Resilient Quantum Computation: Error Models and Thresholds

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Recent research has demonstrated that quantum computers can solve certain types of problems substantially faster than the known classical algorithms. These problems include factoring integers and certain physics simulations. Practical quantum computation requires overcoming the problems of environmental noise and operational errors, problems which appear to be much more severe than in classical computation due to the inherent fragility of quantum superpositions involving many degrees of freedom. Here we show that arbitrarily accurate quantum computations are possible provided that the error per operation is below a threshold value. The result is obtained by combining quantum error-correction, fault tolerant state recovery, fault tolerant encoding of operations and concatenation. It holds under physically realistic assumptions on the errors.

1. Introduction

The discovery that quantum computers can be much more powerful than their classical counterparts (Deutsch 1985, Bernstein 1993, Simon 1994, Shor1994) and recent advances in quantum device technology (Cirac 1995, Monroe 1995) have brought the field of quantum computation into the limelight. However, until recently, the hope of taming quantum systems has been overshadowed by the fragility of quantum information. Not only must it be preserved in memory, but it should not be lost when manipulated. This fragility comes from two seemingly contradictory requirements. The system must be well insulated from the environment to avoid losses, while at the same time we need to interact with it strongly to perform the desired computation. Due to these requirements, it is impossible to completely isolate a physical quantum computer from the environment. As a result, the computer necessarily becomes increasingly entangled with the outside world and any quantum information it contains is apparently lost. Moreover, quantum computation requires the application of precise unitary operations to the system. It is clear that these operations cannot be implemented exactly. For these reasons, some authors have concluded that the theoretical power of quantum computers cannot be harnessed (Landauer 1995, Unruh 1995).

The first indication that early assessments of the practicality of quantum computation might be overly pessimistic was the discovery of quantum error-correcting codes by Shor (1994) and Steane (1996a,b). These codes imply that it is possible to overcome memory or transmission errors provided that the unitary operators required for encoding, decoding and error-correction could be implemented with high accuracy. Subsequent work on quantum error-correction has yielded very efficient codes and generalized much of the classical theory (Calderbank 1995, Laflamme 1996 , Bennett 1996, Gottesman 1996, Calderbank 1996). The requirement of accurate error-correction operations was soon relaxed by Shor (1996) and essentially eliminated in the case of quantum channels in Knill (1996b). To use quantum error-correction for quantum computation requires not only encoding states, but manipulating them in encoded form. That this can be done in some interesting cases was discovered independently by Shor (1996), Kitaev (1996) and two of us (Zurek 1996). Shor's results in particular showed that provided the noisy behavior of quantum memory and gates is stochastic with error probability of the order of $O(\log^{-\alpha n})$ (where n is the total number of noisy operations in the quantum network and α is a positive constant), arbitrarily accurate encoded quantum computation is possible. This was a substantial improvement over the best previous bound of $O(1/n)$ and removed almost all fundamental obstacles to practical quantum computation.

Two obstacles to quantum computation remained. The first is that as the number of elementary quantum operations grow, Shor's fault tolerant implementation still require asymptotically zero error per operation. The second concerns the fact that many if not most error types expected in real devices cannot be represented in the stochastic error model. In particular, unitary over-rotation of operations and small but non-negligible interactions between nearby qubits give rise to such errors. The purpose of this paper is to show that an error threshold exists such that if each gate in a physical implementation of a quantum network has error less than this threshold, it is possible to perform encoded quantum computations with arbitrary accuracy. This result holds under physically realistic error models, thus showing that in principle, unlimited quantum computation with noise is possible.

In the first part of the paper we introduce the different techniques required to implement quantum fault tolerance. We emphasize the various assumptions on errors and then introduce four fundamental elements of fault tolerance: quantum error-correcting codes, fault tolerant error-correction methods, encoded operations and concatenation. We introduce a new method for obtaining a complete set of fault tolerantly encodable operations which greatly simplifies the analysis. We combine the elements to implement a computation fault tolerantly. In the second part, the fault tolerant networks are analyzed to obtain rigorous thresholds for the quasi-independent stochastic and monotonic error models.

2. Methods

(a) *Quantum networks and operations*

We can assume without loss of generality that quantum algorithms are described by means of a quantum network†. A quantum network is a space-time diagram of the operations that are to be applied to each qubit. Recall that a qubit is the prototypical two state system. It's state space is the Hilbert space spanned by $|0\rangle$ and $|1\rangle$. A qubit's time line is represented by a horizontal line, and opera-

† For the purposes of algorithm design this is not necessarily the best choice. Informal quantum programming languages ("quantum pseudocode") are used in Cleve (1996) and Knill (1996c).

tions, that is quantum gates, are denoted by blocks or vertical lines connected to the qubits being acted on. (See the figures for examples). The operations' actions and symbols are defined in Table 1.

A complete set of unitary operations can be formed from controlled-nots and one qubit phase shifts (DiVincenzo 1995, Barenco 1995, Lloyd 1996). A useful set of operations consists of the sign and the bit flip, the controlled-not, the Hadamard transform and the phase shift of $|1\rangle$ by i. These generate the *normalizer group*. To these one could add the Toffoli gate to achieve completeness (Shor 1996a). We choose to complete the set of operations by adding preparation of $|\pi/8\rangle = \cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle$ to our repertoire rather than the Toffoli gate. A circuit which implements the Toffoli gate using this operation is given in Figure 1.

To be able to perform quantum computations requires at least two more operations: preparation of $|0\rangle$ and measurement of $|0\rangle, |1\rangle$ which we call the *classical* basis. Finally, in many cases, classical computation can be used to process measurement outcomes and control future quantum gates. Classical (reversible, if so desired) circuits may be used to represent these computations.

The primary purpose of a quantum algorithm is to produce a desired quantum state from classical input such as $|0\rangle$. Normally, some or all of the output state's qubits are measured to obtain the actual information that is needed. Our main goal is to describe an implementation of quantum algorithms which satisfies that if each gate's error is below a threshold value, the algorithm's computational state is maintained with arbitrary accuracy in an encoded form. Furthermore, the bits of the computational state can be measured and the outcome is consistent with the claimed accuracy of the state.

It is important to realize that the existence and actual value of the threshold depend critically not only on the details of the implementation, but also on the assumptions on error behavior. The values determined below could be substantially improved in a particular physical setting.

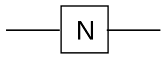
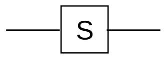
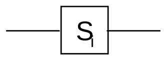
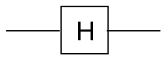
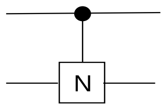
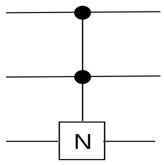
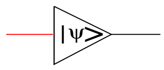
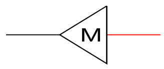
(*b*) *Assumptions and error models*

To describe a noisy quantum network we introduce the notion of *error locations*. Error locations are to be chosen such that one can assume without loss of generality that errors "occur" only at those locations, and that these errors satisfy some independence properties with quantifiable error probabilities or *strengths*.

We consider two types of error locations: operational error locations and memory error locations. The former exists after each gate (including state preparation but not measurements) and extends over all the qubits involved in the gate's operation. The latter exists on each line segment representing a unit time interval of a qubit's time line, provided the qubit is not involved in a gate in this interval. The placement of memory errors depends on the temporal layout of a network and requires partitioning the network into unit time slices. The time units are determined by the maximum execution time of a gate. Figure 2 shows how error locations are placed in one of the component networks required for fault tolerant computing.

We are assuming that sufficiently many gates can be executed in parallel to avoid loss of quantum information in memory. In fact, there is a trade-off between memory error and parallelism. If t is the maximum time a qubit's state can be left in storage without unacceptable loss of information, then the minimum number of operations per unit time must be well above q/t, where q is the number of qubits

Table 1. *Elementary unitary operations on one to three qubits*

Names	Symbols	Action	Gate icon
not, bit flip	$-i\sigma_y, N$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	
sign flip	σ_z, S	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	
i-phase shift	S_i	$\begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$	
Hadamard rotation	H	$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$	
controlled-not, xor	N_2	$\lvert x\rangle\lvert y\rangle \rightarrow \lvert x\rangle\lvert x+y \bmod 2\rangle$	
Toffoli gate	T	$\lvert x\rangle\lvert y\rangle\lvert z\rangle \rightarrow \lvert x\rangle\lvert y\rangle\lvert z+xy \bmod 2\rangle$	
Preparation of $\lvert\psi\rangle$			
Measurement of a qubit			

actively involved in computation. If the memory error per timestep is known and is substantially less than the operational error, this can be exploited to avoid some parallelism.

The utility of error locations comes from the observation that the actual behavior of a quantum network can be represented (non-uniquely) as a mixed sum of networks with linear error operators placed at each error location. Each network in the sum is associated with a state of the environment. We call this an *error expansion* of the network. The final state of the computation can be evaluated by obtaining the states associated with each summand and formally adding them. It is important to realize that the error expansion gives correct input-output be-

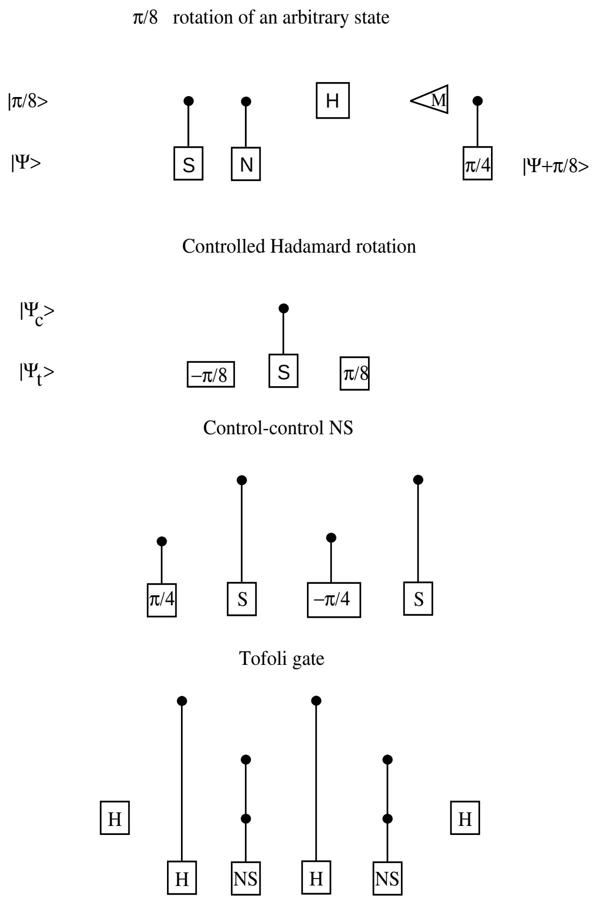


Figure 1. Construction of a Toffoli gate from the primitves given in Table 1. The first figure gives a $\pi/8$ rotation on an arbitrary state. Using this gate we can then construct a control-Hadamard operation with which we can build a control-control-Hadamard. Finally bringing all this together we show how to construct a Toffoli gate for the first 3 qubit, whatever the 4th bit is.

havior of the entire network but need not accurately represent the intermediate physical states of the qubits.

Assumptions on error behavior are conveniently expressed as constraints on allowed error expansions. There are three classes of constraints that can be considered. The first involves the types of error operators that can occur at a given location. We will for the most part assume that these operators do not lead to loss of amplitude from the two dimensional computational subspace in a qubit (*leakage errors*). We will briefly discuss how this assumption can be relaxed by exploiting special gates which reliably return lost amplitudes to the computational subspace. With the assumption that there are no leakage errors, we can, without further loss of generality, take the error operators to be one of the standard errors

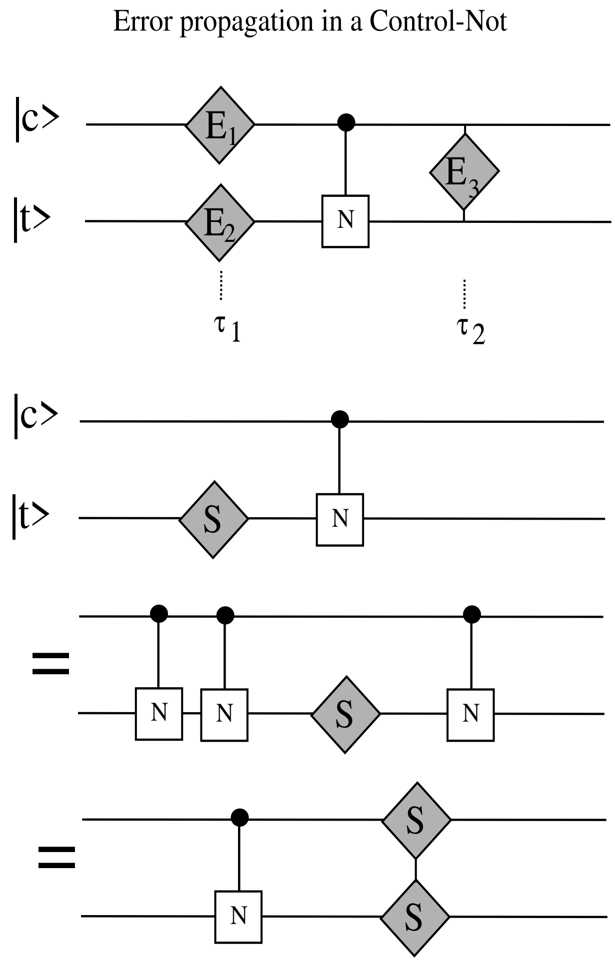


Figure 2. The top part of the figure shows the error location for a controlled-not gate. The second part shows the propagation of a sign flip in the target bit. This type of error propagates from the target to the control bit.

(no error, bit flip, sign flip or both). An additional assumption we will use is that errors in classical computations based on measurement outcomes are error free. This is a good approximation to what is possible in practice. We will return to this assumption in the concluding comments.

The second class of constraints involves the nature of the mixture. The analysis of our methods is simplest when the mixture is obtained by stochastically and independently placing one of the standard errors at each error location. Given our assumption on the error operators, this is equivalent to requiring that the states of the environment associated with each summand of the error expansion are orthogonal. A weaker assumption requires only that a stochastic error expansion exists without making restrictions on the operators involved. In this case stochasticity is defined as having orthogonal environments associated with each summand. For such expansions it makes sense to define the probability of error at a given location as the probability of those summands in the expansion which do not have an identity operator at this location. Note that these probabilities are in principle state dependent. Since the input states are in effect predetermined in a quantum network, the probabilities are fixed but may be difficult to obtain by calculation.

The third class of constraints enforces independence of errors at different locations. With this in mind we define the following error models:

• Independent stochastic errors: The error expansion is obtained by making an independent, random choice at each error location of either the identity operator or a quantum operation. A quantum operation is defined as a stochastic (in the sense defined above) combination of linear operators satisfying a unitarity condition (Knill 1996a, Schumacher 1996). The probability of error at a given location is defined as the probability of making the latter choice. The maximum probability of failure over all error locations is the error probability associated with the model. A convenient strengthening of this error model requires that the quantum operation consists of a random choice of one of the standard errors.

• Independent errors: The error expansion is obtained by assigning a quantum operation to each error location. In this case we can no longer talk of error probabilities. The error *strength* of a quantum operation is defined below. The error strength associated with this model is the maximum error strength of the quantum operations.

• Quasi-independent stochastic errors: An error expansion satisfies the quasi-independent stochastic error model with error probability p if it can be decomposed into a stochastic sum with the property that the probability of all summands which have a non-identity quantum operation at a given k many error locations is at most p^k.

• Quasi-independent errors: Let each summand of an error expansion be associated with a set of *failed* error locations such that all other error locations are instantiated with an identity operator. Such an error expansion satisfies the quasi-independent error model with error strength p if the total strength of the summands for which at least a given k many error locations have failed is at most p^k. It satisfies the quasi-independent monotonic error model with bound C if the total strength of any subset of these summands is at most Cp^k.

The analysis to be presented below establishes thresholds for quasi-independent stochastic and monotonic errors. The analysis for quasi-independent errors without the monotonicity assumption is more complicated and yields a somewhat worse threshold. It will be presented elsewhere.

The strength of a set of linear operators labeled by states of the environment is defined as follows:

$$|\sum_i |e_i\rangle A_i| = \sup_{|\psi\rangle} |\sum_i |e_i\rangle A_i|\psi\rangle|.$$

This is the maximum modulus of the amplitude of the outcome of applying the operation to a (normalized) state. Given a representation of a quantum operation of the form $|e_0\rangle I + \sum_i |e_i\rangle A_i$, its error strength is the strength of the second term. Any quantum operation has a representation of this form with the property that $\mathrm{tr} A_i = 0$ for each $i \geq 1$. By default, the error strength is determined by such a representation In this case, the error strength is independent of the choice of representation with this property. Note that in the case of summands of error expansion, we could, at least in principle, replace the supremum in the definition of error strength by evaluation at the known input state. Error strength is closely related to the standard notion of fidelity (Knill 1996a, Schumacher 1996) but easier to analyze in the present context. For stochastic models, error probability is given by the square of the error strength. Thus noise limits are generally much more stringent when considering non-stochastic models.

The error models described above can be used to bound the probability† of a network's computation failing. This is at most the probability of at least one error location having a non-identity operator in the error expansion. Suppose the network has n error locations. Then for the quasi-independent stochastic or monotone models with error probability p, this is at most np or Cnp, respectively. For the quasi-independent error model with error strength p this is given by $(1+p)^n - 1$ ‡. Thus high probability of success is assured if $p \ll 1/n$.

How realistic are these error models? Since physics is generally described using local interactions, it seems reasonable to assume that error events on qubits are caused by independent environments except when they are intentionally modified by an interaction implementing a multi-qubit operation. This is the independent error model in physical terms. It covers not only independent stochastic errors, but also non-identity unitary operators at each error site. For example, a consistent modification of the internal energies of each qubit by a weak external field is allowed, provided the deviations are small enough. The quasi-independent error model is substantially more general than the independent error model. In addition to other error types, it can also model weak pairwise interactions between adjacent qubits. The quasi-independent monotonic error model makes the reasonable additional assumption that relevant sums of subsets of operators in an error expansion do not conspire to greatly increase the modulus of the amplitude of the input state.

The assumption that no amplitude is lost from the computational systems can be quite unrealistic. For example in the ion trap, amplitude may be lost to any of the other available levels, some of which are in fact needed for some operations and readout. Amplitude loss can be modeled with error expansions by extending the dimension of each system and allowing non-computational error operators. However, the fault tolerant networks implemented here can fail in the presence of these errors. To make them work requires the use of *stop leak gates* as described in the analysis section.

Although the above arguments suggest that the quasi-independence assumptions can be satisfied in principle, it is worth pointing out that on the surface, some proposed implementations such as the ion trap quantum computer, violate this assumption from the beginning by involving a shared bus in every two-qubit operation. To ensure that independence can still be assumed requires reliable dissipation of residual information in the bus qubit between any two operations using it. The need for such dissipation is similar to the need for reliable methods to restore amplitude to the computational state space of a qubit in the case of leakage errors. Dissipation of bus qubit information is used in Cirac (1996a) for controlling bus errors.

† When discussing both stochastic and non-stochastic models, we use the word "probability" to mean either probability or strength, depending on the model.

‡ This bound is obtained by inclusion-exclusion based on the known bounds for summands involving failures at given locations.

(*c*) *Quantum error-correction*

The first element required for fault tolerance is quantum error-correction. In classical communication and computation, error-correction is usually accomplished by redundantly encoding information. The simplest and quite effective if inefficient method involves duplicating the information at least three times and using majority voting to recover it after errors have occurred. This method cannot be straightforwardly applied to quantum computers for three reasons. First it is not possible to clone arbitrary quantum states (Wooters 1982). Second, in order to take a majority vote, it is naively thought that we must first learn the encoded information by measurement. This would destroy any quantum coherence of the state. Finally, only one type of error needs to be considered for classical binary information, the bit flip. Quantum states can be modified by a continuum of possible errors.

Recently Shor (1995) and Steane (1996a,b) discovered how these objections could be overcome. To avoid copying information to introduce redundancy it is possible to exploit highly entangled states supported by additional qubits. Thus a quantum state is unitarily associated with a linear combination of such entangled states on sufficiently many qubits to permit recovering the information after the loss of information in few of the qubits. These linear combinations define the coding subspace. To avoid collapse of the quantum information by measurement in the process of correcting errors, it is possible to make a partial measurement which extracts only error information and leaves the encoded state untouched. This can be arranged by choosing an encoding with the property that the standard errors translate the coding subspace to orthogonal subspaces. To deal with the fact that there is a continuum of possible errors, it is sufficient to recognize that every error can be represented as a linear combination of the standard errors. Together with the observation that linear combinations of correctable errors are also correctable, this allows discretizing the error possibilities. The general theory of error-correction is discussed in Knill (1996a). Standard errors are introduced in Steane (1996a).

To prove the threshold theorem we make use of the one qubit single error-correcting code on seven qubits based on the classical Hamming code (Steane 1996a, Calderbank 1995). One can view this code as a prescription for interpreting the seven qubit system as a pair of abstract particles, the abstract qubit and the syndrome space. Thus $Q \otimes Q \otimes Q \otimes Q \otimes Q \otimes Q \otimes Q \simeq A \otimes S$, where A is the abstract two state particle and S is the syndrome space. The syndrome space is endowed with a basis $|000000_S\rangle, |000001_S\rangle, \ldots, |111111_S\rangle$ (64 orthonormal states). The one qubit state $\alpha|0\rangle + \beta|1\rangle$ is encoded as $|\psi\rangle = (\alpha|0_A\rangle + \beta|1_A\rangle)|000000_S\rangle$ in $A \otimes S$. The representation ensures that if an error operator acts on one of the seven supporting qubits, the state component of $|\psi\rangle$ in A is unchanged and only the syndrome space is affected. This permits recovery of the state $|\psi\rangle$ by measuring S and resetting its state to $|00000_S\rangle$. Provided that this can be done sufficiently frequently and reliably, one can maintain the state of the abstract particle A for prolonged periods, even in the presence of noise acting independently on the seven qubits of the code.

The representation $A \otimes S$ can be specified by supplying observables for A and S. We provide six binary observables for S with eigenvalues ± 1 depending on the value of the corresponding bit in the labeling of the basis. These observables are given by

$$\begin{aligned}
S_1 &= I \otimes I \otimes I \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z, \\
S_2 &= I \otimes \sigma_z \otimes \sigma_z \otimes I \otimes I \otimes \sigma_z \otimes \sigma_z, \\
S_3 &= \sigma_z \otimes I \otimes \sigma_z \otimes I \otimes \sigma_z \otimes I \otimes \sigma_z, \\
S_4 &= I \otimes I \otimes I \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y \otimes \sigma_y, \\
S_5 &= I \otimes \sigma_y \otimes \sigma_y \otimes I \otimes I \otimes \sigma_y \otimes \sigma_y, \\
S_6 &= \sigma_y \otimes I \otimes \sigma_y \otimes I \otimes \sigma_y \otimes I \otimes \sigma_y.
\end{aligned}$$

An observable for A which behaves correctly relative to the errors we wish to protect against is not as easy to provide, due to its dependence on a non-linear syndrome decoding method. Instead we provide the following observable which defines the correct encoding subspace, and for given values of the syndrome observable gives a basis related to the abstract particle's natural basis by one of the standard errors:

$$A_1 = \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z \otimes \sigma_z.$$

Which standard error affects this observable given the state of the syndrome space can be determined from the details of the decoding method.

The fact that this code corrects one qubit errors is established in Steane (1996a). Each of the syndrome's observables can be measured by a simple circuit which determines a parity in either the $|0\rangle, |1\rangle$ or the $|+\rangle|-\rangle$ basis of the qubits. The parity checks yield the syndrome state. Each combination of standard errors applied to the qubit changes the values of the syndrome observables in a deterministic fashion. Each error consisting of at most one bit flip and one sign flip yields a distinct syndrome state. Thus, which of these errors occurred can be learned from the syndrome, and the error can be corrected (up to a global phase) by applying the same operators again.

(*d*) *Fault tolerant error-correction*

Error-correcting codes work under the assumptions that the encoding and recovery (the step consisting of detecting the errors and correcting them) are error free. This can be a good approximation in the case of quantum communication or memory where the errors occur mostly during transmission or idle time and not in the encoding and recovery period. In general, we cannot expect that the operations involved in recovering the encoded state are exact and a method must be devised for implementing them such that any single error (in the case of the 7-qubit code) occurring anywhere in the network can be eliminated. An efficient method for accomplishing this goal is given in Shor (1996) and DiVincenzo (1996). It is based on two basic ideas. The first is to use prepared, verified "cat" states to permit extracting the syndrome information without directly applying multi-qubit operations to the qubits of the code. The verification of the prepared states is used to reliably eliminate pre-existing errors which could adversely affect more than one qubit of the code when the syndrome is extracted. The second idea is to perform the syndrome extraction multiple times to ensure that the syndrome has been measured correctly. This avoids the problem of destroying the state by failing to restore the syndrome to $|000000_S\rangle$. Note that the operators involved in restoring the syndrome act independently on each qubit. A network for fault

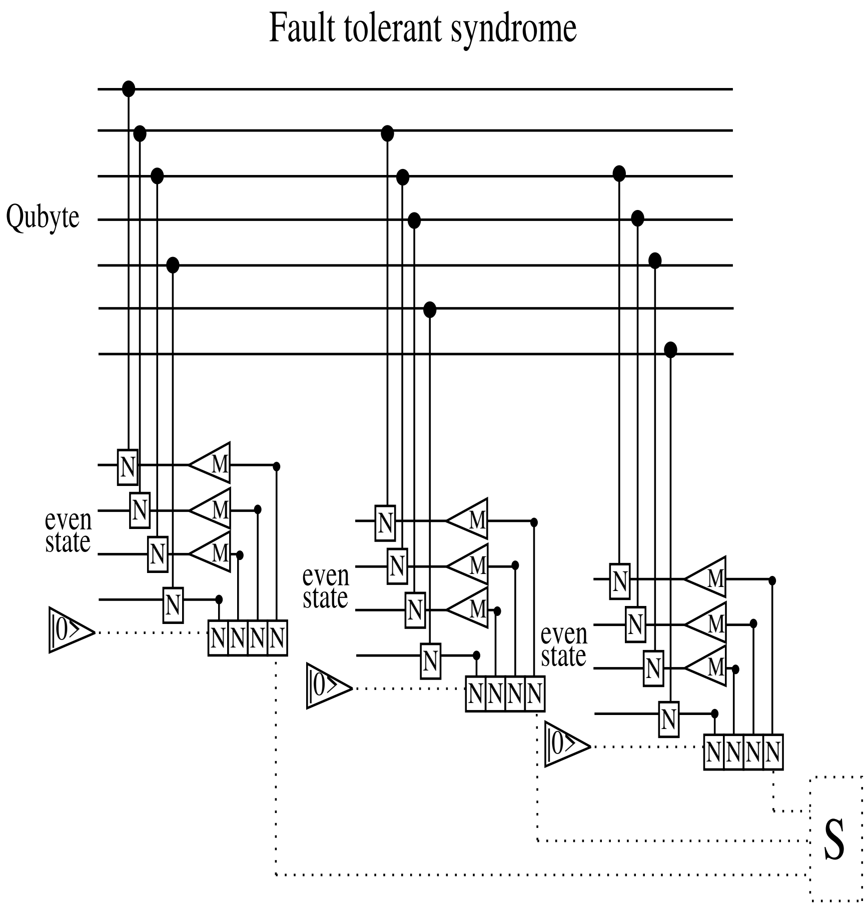


Figure 3. Fault tolerant calculation of the syndrome for the 7 bit code. The even state is obtained from the fault tolerant operation given by Shor (1996)

tolerant recovery suitable for analysis is shown in Figures 3 and 4. To explicitly check that the method described has the desired effect of tolerating at least one error in the error locations of the network requires a detailed study of how errors propagate. Error propagation is discussed in Section 3.

(*e*) *Encoded operations*

Quantum error correcting and fault tolerant methods protect quantum information when the computer is not actively performing logical operations on the computational state. A quantum computation will be required to perform such operations. This can of course be done by decoding the state, applying the desired operation and reencoding it. However, this temporarily removes the protection offered by the encoding, a serious problem if one wishes to compute fault tolerantly. Thus it is necessary to act directly on the encoded state by means of suitably encoded operations. The trick is to do this without having any single operational error introducing multi-qubit errors in (for example) the seven qubits used for

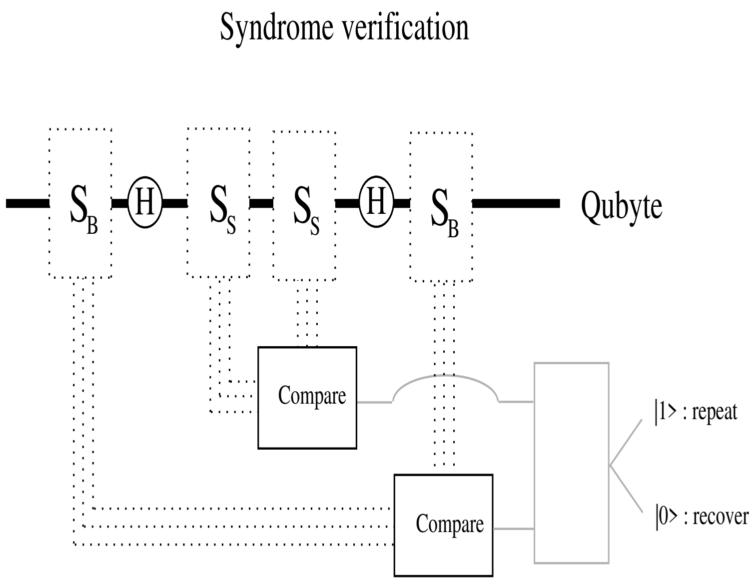


Figure 4. Verification of the syndrome of a qubyte. The syndrome for both bit flips S_B and sign flips S_S are calculated twice to learn if the error occurred during the syndrome operation.

encoding an abstract particle. The simplest method for accomplishing this involves implementing operations *transversally.* Thus we label the qubits of each coded qubit from 1 to 7 and permit interactions between them only if they have the same label. Note that with a suitable choice of labels on the even states, this rule is satisfied by the recovery operator, but not by the even state preparation procedure. In the latter case, dangerous two or more qubit errors are eliminated by the verification step.

What operations on states encoded using the seven qubit code can be performed transversally? In Shor (1996) it is shown that the normalizer group as well as preparation of encoded $|0\rangle$ and classical measurement can be implemented in this fashion. Such a network for the controlled-not is shown in Figure 5. Similar implementations are possible for many other codes as demonstrated by Gottesman (1997).

A useful property is that encoding, decoding and recovery operators can be implemented using only generators of the normalizer group. Measurement is performed by measuring each qubit in the classical basis and decoding the outcome classically. Preparation involves fault tolerantly measuring the observable A_1 in addition to the syndrome and resetting all observables to 0 by applying a suitable combination of standard errors.

The normalizer group is not computationally complete. In Shor (1996), a fault tolerant method for implementing the Toffoli gate is used to achieve completeness and in Knill (1996d) two conceptually simpler techniques involving two-qubit op-

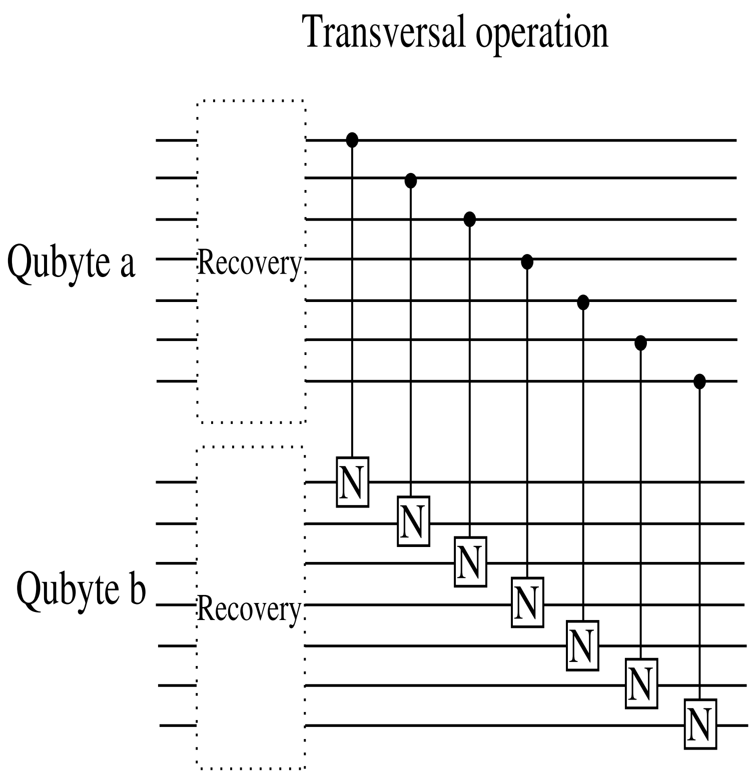


Figure 5. Fault tolerant implementation of a control not between two qubytes. Each qubit of a qubyte interacts at most with one qubit of another qubyte: this is the transversality property.

erations are given. Here we use a third idea which requires only preparation and verification of the $|\pi/8\rangle = \cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle$ state. Figure 6 shows networks for generating a reliable encoded $|\pi/8\rangle$ state given the ability to prepare unencoded $|\pi/8\rangle$ states. The trick is to realize that $|\pi/8\rangle$ is the $+1$ eigenstate of the Hadamard transform. By performing a measurement using Kitaev's techniques (Kitaev 1996) via a controlled Hadamard transform, one can obtain either $|\pi/8\rangle$ or $|5\pi/8\rangle$. The latter can be converted into the former by a $\pi/2$ rotation. The controlled Hadamard transform can be implemented transversally as shown in Figure 6. To eliminate single errors, the measurement is performed twice with an intermediate recovery network. The state is rejected if the two measurements do not agree.

State preparation and measurement are important not only for generating and exploiting $|\pi/8\rangle$, but for computation in general. As shown in Aharanov (1996), without the ability to generate bounded noise initial states at any time during the computation, efficient fault tolerance is unachievable. Intuitively, this can be explained by viewing fault tolerant recovery as a method for maintaining the error "temperature" of the computational state by transferring error information to the cooler states such as the prepared even states. Measurement is useful for accessing the error information and removing its entropy and for input/output.

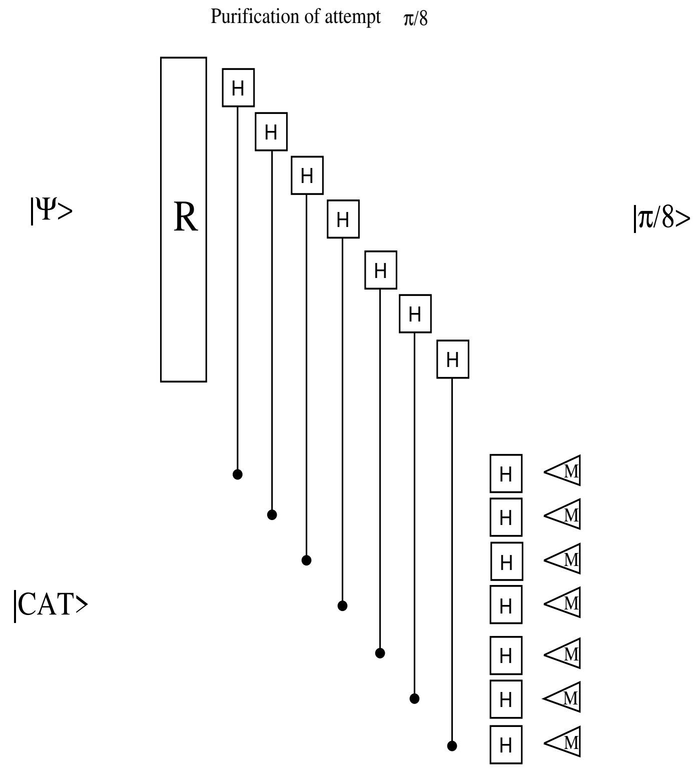


Figure 6. The first part of this picture describes how to rotate any state by $\pi/8$. The second part gives a realization of a control Hadamard gate and finally we show a fault tolerant network giving purified $|\pi/8\rangle$ states using the elements given above.

(*f*) *Concatenation*

The final ingredient required to implement fault tolerance with a threshold error rate is concatenation. The combination of quantum error-correction, fault tolerant error-correction and encoded operations can be viewed as a technique for exploiting qubits on which we can operate with an error p per operation to define abstract (encoded) qubits with a smaller error per operation. With the implementations described above, the effective error probability or strength is reduced (for sufficiently small p) from p to cp^2, where c is to be determined below. Concatenation involves applying the same combination of techniques using the abstract qubits as the starting two state systems for encoding. Figure 7 shows the networks obtained by applying concatenation twice. We shall see that the effective error on the encoded state after h iterations of the concatenation procedure is reduced to $c^{2^h-1}p^{2^h}$.

3. Analysis

There is a simple method for estimating accuracy thresholds associated with concatenation. In many cases, the threshold is obtained by exploiting "single-error elimination" networks, which have the property that if an error occurs at an error location with no other errors present in some region of influence, then this error

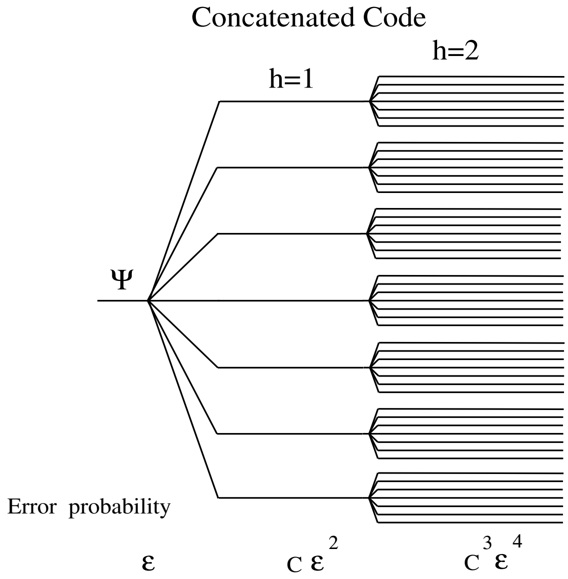


Figure 7. Concatenation of the 7 bit code. If the error rate is ϵ for the qubits, the encoding will gives a rate of $C^{2^h-1}\epsilon^{2^h}$ for the h^{th} level of the hierarchy.

will not affect the computational state. A naive estimate for the probability of error at the next level is given by the number of pairs of errors that can occur within the region of influence times the square of the probability of error at this level. In networks such as those used in this work, the number of error locations in the region of influence is bounded by the number of qubits required for the largest gates (here this is the controlled-not, with 14 qubits), times the number of operations affecting a qubit before an error at a given location can be eliminated (here this is estimated as $7*6*2+2$, where 7 is (close to) the average number of operations contributing to a qubit in the cat state generator, 6 is the number of syndrome bits, 2 is the average number of attempts to measure the syndrome, and the additional 2 is for the correction step and the encoded operation). For our network we get $14*86 = 1204$ operations, which yields a probability of error at the next level of less than 10^6p^2, which is less than p if $p < 10^{-6}$. This is close to what will be established by the more formal arguments below.

Our analysis consists of establishing the error behavior inductively for each level of the concatenation hierarchy. To do so we exploit the properties of stabilizer codes and the quasi-independent error models. We first prove the threshold theorem for the normalizer group and the quasi-independent stochastic error model. We then show how it can be extended the monotone model and to a complete set of operations.

Useful properties of stabilizer codes are:
(i) Recovery operations and encoded normalizer operations are implemented using normalizer operations only.
(ii) The syndrome is completely determined by the standard error operations applied to the qubits.
(iii) If the errors in different error locations during fault tolerant recovery are

$$
\begin{aligned}
SN &= -NS \\
SH &= HN \\
NH &= HS \\
SS_i &= S_iS \\
NS_i &= -iS_iN \\
(I \otimes S)N_2 &= N_2(S \otimes S) \\
(S \otimes I)N_2 &= -N_2(S \otimes I) \\
(I \otimes N)N_2 &= N_2(I \otimes N) \\
(N \otimes I)N_2 &= N_2(N \otimes N)
\end{aligned}
$$

Table 2. *Error propagation identities obtained by conjugation ($EV = V(V^\dagger EV)$).*

instantiated by standard errors and the incoming state's syndrome is given, then the outgoing state's syndrome is determined.
(iv) Standard errors applied to an encoded state whose syndrome is given induce a standard error on the encoded state.

(*a*) *Error propagation*

Recall that an instance of the error behavior of a computation is obtained by specifying one of the standard errors at each error location. The analysis of errors at the next level requires determining the effect of these standard errors on the state of the syndrome between gates. This is accomplished by moving errors from locations internal to the encoded gates to locations between gates by error propagation. The basic technique is to conjugate standard errors through gates as illustrated in Figure 2. A complete list of conjugation relationships for standard errors and normalizer operations is given in Table 2.

It is not possible to propagate errors through measurement operations. Note however that sign flip errors do not affect the measurement outcome, while bit flip errors flip the classical outcome. Given the syndrome before a recovery operation and the standard errors at error locations in the recovery operation, the classical parities that are obtained by the measurements are determined and the effect of the recovery on the state can be computed. In this case all errors in the recovery network can be replaced by errors before and after the recovery network.

Given the standard errors at all error locations, the syndromes of the encoded states between gates can be determined by first computing them for the encoded state preparations and then applying the above property of the recovery network to each gate whenever the input syndromes have been obtained. In principle this induces an error behavior at the next level which can then be analyzed without further reference to the syndromes at this level. Unfortunately, this simple approach does not easily permit computing the relevant error strengths at the next level. To do that we use additional error propagation steps to assure quasi-independent behavior.

(*b*) *Encoded gate failures*

For the purposes of analysis, we refine our error models so that each summand of the error expansion is also associated with a labeling of each gate or memory operation which declares whether or not that operation *failed.* The labeling is assumed to satisfy that any operation whose associated error-location has a non-identity operator has failed, but the converse need not hold. The induction hypothesis requires that the error model (quasi-independent stochastic or monotone) applies to the summands associated with failures at any given k error locations. Each of the error models defined earlier permits a failure labeling such that this holds. The goal of the analysis is to show that the hypothesis holds at the next level for a quadratically smaller error parameter.

Recall that an encoded gate consists of a (extended, for preparation gates) recovery network for each input encoded state and a transversally encoded operation. Our algorithm for determining whether an encoded gate failed and establishing the induced behavior with errors depends only on the failure locations at the current level, not on the actual standard error associated with the location. This implies that the analysis is correct provided that the failure events behave stochastically. Different standard errors associated with the same failure patterns can combine non-stochastically. In principle this permits a similar analysis to be used for situations where the standard errors do not commute with the network operations.

The first step is to determine for each recovery used in a state preparation or preceding a gate, whether it was successful. It is considered successful if after a suitable error propagation step, the correct syndrome has been obtained. Recall that each recovery network consists of two recovery attempts, where the second one is executed only if the first one has failed due to an inconsistency in the measured parities. Each attempt involves two extractions of the syndrome using prepared cat states. The following can be established by a careful analysis of the recovery network.

(i) If no failure occurs in the first recovery attempt, then the correct syndrome has been obtained.

(ii) If only one failure occurs in the first attempt then after forward or backward propagation of the errors, either the correct syndrome has been learned or the extracted parities are inconsistent.

(iii) If the extracted parities in the first attempt are inconsistent, then (i) or (ii) holds for the second pair.

This allows us to label the recovery as having failed if at least one failure occurred in the network for the first attempt and at least two failures occurred in the complete recovery network. (The second (conditional) attempts' failures have no effect if they are not actually executed.) For the purposes of getting a tighter bound on the threshold, we use the following definitions of failure which can be verified to be necessary for the recovery network to identify an incorrect syndrome or to not identify a syndrome at all.

Definition: A syndrome bit extraction has failed if it contains a failure in a location other than the second cat state preparation attempt. A recovery network has failed if it contains at least two failures of syndrome bit extractions satisfying: The two failures are in different halves of the first recovery attempt, or they are in different recovery attempts.

Errors associated with a successful recovery are now propagated to the previous or the following encoded gate as needed to have the successful recovery look like an error-free recovery attempt.

Any encoded gate for which one of the associated recovery networks has failed, is considered a failed gate. After having determined which gates failed because of a failed recovery, we must determine which gates failed because of having introduced more than one error *after* the recovery networks. These are errors which may have arisen during the step which restores the syndrome to 0, or during the encoded operation, or have been propagated from a successful recovery or from a subsequent successful gate.

Definition: An encoded gate has failed if one of the associated recoveries failed, or they succeeded and there are at least two failures among the syndrome bit extractions of the second syndrome measurement of one of its first recovery attempts, among the operations involved in restoring the syndrome to 0, in the encoded operation itself, and in the first recovery attempt of a following successful gate.

Since determining whether a gate failed may require knowing the following gate's status, this definition requires determining failure backwards in time so that when a given gate is considered, the following gate's status is indeed known. Simpler definitions with the property that failure is determined by considering errors only in a small neighborhood of an encoded gate are possible but lead to somewhat worse threshold estimates.

There are two properties of the failure declaration that will be required for the analysis. The first guarantees that if after propagation of errors there is an induced error in the encoded gate, the gate is determined to have failed. The second ensures that the failed gates can be associated with disjoint pairs of error locations which, according to the definition, contributed to the declaration of failure.

(c) *Thresholds for the normalizer group*

We begin with the analysis for the stochastic error model and then explain how it generalizes to the monotonic non-stochastic model. The analysis exploits the subadditivity property of probabilities over unions of events to estimate the probability that a given k gates at the next level are considered failed. To do so requires calculating the number of possible minimal sets of error locations that can cause a given gate to fail. In our case, these sets are pairs of locations. Suppose that the total number of such pairs for a gate is f. By assumption, the probability for a given such pair to have failed is bounded by p^2, where p is the failure probability at the current level. Thus the probability that the next level gate under consideration is declared a failure is bounded by fp^2. Because of the second property of the failure definition, the failure of k next level gates requires $2k$ failures at this level, with k disjoint pairs of these failures contributing to the failure of each of the k gates. Thus the probability of failure of these gates is bounded by $(fp^2)^k$, so the new failure probability for the stochastic error model is bounded by fp^2.

It remains to determine the number f of minimal pairs which can cause a gate to fail. The actual value of f depends to some extent on the gate. We will obtain an upper bound, and consider memory error locations separately from other locations. Excluding the error locations in the second cat state preparation attempts, the number of error locations in the network for extracting a bit of the syndrome is $(19, 10)$ operational and memory error locations, respectively. In the case where memory errors are significant, we assume that all operations are efficiently pipelined and that conditionally executed recovery attempts do not cause excessive delays in other operations. Thus we get a total of $(6 * 19 + 7, 6 *$

$10) = (121, 60)$ locations in a single syndrome extraction network (there are 6 syndrome bits to compute and one encoded Hadamard transform is used in each extraction attempt to switch between the $|0\rangle/|1\rangle$ and the $|+\rangle/|-\rangle$ basis). The number of pairs which can lead to failure of a recovery network is therefore given by $121^2 + (2 * 121)^2 = 73205$ and $181 + (2 * 181)^2 = 163805$ with and without memory locations taken into account, respectively (obtained by counting pairs of locations with one in the first and the other in the second syndrome extraction attempt, and those with the one in the first and the other in the second recovery attempt). For state preparation, this increases to 98000 and 220500, respectively (because of the additional syndrome bit required for each syndrome extraction). For two qubit operations there are two recovery networks, which gives $2 * 73205$ and $2*163805$ minimal failure sets. The number of pairs of error locations that can contribute toward gate failure if the recovery networks succeed can be computed similarly. There are up to $14 + 7 = 21$ error locations associated with the encoded operation and the step which resets the syndrome to 0. (The maximum occurs for the controlled-not, with 7 operations in the encoded controlled-not and up to 14 required to cancel errors determined by the recovery network). By removing error pairs in the following recoveries which would cause those to fail, we get $\binom{6*121+21}{2} - 6 * 121^2 = 190785$ and $\binom{6*181+21}{2} - 6 * 181^2 = 415605$ for two-qubit gates. (The pair has to come from the second syndrome extraction attempt of the gate's recovery networks, from the error-correction, the encoded operation or the first recovery attempt of a subsequent gate. Those pairs which arise from the subsequent recoveries and cause those to fail have been subtracted, as well as those which introduce only one error in each qubyte.). Other gates have fewer such error pairs. By adding the values for the two qubit gates, we get a bound of $f \leq 337195$ and $f \leq 743215$, respectively. This gives threshold error bounds of about $3.0\ 10^{-6}$ and $1.3\ 10^{-6}$, respectively.

The monotonic error model requires that for some C and error parameter p, any summand of those error operators contributing to the failure of a given k gates has strength at most Cp^k. Because of this assumption, the stochastic analysis above can be used almost verbatim for this model. This is based on the observation that all the estimates given there are based on bounding the probability of a set of error events by representing it as a union of events, each of which fits the requirements for satisfying a bound of the form Cp^k. Thus, the strength of the sum of events which contributes to the failure of a given k next level gates is bounded by $C(fp^2)^k$. It can be seen that the requisite monotonicity condition is also preserved at the next level. Consequently, the bound for the threshold has the same value, but for strength rather than probabilities. The quantity C only affects the overhead required to achieve a desired accuracy of the computation at the uppermost level.

(*d*) *Extension to a complete set of operations*

To obtain a complete set of operations it suffices adding the encoded $|\pi/8\rangle$ preparation gate where needed at the uppermost (computational) level. To obtain the threshold requirements for this gate we bound its error given the error behavior of the $|\pi/8\rangle$ gates at the previous level. The $|\pi/8\rangle$ preparation step fails if one of its two recovery networks fail, or if there are at least two errors introduced into the encoded state by propagation, potentially from a subsequent successful gate. Conservatively, failure not attributable to the two recovery networks can be reduced to errors in the first attempts at generating the two cat states for the controlled encoded Hadamard transforms, the implementation of the controlled Hadamard transforms (including it's two required $|\pi/8\rangle$ states), the second syndrome extraction in the first attempts at recovery in each recovery network, or the first attempt at recovery in the subsequent gate. The computations are similar to the ones given previously. The number of pairs of errors in this set not leading to failure of a recovery network is 313894 and 838831, with memory errors taken into account. The number of pairs of errors leading to failure of one of the two recovery networks is 146410 and 327610, respectively. Adding these leads to slightly worse thresholds of $2.2\ 10^{-6}$ and $0.9\ 10^{-6}$, respectively.

(*e*) *Leakage errors*

The fault tolerant networks analyzed above are not guaranteed to be able to suppress leakage errors. To do that one can use a stop leak gate for qubits which (in the absence of noise) has the property that a state in the computational space is untouched, while any amplitude outside this space is irreversibly returned to the computation. Such stop leak gates must be inserted before each attempt at extracting the syndrome bits at the physical level, thus introducing a number of additional error locations that must be accounted for. Using these gates at the physical level changes the interpretation of an encoded qubit and the behavior of the encoded gates. The abstract particle definition of the encoded qubit is modified so that single leakage events do not destroy the encoded information. The encoded gates act correctly on states where at most one supporting qubit has lost its amplitude. If more qubits lose amplitude, that constitutes a leakage event at the next level. However, stop leak measures are no longer explicitly needed, since each gate's action has been extended so that (in the absence of an excess of internal errors after error propagation) it also returns the amplitude to the encoded computational space. Note that although error propagation cannot be handled quite as algebraically as is possible with the standard error group, it is still possible to move leakage errors to other locations, but with the new failure operator no longer explicitly determined. Alternatively, one can linearly represent all possible operators using standard errors for higher dimensional spaces. One such approach involves artificially splitting an extension of the full Hilbert space available to a physical qubit into $Q \otimes L$ such that $Q \otimes |0\rangle$ is the computationally useful space. This is akin to the abstract particle representation of the encoded information, with L playing the role of the syndrome space. The stop leak gate behaves like a 0-error correcting recovery operator.

(*f*) *Overheads*

Since we have made no attempt at optimizing the overheads involved with coding and concatenation we only provide asymptotic estimates for the method described. The implementation of each encoded gate requires a constant amount of resources at the previous level. Let K be a bound on the total number of qubits and steps at the previous level per encoded gate. The physical resources required for a computational gate is bounded by K^h, where h is the number of levels in the concatenation. If the number of computational gates required is n, and the desired probability of failure of the computation is q, then the number of levels is sufficient provided that $n(fp)^{2^h} < q$, where p is the physical probability of failure.

This gives $h > \log_2(\log_{1/(fp)}(n/q))$ and an overhead per computational operation of at least $K^h > (\log_2(n/q)/\log_2(1/(fp)))^{\log_2(K)}$, which is polylogarithmic in n and $1/q$. With the assumption that measured qubits can be freely reused this is also a bound on the number of qubits required to support each computational qubit. Otherwise, this overhead can increase by an additional factor related to the the number of gates the qubit is involved in. In our case $\log_2(K)$ is about 10, which can be quite daunting in practice. Substantial asymptotic improvements can be obtained by using different codes at higher levels and by encoding multiple qubits using single blocks.

4. Conclusion

We have demonstrated that quantum computation with classical input can be performed arbitrarily accurately provided that the noise per operation is sufficiently small and satisfies suitable independence assumptions. The implementation of fault tolerant quantum computation is straightforward. Since the overheads are asymptotically well behaved, the threshold results demonstrate that quantum computation is possible in the presence of physically reasonable sources of noise.

Threshold results have been obtained independently by Kitaev (1996) and Aharonov (1996). Both Kitaev and Aharonov & Ben-Or analyze independent stochastic error models and obtain completeness of operations by adopting Shor's implementation of the Toffoli gate. Kitaev uses a different method for fault tolerantly extracting the syndrome. His method is less efficient and consequently yields substantially worse thresholds. Aharonov and Ben-Or provide an analysis which does not require accurate classical computation for syndrome calculations and estimate a threshold of around 10^{-6} for the independent stochastic error model. There may be some cases where this extension is needed, for example when performing stochastically parallel quantum computations such as those envisioned for NMR quantum computation (Cory 1996, Gerschenfeld 1997). Our techniques for generalizing arguments from stochastic error models to coherent error models can in principle be used to extend their analysis.

Not only do the threshold theorems show that quantum computation is possible in principle, but they demonstrate that the apparent distance limitations of quantum cryptography can be overcome. It suffices to be able to transmit the state of qubits over some reasonable distance before recovery operations must be applied to avoid loss of encoded information. This does of course require compatible physical realizations of qubits for transmission over channels and for manipulation in a quantum computer.

The actual values of the thresholds we have obtained are rigorous, but overly pessimistic in several ways. First, the error models used are the most adversarial still satisfying independence assumptions. We assume that the error behavior at each location is the worst possible for the network subject only to bounds on the strength of correlations. In practice, we need not worry about such adversarial error behavior and the actual error types at the physical level are likely to be much more constrained than assumed by our error-blind analysis. That known error behavior can be exploited to reduce error has been demonstrated in a specific example by Pellizzari, Cirac, Pellizzari and Zoller (Cirac 1996). In addition, simulations due to Preskill's team (private communication) and Zalka (1996) suggest that for the depolarizing channel, thresholds are substantially better than suggested by our calculations. Second, we have made no attempt to optimize the implementation of fault tolerance. Suggestions for optimization can be found in Knill (1996d) and Zalka (1996). Finally, it is clear that our failure definitions are excessively conservative, and a substantial fraction of error pairs which lead us to consider a gate failed in fact do not induce an error at the next level. Nevertheless, the results suggest that over-rotation and other errors not representable stochastically must be controlled more carefully than stochastic noise.

Whether fault tolerant quantum computation can be implemented in practice remains to be seen. However, the results obtained here show that in principle noise of a level below the error threshold is not an obstacle for quantum computation.

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