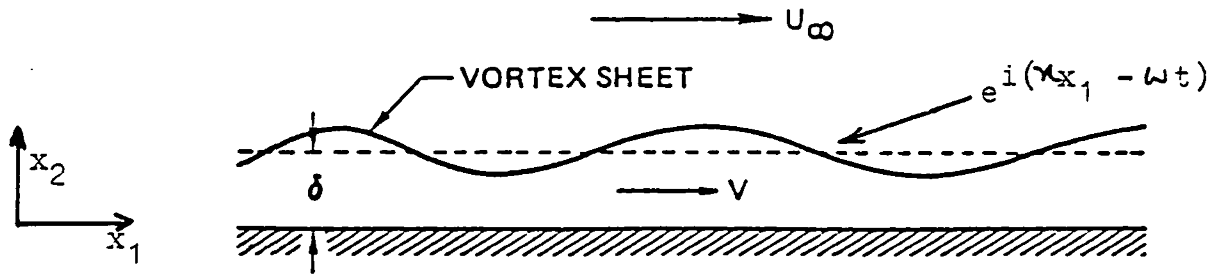
Note: This is actually an exact equality for components of the field variables whose length scale is large compared with the boundary layer width. Indeed, neglecting compressibility in the boundary layer, we have

$$\nabla^2 B = -\text{div}(\underline{\omega}\wedge\underline{v}).$$

Integrate across the boundary layer and invoke the boundary layer approximation:

$$\frac{\partial v}{\partial t} = -\left(\frac{\partial B}{\partial x_n}\right)_\delta = -\left(\frac{\partial B}{\partial x_n}\right)_0 + \Big[(\underline{\omega}\wedge\underline{v})_n\Big]_0^\delta + \int_0^\infty \frac{\partial}{\partial x_\alpha}(\underline{\omega}\wedge\underline{v})_\alpha dx_n = -\left(\frac{\partial v_n}{\partial t}\right)_0 + \frac{\partial}{\partial x_\alpha}\int_0^\infty (\underline{\omega}\wedge\underline{v})_\alpha dx_n, \text{ etc}$$

<u>Idealized boundary layer model for stable, long wavelength disturbances</u> (Tollmien-Schlichting waves)



When $\varkappa\delta \ll 1$ the exact equations reduce to

$$\begin{cases} v_1 = V, \quad v_2 = 0 & \text{for} \quad x_2 < \delta \\ \varkappa = \omega/V & \text{(wave propagates at velocity V)} \end{cases}$$

Denote displacement of vortex sheet: $\zeta = \text{const.}e^{i\varkappa(x_1 - Vt)}$.

displacement velocity: $\upsilon = \left(\frac{\partial}{\partial t} + U_\infty \frac{\partial}{\partial x_1}\right)\zeta \qquad = -i\varkappa(V - U_\infty)\zeta$

<u>Vorticity</u> : $\underset{\sim}{\omega} = (0, 0, \omega_3)$,

where $\omega_3 = (V - U_\infty - v_{1+})\delta\left\{x_2 - \delta - \zeta\right\}$.

(v_{1+} being the x_1-component of $\underline{v}$ just above the vortex sheet)

<u>Vorticity convection velocity</u>

$$\underline{v} = (\tfrac{1}{2}(U + v_{1+} + V), \partial\zeta/\partial t, 0).$$

On the basis of linear theory:

$$(\underset{\sim}{\omega} \wedge \underline{v})_1 = i\omega\zeta(V - U_\infty)\delta(x_2 - \delta)$$

$$(\underset{\sim}{\omega} \wedge \underline{v})_3 = 0$$

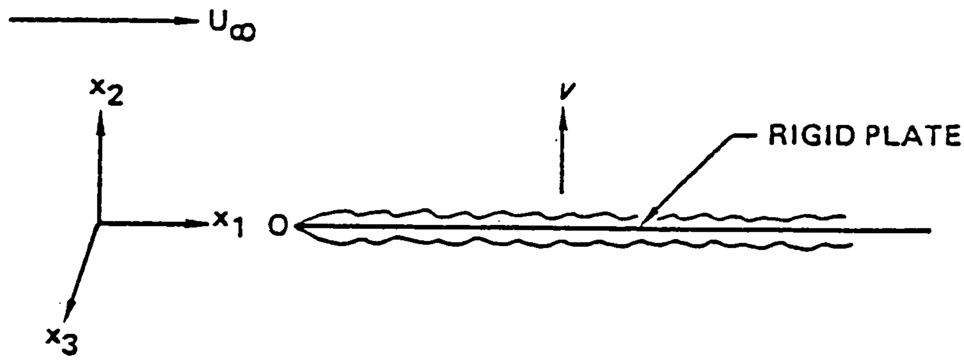
Hence:

$$\frac{\partial}{\partial x_1}\int_0^\infty (\underset{\sim}{\omega} \wedge \underline{v})_1 dx_2 = -\omega\varkappa\zeta(V - U_\infty)\int_0^\infty \delta\left\{x_2 - \delta\right\} dx_2$$

$$= -\omega\varkappa\zeta(V - U_\infty) \equiv \frac{\partial \upsilon}{\partial t}$$

in agreement with the general relation on page 5.7

Application of Liepmann's method near edges



Approximate form of Green's function for sources near the edge:

$$G(\underline{x}, \underline{y}; t, \tau) = f(\underline{x}, y_3; t, \tau) \left\{ (y_1^2 + y_2^2)^{\frac{1}{2}} + y_1 \right\}^{\frac{1}{2}} .$$

smoothly varying as function of y_3

Principal contribution to $B(\underline{x},t)$ from edge provided by $(\underline{\omega}_\wedge \underline{v})_1$, $(\underline{\omega}_\wedge \underline{v})_2$. Assume: $\longrightarrow$

$$\begin{cases} (\underline{\omega}\wedge\underline{v})_1 = A_1(y_2, y_3, \tau)e^{i\kappa y_1} ; \\ (\underline{\omega}\wedge\underline{v})_2 = A_2(y_2, y_3, \tau)e^{i\kappa y_1} \end{cases}$$

κ = hydrodynamic wavenumber, A_1, A_2 taken to be of same order.

Then

$$B_1 = -\tfrac{1}{2}\int dy_2 dy_3 d\tau f(\underline{x}, y_3; t, \tau)A_1(y_2, y_3, \tau)\times$$

$$\times \int_0^\infty \frac{\{(y_1^2 + y_2^2)^{\frac{1}{2}} + y_1\}^{\frac{1}{2}}\ e^{i\kappa y_1}dy_1}{(y_1^2 + y_2^2)^{\frac{1}{2}}}$$

$$B_2 = -\tfrac{1}{2}\int dy_2 dy_3 d\tau f(\underline{x}, y_3; t, \tau)A_2(y_2, y_3, \tau)\times$$

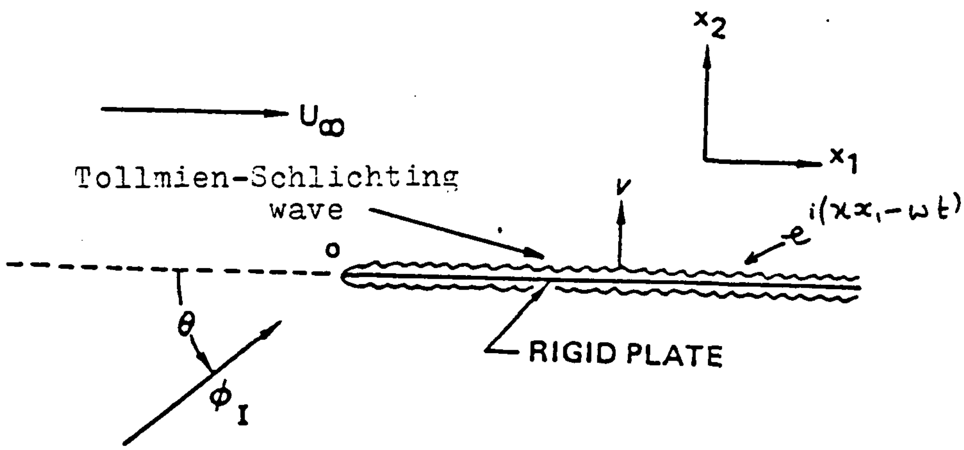
$$\times \int_0^\infty \frac{y_2 e^{i\kappa y_1}dy_1}{(y_1^2 + y_2^2)^{\frac{1}{2}}\ \{(y_1^2 + y_2^2)^{\frac{1}{2}} + y_1\}^{\frac{1}{2}}}$$

i.e.,

$$\frac{B_1}{B_2} \simeq \frac{J_{-\frac{1}{2}}(-i\kappa\delta) - \underline{J}_{-\frac{1}{2}}(-i\kappa\delta)}{J_{\frac{1}{2}}(-i\kappa\delta) - \underline{J}_{\frac{1}{2}}(-i\kappa\delta)} \quad \begin{array}{ll} \simeq O(1/(\kappa\delta)^{\frac{1}{2}}) & \text{as } \kappa\delta \to 0: \\ \simeq O(1) & \text{as } \kappa\delta \to \infty. \end{array}$$

$\therefore$ $(\underline{\omega}_\wedge \underline{v})_1$ dominant when $\kappa\delta \ll 1$. This implies that Liepmann's method is valid near the edge provided that $\kappa\delta \ll 1$.

Acoustic/Mean Flow interaction at a Leading Edge



Incident sound wave:

$$\phi_I = \phi_0 \, e^{i\{k_0(x_1\cos\theta + x_2\sin\theta) - \omega t\}}$$

Surface conditions: $v_2 = \partial\phi/\partial x_2 = v_\pm \, e^{i(\varkappa x_1 - \omega t)}$

for $x_2 = \pm 0$ respectively and $x_1 > 0$.

The amplitudes $v_\pm$ of the Tollmien-Schlichting wave displacement velocities will be determined from a leading edge Kutta condition.

Now

$$v_2(x_1,+0) = \left\{ \tfrac{1}{2}(v_+ + v_-) + \tfrac{1}{2}(v_+ - v_-)\right\} e^{i(\varkappa x_1 - \omega t)}$$

$$v_2(x_1,-0) = \left\{ \tfrac{1}{2}(v_+ + v_-) - \tfrac{1}{2}(v_+ - v_-)\right\} e^{i(\varkappa x_1 - \omega t)}$$

The "pumping" motion around the leading edge is produced by the asymmetric part $\tfrac{1}{2}(v_+ + v_-)$. i.e., $v_+ - v_-$ is indeterminate,

$\therefore$ take $\qquad v_+ = v_-$

in the boundary conditions.

This result is equivalent to applying the condition that the wall layers cannot be a net source/sink of fluid($\oint \underline{v}.dS = \int^{\infty} (v_+ - v_-) e^{i(\varkappa x_1 - \omega t)} dx_1 = 0$)

$\boxed{\text{Motion on opposite sides of plate } 180^\circ \text{out of phase}}$

The calculation of ϕ accordingly constitutes the following diffraction problem:

Find $\phi = \phi_s + \phi_I$

where (i) $\partial\phi/\partial x_2 = v_0 e^{i(\varkappa x_1 - \omega t)}$ $x_1 > 0, x_2 = 0$;

(ii) ϕ, $\partial\phi/\partial x_2$ are continuous for $x_1 < 0$, $x_2 = 0$;

(iii) scattered field must satisfy the radiation condition.

This is a standard type of Wiener-Hopf problem. Application of the Kutta condition leads to

$$v_0 = i\sqrt{2k_0\varkappa}\,\phi_0 \sin\left(\tfrac{1}{2}\theta\right)$$

Near the leading edge the diffracted component of the <u>acoustic</u> particle velocity on the "surface" of the plate is

$$u_1(x_1, \pm 0) = \mp\sqrt{\frac{2k_0}{\pi x_1}}\,\phi_0 \sin\left(\tfrac{1}{2}\theta\right) e^{-i(\omega t + \pi/4)} \qquad \text{(A)}$$

$$u_2(x_1, 0) = 0.$$

This can be used to work out the dissipation of acoustic energy (see page 4.2) per unit span:

$$\Pi = \rho_0 \int \underset{\sim}{\omega} \wedge \underset{\sim}{v} \cdot \underset{\sim}{u}\, dx_1 dx_2$$

$$\simeq 2\rho_0 \int_0^\infty u_1(x_1, +0) \int_0^\infty (\underset{\sim}{\omega} \wedge \underset{\sim}{v})_1 (x_1, x_2)\, dx_2 dx_1 \qquad \text{(B)}$$

Making use of the general low Strouhal number result (page 5.7)

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x_1} \int_0^\infty (\underset{\sim}{\omega} \wedge \underset{\sim}{v})_1\, dx_2$$

we find

$$\int_0^\infty (\underset{\sim}{\omega} \wedge \underset{\sim}{v})_1\, dx_2 = -\frac{\omega v_0}{\varkappa} e^{i(\varkappa x_1 - \omega t)}$$

$$= -i\omega\left(\frac{2k_0}{\varkappa}\right)^{1/2} \phi_0 \sin\left(\tfrac{1}{2}\theta\right) e^{i(\varkappa x_1 - \omega t)} \qquad \text{(C)}$$

To evaluate Π from (B) the REAL parts of (A), (C) must be used; after averaging over a wave period $2\pi/\omega$ we obtain:

$$\Pi = -2\rho_0 |k_0| |\phi_0|^2 \sin^2(\tfrac{1}{2}\theta)\, \mathrm{Re}(\omega/\varkappa) < 0$$

NEGATIVE! since $\mathrm{Re}(\omega/\varkappa)$ must be positive because the Tollmien-Schlichting wave propagates in the positive x_1-direction.

Note: The acoustic particle velocity of the incident wave makes no contribution to Π because of the asymmetry of the boundary layer waves.

i.e.,

$$\boxed{\text{ACOUSTIC ENERGY IS EXTRACTED FROM THE MEAN FLOW AT A LEADING EDGE}}$$

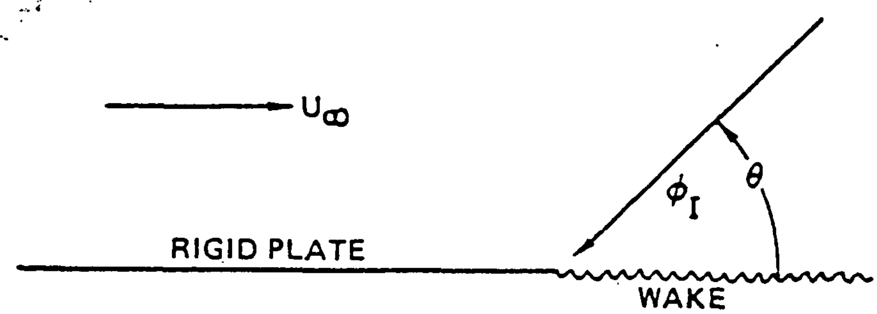
We can define an emission cross-section:

$$\sigma_+ = \frac{\text{edge generated acoustic power}}{\text{incident power flux}}$$

$$= \frac{4\sin^2(\tfrac{1}{2}\theta)}{\mathrm{Re}(\varkappa)}$$

Maximum energy is extracted for $\theta = \pm\pi$. No energy is transferred at $\theta = 0$ since no surface waves are excited.

The Trailing Edge Problem



In this case no Tollmien-Schlichting waves are involved in the interaction, although a vortical wake is formed in which(on linear theory) vorticity convects downstream at the mean flow velocity U

The principal component of $\underset{\sim}{\omega} \wedge \underline{v}$ is found to be

$$(\underset{\sim}{\omega} \wedge \underset{\sim}{v})_2 = -2i(2k_0\varkappa_0)^{1/2} U_\infty \Phi_0 \sin(\tfrac{1}{2}\theta)\, \delta(x_2)\, e^{i(\varkappa_0 x_1 - \omega t)}$$

$$\varkappa_0 = \omega / U_\infty ,$$

and the relevant component of the acoustic particle velocity for use in the energy transfer integral of page 4.2 is

$$u_2(x_1, 0) = \left(\frac{2k_0}{\pi x_1}\right)^{1/2} \Phi_0 \sin\left(\tfrac{1}{2}\theta\right) e^{-i(\omega t + \pi/4)} \quad (x_1 > 0).$$

Use of the real parts of these expressions in $\Pi = \rho_0 \int \underset{\sim}{\omega} \wedge \underset{\sim}{v} \cdot \underline{u}\, d^2\underset{\sim}{x}$ and an average over a wave period yields:

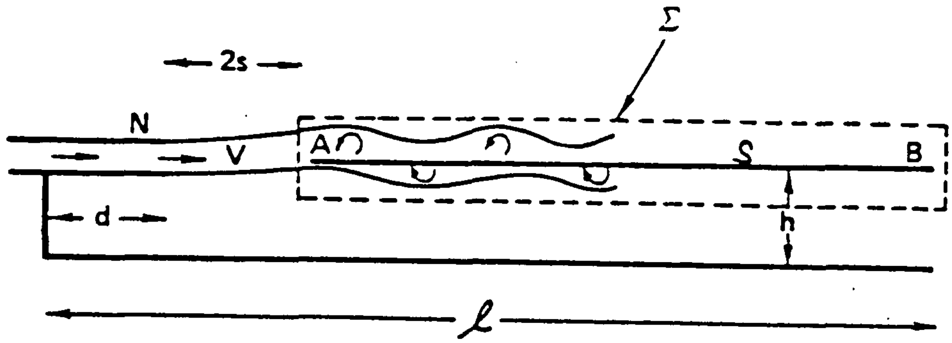
$$\Pi = 2\rho_0 |k_0| |\Phi_0|^2 \sin^2(\tfrac{1}{2}\theta) . (\omega / \varkappa_0)$$

$$\equiv 2\rho_0 U_\infty |k_0| |\Phi_0|^2 \sin^2(\tfrac{1}{2}\theta) > 0$$

i.e., **ACOUSTIC ENERGY IS DISSIPATED AT A TRAILING EDGE**

Absorption cross-section: $\sigma_- = \dfrac{4\sin^2(\frac{1}{2}\theta)}{|\varkappa_0|}$

The Jet-Drive Mechanism of the Organ Pipe



Pipe has square cross-section

$$2s \ll d, h \ll \ell$$

Linear theory will be applied to determine the threshold velocity V_n $(n = 1,2,3,\ldots)$ of the eigenfrequencies ω_n.

From the general theory of page 5.4:

$$B(\underline{x},t) = -\int_{\text{jet}} \frac{\partial G}{\partial y_j}(\underline{x},\underline{y};t,\tau)(\underline{\omega}\wedge\underline{v})_j\, d^3\underline{y}\, d\tau + \frac{\partial}{\partial t}\oint_{\Sigma} G(\underline{x},\underline{y};t,\tau)\, v_n(\underline{y},\tau)\, d\Sigma\, d\tau$$

Approximation: For a thin symmetric jet, the volume integral can be neglected, i.e., the effect of jet instabilities is of minimal importance.

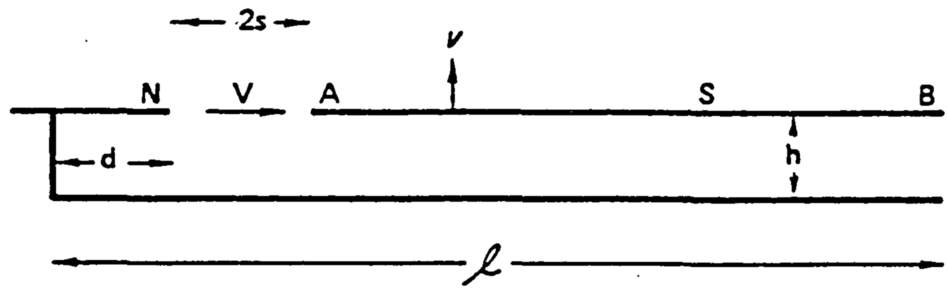
Energy is extracted from the mean flow to maintain the acoustic oscillations in the pipe, and this occurs through the action of the wall jets downstream of the lip A (c.f., result on page 5.12 for a leading edge).

In the Liepmann approximation, and setting $B = -\partial\phi/\partial t$, we can write

$$\phi(\underline{x},t) = -\oint_S G(\underline{x},\underline{y};t,\tau)\, v(\underline{y},\tau)\, dS\, d\tau$$

displacement velocity
(determined from leading edge
Kutta condition)

Simplified analytical model:



The analytical problem is equivalent to:

Find ϕ such that:

(i)

$$\frac{\partial \phi}{\partial x_2}(x_1, \pm 0, x_3) = v = v_0 e^{i(\varkappa x_1 - \omega t)} \quad \text{for} \quad x_1 > s, \ |x_3| < \tfrac{1}{2}h \qquad \text{(A)}$$

where $\varkappa$ = wavenumber of displacement thickness waves on AB.

(ii) In the mouth:

$$\left(\frac{\partial \phi}{\partial x_2}\right)_{x_2=-0} = \left(\frac{\partial \phi}{\partial x_2}\right)_{x_2=+0} = q(x_1)e^{-i\omega t} \quad \text{for} \quad |x_1| < s, |x_3| < \tfrac{1}{2}h \qquad \text{(B)}$$

(iii) Pressure is continuous across the jet:

$$\phi_{x_2=+0} = \phi_{x_2=-0} \quad \text{for} \quad |x_1| < s, \ |x_3| < \tfrac{1}{2}h \qquad \text{(C)}$$

(iv) Kutta condition:

$q(x_1)$ is finite as $x_1 \longrightarrow s - 0$.

Note: explicit dependence on time factor $e^{-i\omega t}$ will be suppressed.

Conditions (A), (B) define effective monopole source distributions on interior/exterior wall AB.

Introduce time-harmonic Green's functions $\mathcal{G}_{\pm}(\underline{x}, y_1)$ which satisfy:

$$(k^2 + \nabla^2)\mathcal{G}_{\pm} = 0 \quad ; \quad \partial \mathcal{G}_{\pm}/\partial x_2 = \pm\delta(x_1 - y_1)$$

on respectively the sides $x_2 = \pm 0$, $|x_3| < \frac{1}{2}h$ of the upper wall of the pipe. ($k = \omega/c$).

Potentials generated by the source distributions:

$$\phi_{\pm}(\underline{x}) = \pm \underbrace{\int \Sigma(y_1)\mathcal{G}_{\pm}(\underline{x}, y_1)\,dy_1}_{\text{mouth}} \pm \underbrace{\int_s^{\ell-d-s} v_0\, e^{i\varkappa y_1}\mathcal{G}_{\pm}(\underline{x}, y_1)\,dy_1}_{\text{displacement thickness effect}} \qquad \text{(D)}$$

Condition (C) across the mouth becomes:

$$\phi_{+}(x_1, +0, x_3) + \phi_{-}(x_1, +0, x_3) = \phi_{+}(x_1, -0, x_3) + \phi_{-}(x_1, -0, x_3)$$

But:
$$\phi_{-}(x_1, +0, x_3) \ll \phi_{+}(x_1, +0, x_3)$$
$$\phi_{+}(x_1, -0, x_3) \ll \phi_{-}(x_1, -0, x_3)$$

Hence, the mouth condition is taken as:

$$\phi_{+}(x_1, +0, x_3) = \phi_{-}(x_1, -0, x_3) \qquad \text{(E)}$$

for $|x_1| < s$, $|x_3| < \frac{1}{2}h$.

Now when x_1 lies within the mouth $(|x_1| < s)$:

$$G_\pm \equiv G_\pm(\xi, \eta) = \frac{1}{\pi} . \ln|\xi - \eta| + a_\pm$$

where $\xi = x_1/s$, $\eta = y_1/s$, and

$$a_+ = \frac{1}{\pi} . \ln(se/2h) - ik_o h/4\pi ,$$

$$a_- = \frac{1}{\pi} . \ln\left\{(2\pi s/h)\sinh(\pi d/h)\right\} - \frac{1}{k\ h} . \tan\left\{k\ (\ell + \lambda) + i(k\ h)^2/4\pi\right\}$$

in which λ is the "end-correction" of the open end B.

Using equation (D) in (E) we find:

$$\int_{-1}^{1} q(\eta) \ln|\xi - \eta| d\eta = F(\xi), \quad (|\xi| < 1)$$

$$F(\xi) = -v_o \int_1^\infty e^{i\varepsilon\eta} \ln(\eta - \xi) d\eta - \frac{\pi}{2} \left(Q + \frac{iv_o e^{i\varepsilon}}{\varepsilon}\right) \cdot (a_+ + a_-)$$

where $Q = \int_{-1}^{1} q(\eta) d\eta$, and $\epsilon = \varkappa s$, the reduced frequency.

The general solution of this singular integral equation is:

$$q(\xi) = \frac{1}{\pi^2(1-\xi^2)^{\frac{1}{2}}} \left[\fint_{-1}^{1} \frac{(1-\eta^2)^{\frac{1}{2}}}{\eta - \xi} F'(\eta) d\eta - \frac{1}{\ln(2)} \int_{-1}^{1} \frac{F(\eta) d\eta}{(1-\eta^2)^{\frac{1}{2}}} \right]$$

This solution involves the two unknown constants Q, v_o. They are eliminated as follows:

Integrate over the mouth $(|\xi| < 1)$:

$$Q + \frac{1}{\pi \ln(2)} \int_{-1}^{1} \frac{F(\eta)\, d\eta}{(1-\eta^2)^{\frac{1}{2}}} = 0,$$

Apply the Kutta condition that $q(\xi)$ is finite as $\xi \longrightarrow 1-0$:

$$Q - \frac{1}{\pi} \int_{-1}^{1} F'(\eta) \left(\frac{1+\eta}{1-\eta}\right)^{\frac{1}{2}} d\eta = 0.$$

Evaluating these integrals we obtain the two linear equations in Q, v_o:

$$\left(Q + \frac{i v_o e^{i\varepsilon}}{\varepsilon}\right)\left(\ln(2) - \tfrac{1}{2}\pi(a_+ + a_-)\right) + \frac{\pi v_o H_o^{(1)}(\varepsilon)}{2\varepsilon} = 0$$

$$\left(Q + \frac{i v_o e^{i\varepsilon}}{\varepsilon}\right) - \frac{i\pi v_o}{2} \cdot \left\{H_o^{(1)}(\varepsilon) + iH_1^{(1)}(\varepsilon)\right\} = 0 .$$

The condition for a non-trivial solution yields the characteristic equation giving $\omega = \omega(V)$

$$F(\varepsilon) + \ln(2) - \tfrac{1}{2}\pi(a_+ + a_-) = 0,$$

where

$$F(\varepsilon) = -iH_o^{(1)}(\varepsilon)/\varepsilon\{H_o^{(1)}(\varepsilon) + iH_1^{(1)}(\varepsilon)\}.$$

Substituting for a_+, a_-:

$$\sin\{k\,L + (2k\,h/\pi)F(\varepsilon) + i(k\,h)^2/2\pi\} = 0,$$

and therefore,

$$k\,L + (2k\,h/\pi)F(\varepsilon) + i(k\,h)^2/2\pi = n\pi \quad (n = 1,2,3,..)$$

where

$$L = \ell + \lambda + \frac{2h}{\pi} \cdot \ln\{4(h/s)^2/e\pi\sinh(\pi d/h)\}$$

Note that L defines the effective length of the pipe inclusive of end-corrections.

Hence we have:
$$\omega = \omega_n \left\{ 1 - \frac{i\omega_n h^2}{2\pi c L} - \frac{2h}{\pi L} F(\varepsilon_n) \right\} \qquad (*)$$

where

$$\omega_n = n\pi c/L, \qquad \varepsilon_n = \kappa_n s,$$

in which $\varkappa_n$ = wavenumber of the displacement thickness wave at frequency ω_n.

The second two terms in the curly brackets represent respectively radiation damping and displacement thickness effects. Interior boundary layer effects are important in practice, and these can be formally incorporated by adding on to the RHS of (*)

$$\frac{-i}{h}(2\omega_n)^{1/2}\left\{ \nu^{1/2} + (\gamma-1)\chi^{1/2} \right\}.$$

We can now write: $\omega = \omega_n + i\delta(\omega_n, V)$,
in which δ is real. It follows that the nth mode can be excited provided that $\mathrm{Im}(\omega) > 0$, i.e., that

$$\delta > 0.$$

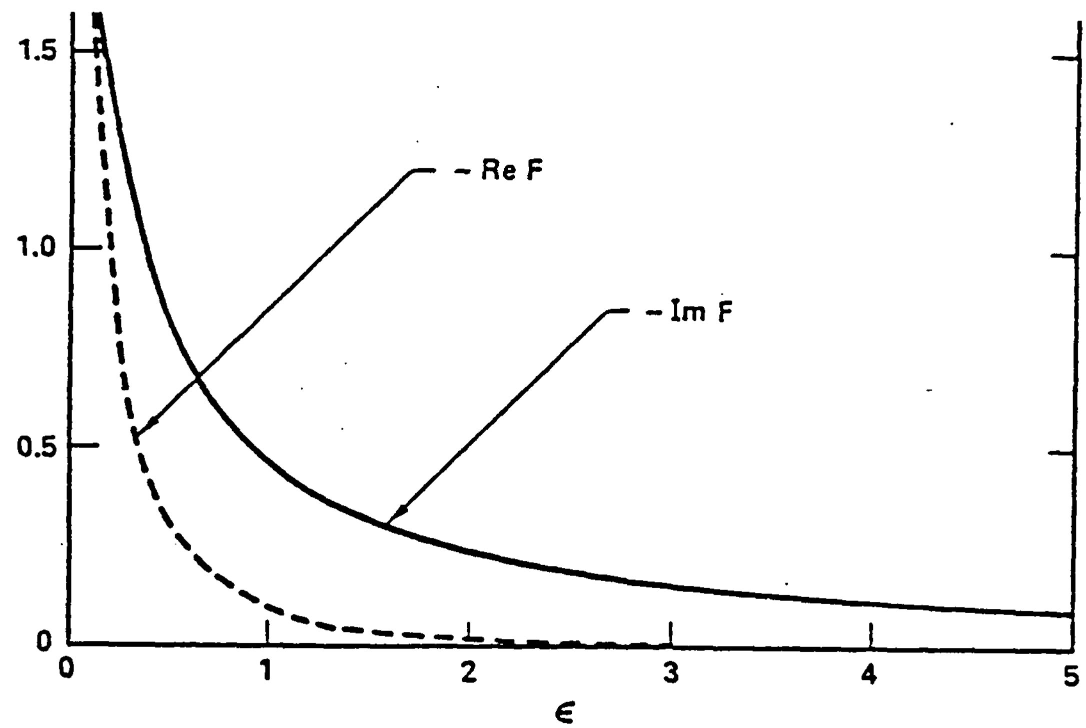
in other words if:

$$-\mathrm{Im}\, F(\varepsilon_n) \geq \frac{n\pi h}{4L} + \frac{L}{2h^2}\left(\frac{2\pi L}{nc}\right)^{\frac{1}{2}} \left\{ \nu^{\frac{1}{2}} + (\gamma - 1)\, \chi^{\frac{1}{2}} \right\}$$

To use this result we must express $\varkappa_n$ in terms of the radian frequency ω_n. The simplest hypothesis is that the wall jets on AB have a top-hat profile, in the long wavelength limit we then have (c.f., page 5.9):

$$\varkappa_n = \omega_n / V$$

Dependence of $F(\epsilon)$ on real ϵ :



For $\epsilon > 0.8$ $\quad -\mathrm{Im}(F) = 0.46/\epsilon$,
hence the threshold velocities V_n are given by

$$\left(\frac{V}{c} > \right) \qquad V_n/c = \frac{n^2\pi^2 sh}{1.84L^2} + \frac{\pi^{3/2} s}{0.92h}\left(\frac{2nL}{ch^2}\right)^{\frac{1}{2}}\left\{\nu^{\frac{1}{2}} + (\gamma - 1)\chi^{\frac{1}{2}}\right\}$$

Example (Coltman 1976, JASA 60, 725):

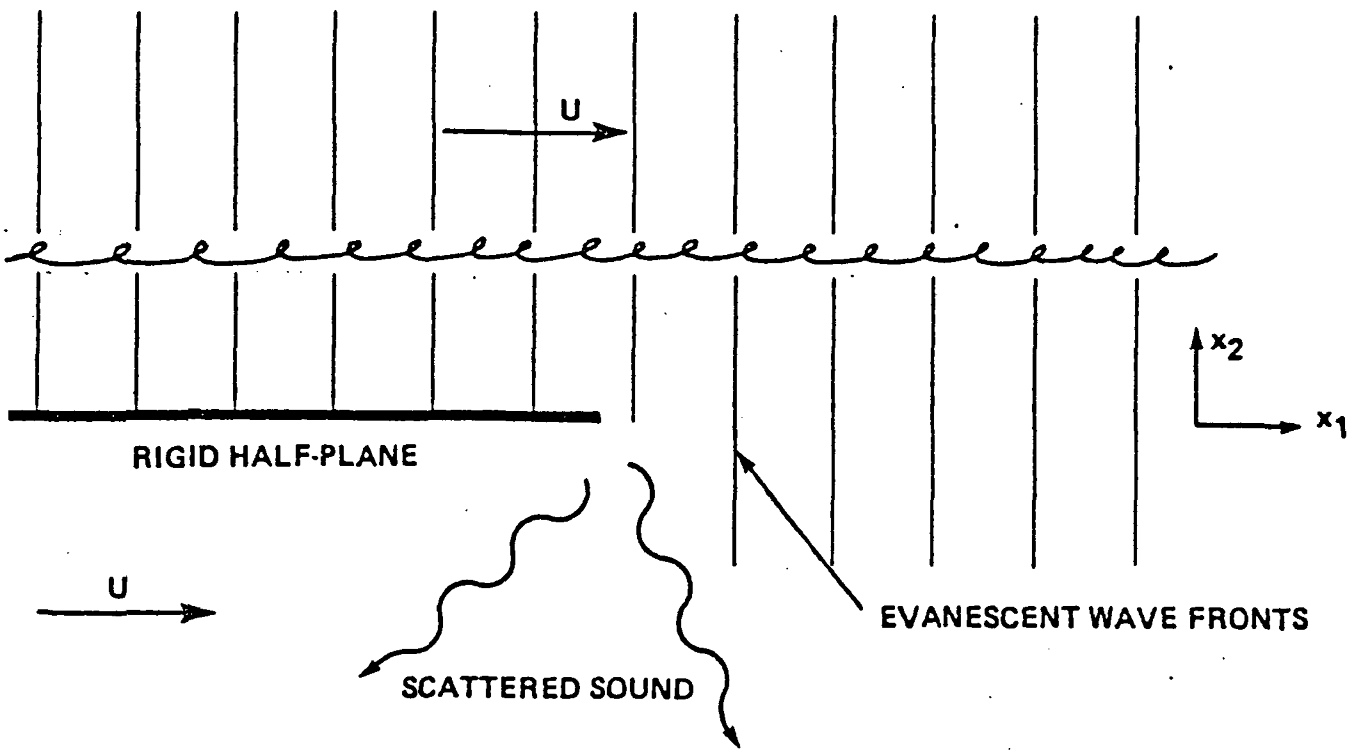
$s = 0.35$ cm, $h = 1.8$ cm, $L = 68$ cm, $c = 34000$ cm sec^{-1}
$\nu = 0.15$ cgs, $\chi = 0.21$ cgs, $\gamma = 1.4$

$$\therefore \quad V_n = 802\, n^{\frac{1}{2}}(1 + 0.031 n^{3/2}) \quad \text{cm sec}^{-1}$$

n	frequency Hz	V_n predicted cm.sec^{-1}	V_n measured by Coltman (1976) cm.sec^{-1}
1	250	827	640
2	500	1234	1200
3	750	1613	1700
4	1000	2002	-

Displacement thickness theory of trailing edge noise

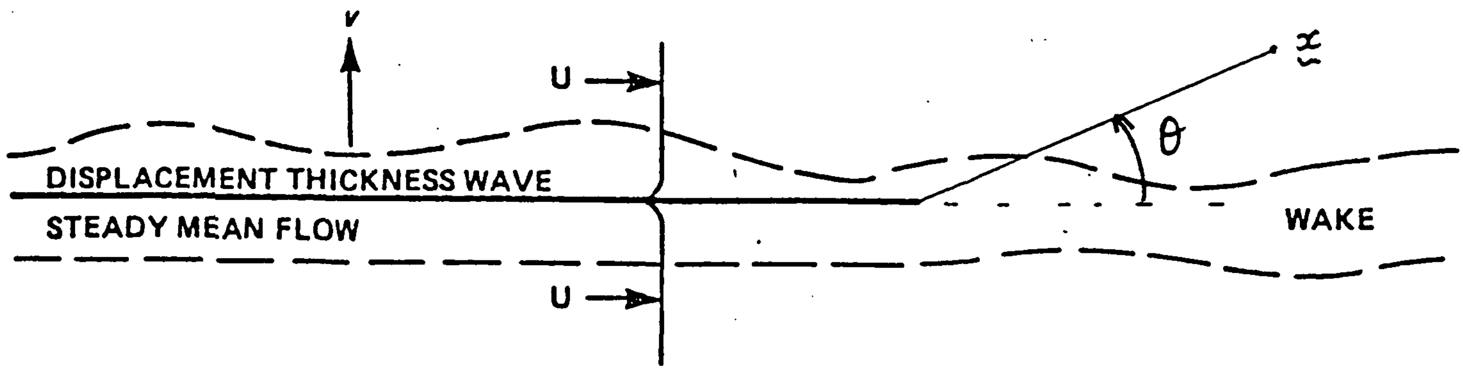
Conventional estimates of the edge noise are usually based on the so-called evanescent wave theory:



Schematic illustration of the evanescent wave theory of trailing edge noise. Turbulent fluctuations translating to the right would generate exponentially decaying, potential flow disturbances beneath the boundary layer if the plate were absent. The sound is calculated by a consideration of the diffraction of these waves at the edge.

This takes no account of changes in the large scale turbulence characteristics as it translates past the edge.

Representation of a low Strouhal number boundary layer disturbance by means of a displacement thickness surface wave.



A low Strouhal number ($\omega\delta^*/U \ll 1$) boundary layer disturbance can be modelled as a displacement thickness surface wave. This is a valuable approximation, because the dominant surface pressure fluctuations occur in $0.1 \leq \omega\delta^*/U \leq 1$. In the wake there can exist two types of wave modes:

1. Asymmetric waves, for which v_2 is continuous and the pressure p is continuous,

2. Symmetric (or pulsational) modes, for which v_2 changes sign, but p is still continuous.

When a low Strouhal number disturbance in the boundary layer is incident on the edge it is transformed into a combination of these wake modes. The result is that, in addition to the usual edge noise radiation (predicted by evanescent wave theory), a weaker dipole component is also present, the axis of the dipole being aligned with the mean flow direction. The dipole arises from the net radiation from two equal and opposite monopole sources. The first has strength

$$m = \int_{-\infty}^{0} v_I e^{i\varkappa_I x_1} \, dx_1,$$

where v_I is the displacement velocity amplitude of the surface wave, and $\varkappa_I$ is its wavenumber.

The second monopole is caused by the pulsational mode (2.)

$$\propto e^{i\varkappa_S x_1} \quad \text{, say,}$$

in the wake. The net radiation from these equal and opposite monopoles is a dipole whose strength is proportional to

$$\varkappa_I - \varkappa_S \quad ,$$

and which therefore vanishes if the properties of the turbulence do not change at the edge.

In the acoustic far field:

$$p(r,\theta) = \frac{-p_I}{2(\pi\varkappa_I r)^{\frac{1}{2}}}\left[\frac{1 + M - M_I}{1 + (M - M_I)\cos\theta}\right]\left(\frac{1 - M + M_I}{1 - M^2}\right)^{\frac{1}{2}} e^{i(k r + \pi/4)} \times$$

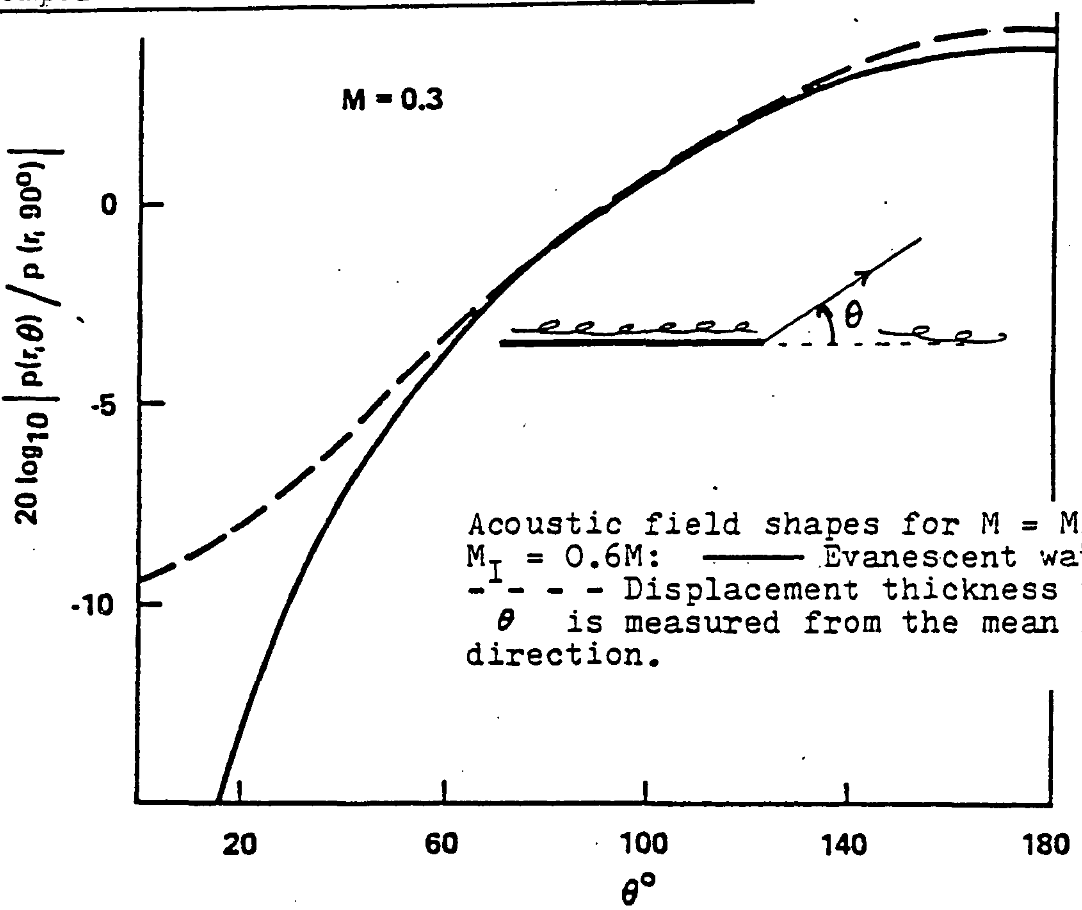
$$\times \left\{ \underbrace{\sin(\tfrac{1}{2}\theta)}_{\text{usual evanescent wave prediction}} + i(\tfrac{1}{2}M_I)^{\frac{1}{2}}\left(\frac{M_S - M_I}{M - M_I}\right)\left(\frac{1 - M^2}{1 - M + M_I}\right)^{\frac{1}{2}} \cdot \underbrace{\frac{\cos(\theta)}{(1 + M\cos\theta)^{\frac{3}{2}}(1 + (M - M_S)\cos\theta)}}_{\text{additional dipole contribution}} \right\}$$

where:

$M = U/c$; M_I = convection Mach number of incident surface wave;

M_S = convection Mach number of symmetric wake mode.

Comparison with evansecent wave theory:



Acoustic field shapes for $M = M_S = 0.3$, $M_I = 0.6M$: ——— Evanescent wave theory; - - - - Displacement thickness theory. θ is measured from the mean flow direction.