## Summer Teach-in Cosmology: 2 George F. Smoot

Cosmological Lecture 2 includes:

- Potential Cosmology Science
- Review of Eric Linder's Main Points
  - Isotropy and Homogeneity
  - plus Riemannian Geometry yields Roberston-Walker Metric
  - Perturbations: symmetry yields:
    - Scalar - energy density perturbations
    - Vector - vorticity and shear
    - Tensor - gravitational waves
  - Need for Inflation - or special conditions
    - Flatness - attractor vs. repulsor
    - Horizon
    - Garbage - Solution to Pollution is Dilution
    - Perturbations
  - Kinematics/Dynamics
    - Scale Factor depends upon time
    - Contents of Universe determine Scale Factor
      - Friedmann Equations - Newtonian Physics
      - Equation of State & Evolution
- Comological Models and Tests
- History of the Universe

## The Geometry of the Universe

Geometry of Universe is that of a 4-D Riemannian space.

The differential distance is then the generalize Pythagorean

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu} dx^\mu dx^\nu \equiv g_{\mu\nu} dx^\mu dx^\nu$$

*The Cosmological Principle* the assumption that on sufficiently large scales the Universe is homogeneous and isotropic.

- 1967: J. Ehlers, P Geren, R.K. Sachs (J. Math. Physics 9, 1344) proved if all observers see an isotropic background radiation, then the metric is Robertson-Walker.
- 1994: W. Stoeger, R. Maartens, G.F.R. Ellis proved the almost theorem.
- 1992-6: COBE DMR shows that CMB isotropic to part in $10^5$.
- Thus: we can then approximate the metric for the universe as:

$$g_{\mu\nu} = (g^{RW})_{\mu\nu} + h_{\mu\nu}$$

where $g^{RW}$ is the background Robertson-Walker metric and $h_{\mu\nu}$ are the small deviations.

## The Choice of Riemannian Geometry

For Riemannian Geometry the invariant differential distance is given by

$$ds^2 = \sum_{\mu\nu} g_{\mu\nu} dx^\mu dx^\nu \equiv g_{\mu\nu} dx^\mu dx^\nu$$

"On the Facts Which lie at the Foundations of Geometry," Helmholtz 1868

- Continuous and first derivative
- Homogeneous in first degree in $dx_i$; (Finsler metric)
  - $ds = F(x_i, dx_i)$
  - $F(x_i, \lambda dx_i) = \lambda F(x_i, dx_i) \qquad \lambda > 0$
- Small rigid body can be rotated freely about a fixed point
  - Metric function in which three-parameter group of transformations are possible
  - This transformation invariance requirement leads to Riemannian metric

## Perturbations - Aside

*The Synchronous Gauge* This gauge confines any perturbations from Minkowski (or Robertson-Walker) spacetime to the spatial part of the metric:

$$g^{0\mu} = (1, 0, 0, 0)$$

(or $ds^2 = c^2 d\tau^2 - a(t)^2 \left[dr^2 + S_k^2 d\Omega^2\right] + h_{ij} dx^i dx^j$ where $i$ and $j$ range from 1 to 3 rather than from 0 to 3. This gauge is commonly used in the study of cosmological perturbations.

*The Newtonian Gauge* The deviation of the metric are expressed in terms of a function that looks like the Newtonian potential, $\Phi$. The metric perturbation is purely diagonal
$h^{\mu\nu} = h$ diagonal(1,1,1,1):

$$c^2 d\tau^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right)(dx^2 + dy^2 + dz^2). \qquad (1)$$

The Newtonian gauge does not allow gravitational waves, but this is the correct choice of metric for weak gravitational fields in a Minkowski background.

## Perturbations - Aside

*Complete Expansion* Since $g_{\mu\nu}$ is symmetric in 3+1 dimension there are 10 independent degrees of freedom in the metric: $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. A convenient scheme that captures these possibilities is to write the cosmological metric as

$$c^2 d\tau^2 = a^2(\eta)\{(1+2\phi/c^2)d\eta^2+2w_i d\eta dx^i-\left[(1-2\psi/c^2)\gamma_{ij}+2h_{ij}\right]dx^i dx^j\} \tag{2}$$

where $\eta$ is the conformal time, and $\gamma_{ij}$ is the comoving spatial part of the Robertson-Walker metric. The total number of degrees of freedom is 2 (scalar fields $\phi$ and $\psi$), 3 vector fields ($w_i$) and 6 (the symmetric 3-tensor $h_{ij}$ which totals 11. To obtain the right number of 10, the tensor $h_{ij}$ is required to be traceless, $\gamma^{ij}h_{ij} = 0$. Thus the perturbations can thus be split into three classes: **scalar perturbations**, which are described by scalar functions of spacetime coordinates and which correspond to the growing density perturbations, **vector perturbations**, which correspond to vorticity perturbations, and **tensor perturbations**, which correspond to gravitational waves.

## The Roberston-Walker Metric

The Robertson-Walker metric is the metric for a 3+1 dimensional space that is isotropic and homogeneous (maximally symmetric).

Can derive easily from assumption of constant spatial curvature.

Two-dimensional example is sphere Embed a three dimensional sphere in a 4-D Euclidean space

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = R^2$$

$$d\ell^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

$$d\ell^2 = dx_1^2 + dx_2^2 + dx_3^2 + \frac{(x_1 dx_1 + x_2 dx_2 + x_3 dx_3)^2}{R^2 - x_1^2 - x_2^2 - x_3^2}$$

$$d\ell^2 = dx_1^2 + dx_2^2 + dx_3^2 + \frac{(x_1 dx_1 + x_2 dx_2 + x_3 dx_3)^2}{R^2 - r^2}$$

Set into polar coordinates

$$x_1 = rsin\theta cos\phi \qquad x_2 = rsin\theta sin\phi \qquad x_3 = rcos\theta$$

$$d\ell^2 = \frac{dr^2}{1 - r^2/R^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2$$

Robertson-Walker metric form 1

$$(cd\tau)^2 = (cdt)^2 - \frac{dr^2}{1 - r^2/R^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2$$

## The Roberston-Walker Metric

A form that is manifestly isotropic and shows that the comoving coordinates are fixed for fundamental observers (those that see an isotropic universe as moving observers have aberration) comes from generalizing polar coordinates to 4 dimensions Set into polar coordinates

$$x_1 = Rsin\chi sin\theta cos\phi \qquad x_2 = Rsin\chi sin\theta sin\phi \qquad x_3 = Rsin\chi cos\theta$$

$$x_4 = Rcos\chi \qquad dx_4^2 = R^2 sin^2\chi d^2\chi$$

$$dx_1^2 + dx_2^2 + dx_3^2 = R^2 cos^2\chi d\chi^2 + sin^2\chi \left[d\theta^2 + sin^2\theta d\phi^2\right]$$

$$d\ell^2 = R^2 \left[d\chi^2 + \{ \begin{array}{c} sin^2\chi \\ \chi^2 \\ sinh^2\chi \end{array} \} \left(d\theta^2 + sin^2\theta d\phi^2\right)\right]$$

The Robertson-Walker form

$$(cd\tau)^2 = (cdt)^2 - R(t)^2 \left[d\chi^2 + \{ \begin{array}{c} sin^2\chi \\ \chi^2 \\ sinh^2\chi \end{array} \} \left(d\theta^2 + sin^2\theta d\phi^2\right)\right]$$

## Conformal Time

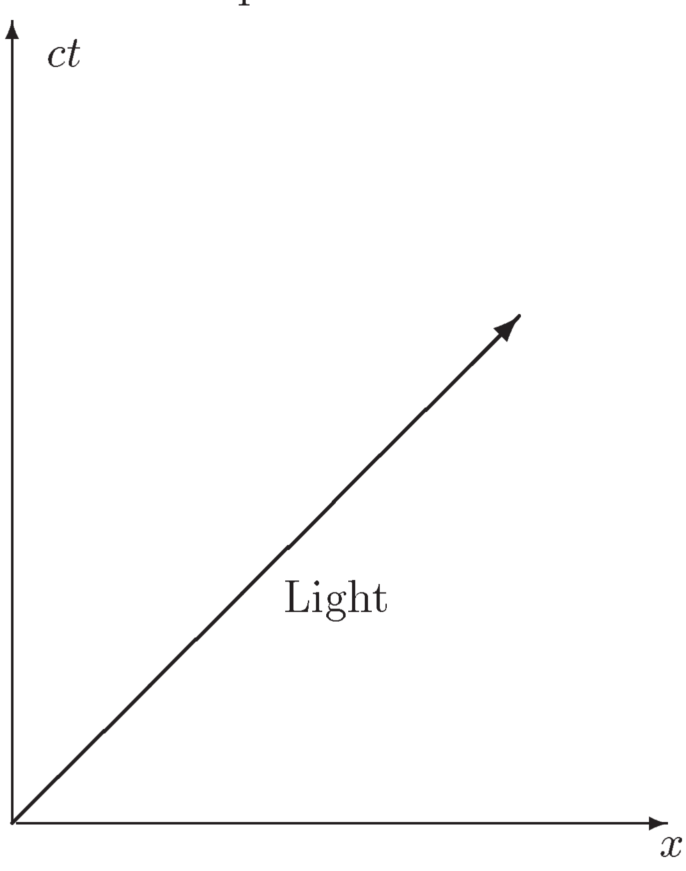
Define conformal time: $\eta$

$$d\eta \equiv cdt/R(t) \qquad cdt = R(t)d\eta$$

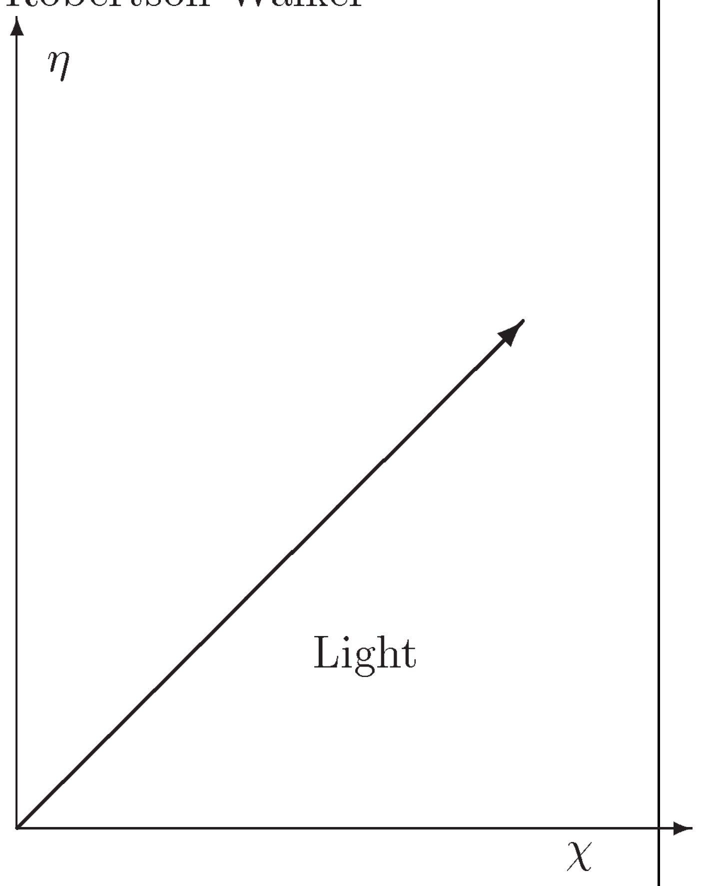
Now light travels on $45°$ in the $\chi$ and $\eta$ plane just like Minkowski ($x$ and $ct$).

$$(cd\tau)^2 = R(\eta)^2 \left[ d\eta^2 - d\chi^2 - \left\{ \begin{matrix} sin^2\chi \\ \chi^2 \\ sinh^2\chi \end{matrix} \right\} \left( d\theta^2 + sin^2\theta d\phi^2 \right) \right]$$

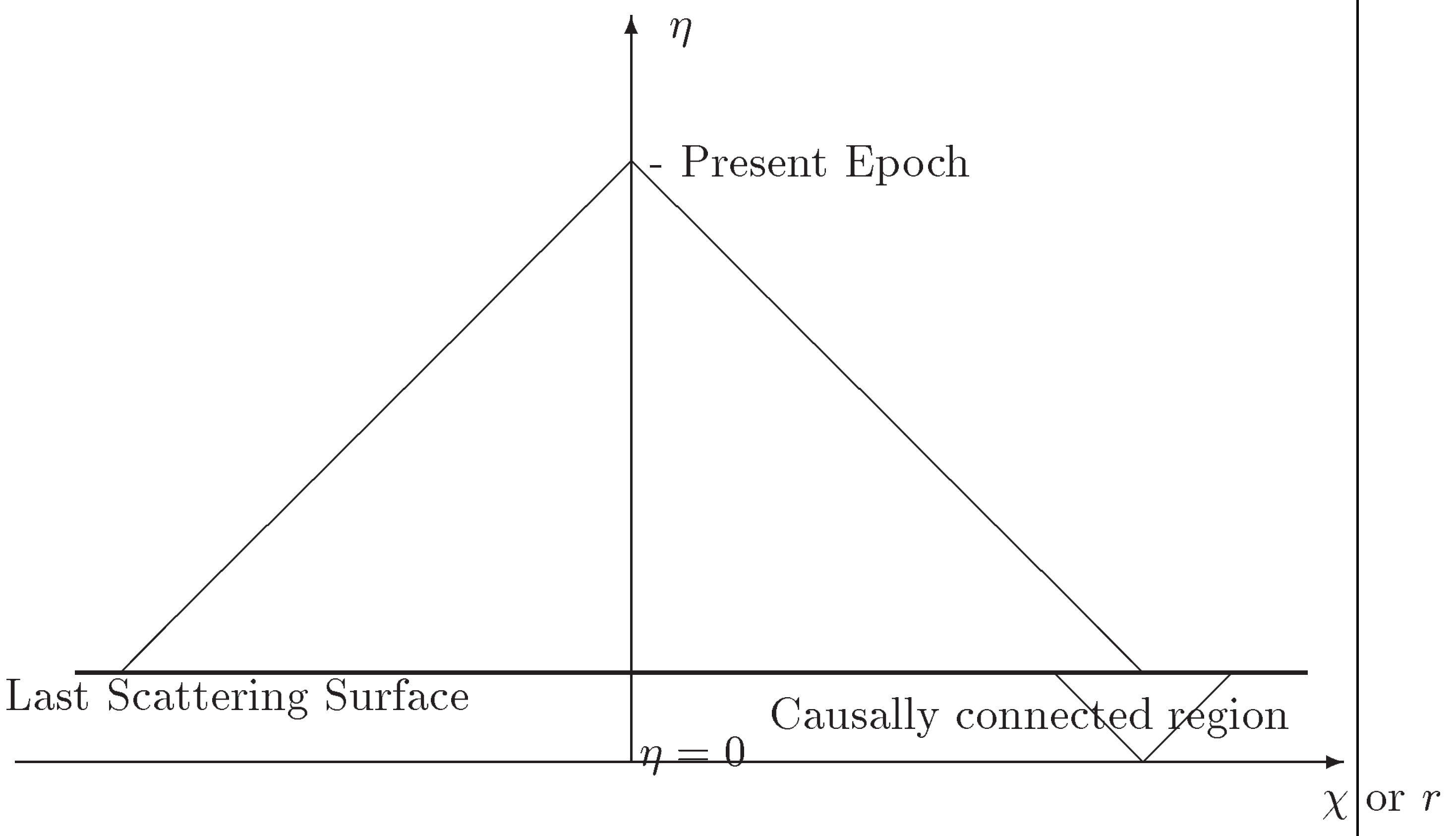
Minkowski Space

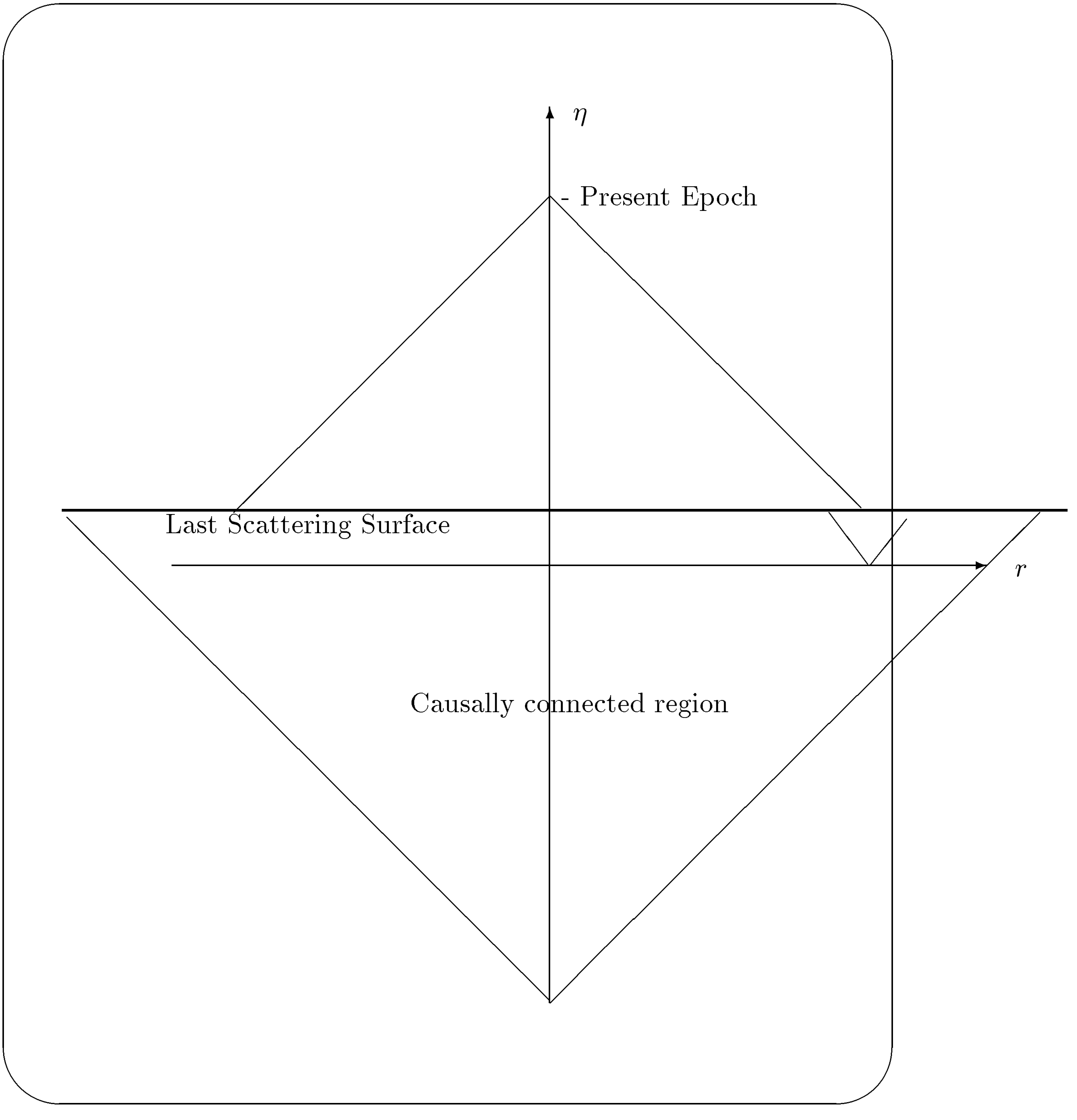



Robertson-Walker



## Causality and the Horizon Problem

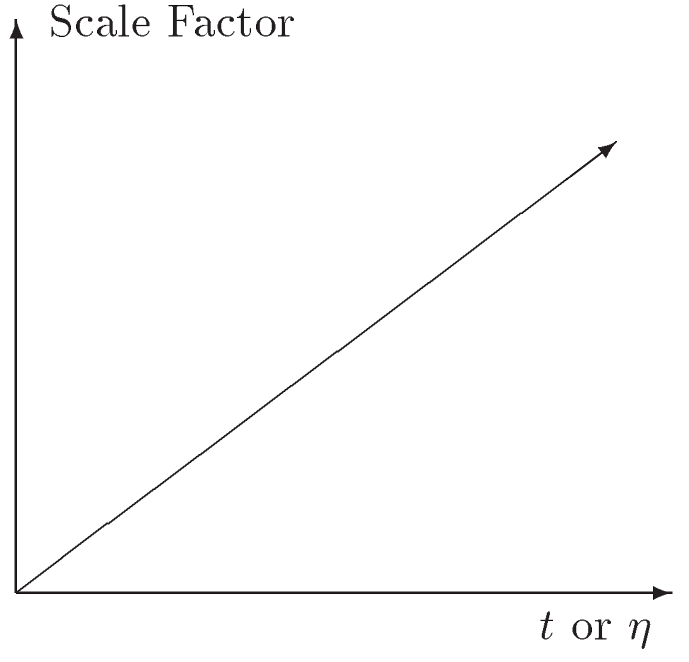


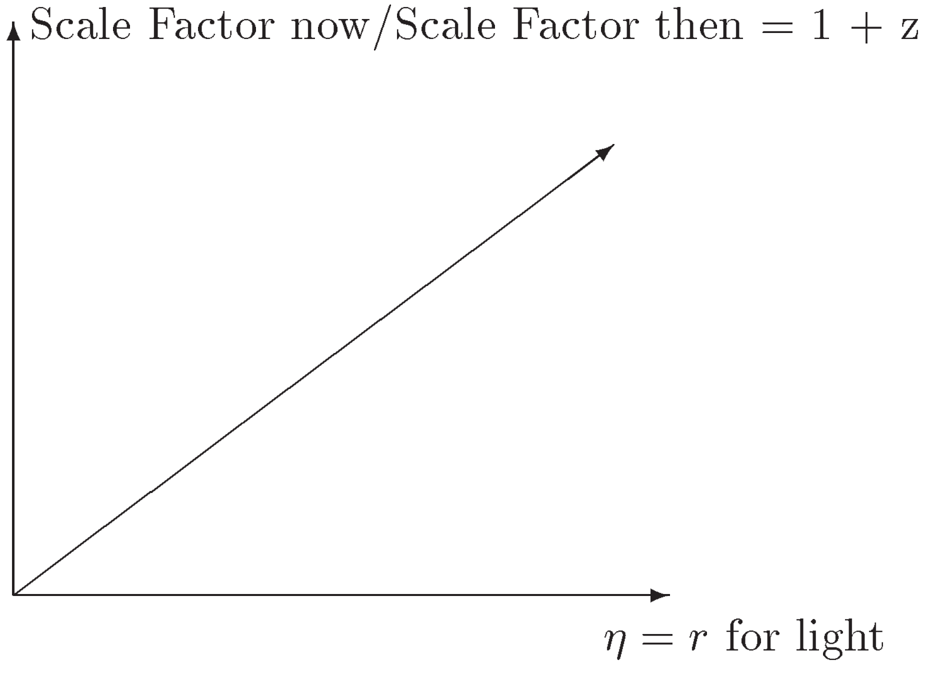


$\eta$
- Present Epoch
Last Scattering Surface
$r$
Causally connected region

## The Kinematics - $R(t)$ or $a(\eta)$

The kinematics of the large scale Universe is the Robertson-Walker metric scale factor time behavior.



## Observed Magnitude-Red Shift Relation

Can one determine the function $R(t)$ or $a(\eta)$ of the Robertson-Walker metric from observing the magnitude versus redshift?

Redshift - $z$ Expansion of the Universe stretches wavelengths

$$1 + z \equiv \frac{\lambda_0}{\lambda_1} = \frac{R(t_0)}{R(t_1)}$$

Luminosity Distance

$$d_L^2 \equiv \frac{L}{4\pi F} = R(t_0)^2 r^2 (1+z)^2$$

$$d_L = R(t_0) r (1+z) = d_{proper}(1+z)$$

where $L$ is the luminosity and $F$ is the flux.

The last equality can be understood as the result of factors:

- Fraction of the sphere covered by detector is $dA/4\pi R(t_0)^2 r^2$
- Expansion decreases the energy crossing the spherical surface per unit time by $(1+z)^2$; one factor of $(1+z)$ arises from the energy due to red shift and another factor of $(1+z)$ arising from the increased time interval.

Thus red shift versus luminosity density is equivalent to relative scale sizes $R(t_0)/R(t_1)$ versus distance which is equivalent to time.

## Kinematics/Dynamics a look ahead

We estimate that the Universe has a moderately complicated $R(t)$ ($a(\eta)$) history:

| $Epoch(Dominator)$ | $R(t)$ | $a(\eta)$ |
|---|---|---|
| $Stuff$ | $R(t) \propto t^{2/3(1+w)}$ | $a(\eta) \propto \eta^{2/(1+3w)}$ |
| $Pre-Inflationary$ | ? | ? |
| $Inflationary$ | $R(t) \propto e^{Ht}$ | $a(\eta) \propto -\eta^{-1}$ |
| $Radiation$ | $R(t) \propto t^{1/2}$ | $a(\eta) \propto \eta$ |
| $Matter$ | $R(t) \propto t^{2/3}$ | $a(\eta) \propto \eta^2$ |
| $DarkEnergy$ | $R(t) \propto t^2$ to $e^{H_{DE}t}$ | $a(\eta) \propto -\eta^{-2}$ to $-\eta^{-1}$ |
| | | |
| $curvature$ | $R(t) \propto t$ | $a \propto e^{\eta}$ |

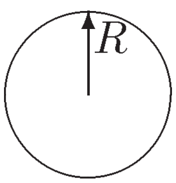
where $w = p/\rho$ is from the equation of state.

## The Dynamics - $R(t)$ or $a(t)$

The dynamics of the large scale Universe is the physics that determine the Robertson-Walker metric scale factor time behavior. These are usually expressed in terms of the Friedmann Equations.

Can determine these correctly by working only in the Newtonian limit.

Consider a uniform distribution of matter (and energy) and focus on the distance $R$ between a test mass (object) and a randomly chosen origin.




We can ignore the gravitational effect of material outside of the spherical cavity of radius $R$ (Newton proved and Birchoff for GR). The Newtonian acceleration of the test particle due to the gravitational attraction of the matter is

$$\frac{d^2R}{dt^2} = -\frac{GM}{R^2} = -\frac{4\pi}{3}G\rho R$$

where $M$ is the mass included in radius $R$ and the last equality holds when the mass is sufficiently uniformly spread that we can treat it as a constant density $\rho$.

Multiply through by $\dot{R} \equiv dR/dt$ to get

$$\dot{R}\frac{d^2R}{dt^2} = -\frac{GM}{R^2}\dot{R}$$

$$\frac{d}{dt}(\frac{1}{2}\dot{R}^2) = -\frac{d}{dt}(\frac{GM}{R})$$

or equivalently

$$\frac{d}{dt}[\frac{1}{2}(\dot{R}^2 + K)] = \frac{d}{dt}(\frac{GM}{R})$$

where $K$ is a constant of integration. (Later we will see that $K$ is the curvature of the universe and it is also equal $-2E/m =$ minus twice the fractional binding energy of a particle.) Integrating this equation we obtain

$$\frac{1}{2}(\dot{R}^2 + K) = \frac{GM}{R}$$

or

$$\frac{1}{2}\dot{R}^2 - \frac{GM}{R} = -\frac{K}{2}$$

We could have found this same formula by writing down the equation for the total energy and dividing through by the mass.

$$\frac{1}{2}\dot{R}^2 - \frac{GM}{R} = \frac{E_{total}}{m}$$

So that the constant of integration is $K = -2E_{total}/m$.

Now we convert included mass to mean density to get an interesting formula:

$$\frac{1}{2}\dot{R}^2 - \frac{4\pi}{3}G\rho R^2 = -\frac{K}{2}$$

Divide this through by $R^2$ and multiply by 2 to find that

$$(\frac{\dot{R}}{R})^2 - \frac{8\pi}{3}G\rho = -\frac{K}{R^2}$$

We can also express this in terms of our comoving coordinates $R = a(t)r$ where for uniform expansion a la the Hubble law $r$ is a constant. $\dot{R} = \dot{a}r$ so that

$$\frac{\dot{R}}{R} = \frac{\dot{a}}{a(t)}$$

Thus we readily get the equation

$$\mathbf{(\frac{\dot{a}}{a})^2 - \frac{8\pi}{3}G\rho = -\frac{K}{a^2}}$$

This is the first Friedmann equation, if we use the Equivalence Principle and replace the mass density with the energy density $T_{00}$. We can find the value for $K$ by using the present epoch values where $H_0 \equiv \frac{\dot{R}}{R}$ is the current value for Hubble constant in the Hubble's law $v = \dot{R} = H_0 R$ and the present mean density $\rho_0$ of the Universe.

$$K = (H_0^2 - \frac{8\pi}{3}G\rho)R_0^2$$

which we can put back into the original equation

$$(\frac{\dot{R}}{R})^2 = \frac{8\pi}{3}G\rho - \frac{K}{R^2} = \frac{8\pi}{3}G\rho - \frac{8\pi}{3}G(\frac{R_0}{R})^2[\rho_0 - \frac{3}{8\pi}\frac{H_0^2}{G}]$$

If we define the current critical density as

$$\rho_{c0} = \frac{3}{8\pi}\frac{H_0^2}{G}$$

which is often called just $\rho_c$ dropping the zero subscript and is roughly

$$\rho_{c0} \cong 2 \times 10^{-29} h_{100}^2 \text{ g cm}^{-3}$$

where $h_{100} = H_0/100 kms^{-1}Mpc^{-1}$. Notice that $\rho_c$ is the density that is the dividing line between having the test particle escape or be bound to the central mass. Thus it marks the division between a universe that will expand forever and on that will slow down enough to turn around and collapse.

We also define the parameter $\Omega = \rho/\rho_c$ which is the ratio of the density of the Universe to the critical density. Now we can recast the formula as one for the expansion rate

$$(\frac{\dot{R}}{R})^2 = H^2 = H_0^2\Omega - H_0^2(\Omega_0 - 1)(\frac{R_0}{R})^2 = H_0^2\Omega_0 a^{-3} - H_0^2(\Omega_0 - 1)a^{-2}$$

The last equality holds for conserved massive particles where $\Omega R^3 = \Omega_0 R_0^3$ and we have set $a(0) = 1$, that is the present scale factor to comoving coordinates is one. Note that $R/R_0 = a/a_0$. In this case the equation reduces to the simple form

$$\dot{a}^2 = H_0^2\Omega a^2 - H_0^2(\Omega_0 - 1) = H_0^2\Omega_0/a - H_0^2(\Omega_0 - 1)$$

(where the last equality holds for the matter dominated case).

We pause a moment to reflect that if $\Omega_0 < 1$ then $\dot{a}^2 \rightarrow H_0^2(\Omega_0 - 1)$ and $a \propto t$. The universe will keep expanding forever.

If $\Omega_0 > 1$ then $\dot{a}^2 \rightarrow 0$ and then later and $< 0$. This means that the universe will stop expanding and collapse.

The rate at which the expansion is slowing down $d\dot{a}/dt$ is proportional to the density of the universe so that the time back to $a \cong 0$ is going to be a function of $\Omega_0$ and the scale is set by $H_0^{-1}$, i.e.

$$t_u = f(\Omega_0) H_0^{-1}$$

It is clear that if $\Omega_0 = 0$, then there is no decelleration and $f(0) = 1$ . We can integrate the equations easily, if $\Omega_0 = 1$, since for a matter dominated case reduces to

$$\dot{a}^2 = H_0^2 / a$$

or taking the square root

$$a^{1/2} \dot{a} = H_0$$

or

$$\frac{2}{3} a^{3/2} = H_0 t \quad \quad or \quad \quad t = \frac{2}{3} a^{3/2} H_0^{-1}$$

where the constant of integration is taken care of by defining the origin of time $t = 0$ as the time $a = 0$. Since we have defined $a = 1$ as the present, we are at $t_u = 3/2\ H_0^{-1}$. Thus $f(\Omega_0 = 1) = 2/3$ and thus $t_u(\Omega_0 = 1) \approx 6.7 \times 10^9 h_{100}^{-1}$ years. Note also that the scale factor for a matter dominated universe is proportional to $a \propto t^{2/3}$. (Actually $a = (3/2\ H_0 t)^{2/3}$.)

## The Dynamics continued

the internal energy $U$ of a gas of particles is $U = \rho V$. Thus

$$dU = -PdV = \rho dV + Vd\rho$$

so that

$$Vd\rho = -(\rho + P)dV$$

so we get the continuity equation

$$\dot{\rho} = -(\rho + P)\frac{\dot{V}}{V} = -3(\rho + P)\frac{\dot{a}}{a}$$

where the last equality comes from the relationship $V \propto a^3$. Now the Newtonian equation of motion has an additional term coming from the pressure which we can see from differentiating the energy conservation equation and using the continuity equation. First recall the equation

$$(\frac{\dot{a}}{a})^2 - \frac{8\pi}{3}G\rho = -\frac{K}{a^2}$$

multiply by $a^2$ and differentiate.

$$2\dot{a}\ddot{a} = 2\frac{8\pi}{3}G\rho a\dot{a} + \frac{8\pi}{3}G\dot{\rho}a^2$$

where $K$ is a constant so that the derivative is zero. Dividing through by $2\dot{a}$

$$\ddot{a} = \frac{8\pi}{3}G\rho a + \frac{4\pi}{3}G\dot{\rho}\frac{a^2}{\dot{a}}$$

The continuity equation $\dot{\rho} = -3(\rho + P)\frac{\dot{a}}{a}$ can be used to eliminate $\dot{\rho}$.

$$\ddot{a} = \frac{8\pi}{3}G\rho a - \frac{4\pi}{3}G3(\rho + P)a$$

gathering terms we get the other Friedmann equation

$$\mathbf{\ddot{a} = -\frac{4\pi}{3}G(\rho + 3P)a}$$

Note that if $\rho + 3P > 0$ is positive then the universe is decellerating - that is the rate of expansion is slowing.

## The Dynamics continued - Stuff

Consider a simple equation of state: $P = w\rho$. If $w =$ constant, i.e. independent of time, then we can use the internal energy relation

$$dU = d(\rho V) = -PdV \propto -Pd(a^3)$$

which implies

$$d[a^3(\rho + P)] = a^3 dP$$

putting in the equation of state

$$d[a^3 \rho(1 + w)] = a^3 w d\rho$$

from which we can conclude that

$$\rho \propto a^{-3(1+w)}$$

Solving for $a$ we have $a \propto t^{2/[3(1+w)]}$ or for $w = -1$, $a \propto e^{Ht}$.

| | | | | |
|---|---|---|---|---|
| $Stuff$ | $P = w\rho$ | $\rho \propto a^{-3(1+w)}$ | $a \propto t^{2/[3(1+w)]}$ | $t_0 = \frac{2}{3(1+w)} H_0^{-1}$ |
| $Radiation$ | $P = 1/3\ \rho$ | $\rho \propto a^{-4}$ | $a \propto t^{1/2}$ | $t_0 = \frac{1}{2} H_0^{-1}$ |
| $Matter$ | $P = 0$ | $\rho \propto a^{-3}$ | $a \propto t^{2/3}$ | $t_0 = \frac{2}{3} H_0^{-1}$ |
| $Curvature$ | $-1/3$ | $\rho \propto a^{-2}$ | $a \propto t$ | $t_0 = H_0^{-1}$ |
| $Vacuum\ Energy$ | $P = -\rho$ | $\rho = constant$ | $a(t) \propto e^{Ht}$ | $t_0 = \infty$ |

## The Dynamics continued Energy Conservation

$$\begin{aligned}
\text{Kinetic Energy} - \text{Potential Energy} &= \text{Constant} \\
\frac{1}{2}mv^2 - \frac{GMm}{R} &= E_{total} \\
\frac{1}{2}mv^2 - \frac{G4\pi\rho}{3}R^2 m &= E_{total} \\
v^2 - \frac{8\pi G}{3}\rho R^2 &= \frac{2E_{total}}{m} \\
H^2 R^2 - \frac{8\pi G}{3}\rho R^2 &= \frac{2E_{total}}{m} \\
H^2\left(1 - \frac{\rho}{(3H^2/8\pi G)}\right)R^2 &= \frac{2E_{total}}{m} \\
\mathbf{H^2\left(1 - \Omega\right)R^2} &= \mathbf{k} \qquad (3)
\end{aligned}$$

Another way to write this is

$$\begin{aligned}
H^2(1-\Omega) &= \frac{k}{R^2} \\
\Omega_k &\equiv \frac{k}{R^2} \\
H^2(1-\Omega-\Omega_k) &= 0 \\
\Omega + \Omega_k &= 1 \qquad (4)
\end{aligned}$$

## The Dynamics continued - Attractor

$$(\Omega(z) - 1)a(z)^2H(z)^2 = (\Omega_0 - 1)a_0^2H_0^2 = k$$
$$|\Omega - 1| \cong (\frac{a}{a_o})^{(1+3w)}$$

Thus for matter and radiation dominated universes $|\Omega - 1|$ is proportional to $(1 + z)^{-1}$ and $(1 + z)^{-2}$, respectively. For a cosmological constant $|\Omega - 1|$ is proportional to $(1 + z)^2$

**Going backwards in a matter or radiation dominated universe results in rapid approach to $\Omega = 1$. Going forward in an accelerating universe results in rapid approach to $\Omega = 1$.**

**$\rightarrow$ Need at least as many e-foldings of accelerating phase as decellerating phase to keep $\Omega \sim 1$.**

The General Relativity derivation of the Friedmann equations that the 3-D space curvature ${}^3R = 6k/a^2$ We now have the relation

$${}^3R = \frac{6k}{a^2} = 6H^2(\Omega - 1)$$

The Gaussian curvature ${}^3R = 1/R^2_{curvature}$, where $R_{curvature}$ is the 3-space radius of curvature. For the Universe

$$R_{curvature} = \frac{cH^{-1}}{|\Omega - 1|^{1/2}}; \qquad R_{curvature} = \frac{cH^{-1}}{(\Omega - 1 + \Lambda/3H^2)^{1/2}}$$

Clearly $\Omega = 1$ means space is flat.

## Solving the Horizon Problem

Another problem solved by Inflation (a sustained period of accelerating universe) is moving (postponing) curvature problems outside the horizon

Fit to simple power law expansion

$$\begin{aligned} a &= a_0 \left(\frac{t}{t_0}\right)^n \\ \frac{da}{dt} &= n a_0 \left(\frac{t}{t_0}\right)^{n-1} \\ \frac{d^2 a}{dt^2} &= n(n-1) a_0 \left(\frac{t}{t_0}\right)^{n-2} \end{aligned} \tag{5}$$

Clearly, if $0 < n < 1$, then the Universe is decellerating.

If $n > 1$, then the Universe's scale factor is accelerating.

In terms of consituents equation of state

$$a = a_0 \left(\frac{t}{t_0}\right)^{2/3(1+w)}$$

So $w < -1/3$ is the break point for accelerating universe.

## Solving the Horizon Problem - 2

In order to overcome the horizon problem we need for inflation (accelerating universe) to go on for at least 60 e-foldings. (Rough number of e-folding since the GUT scale.)

Now to show that this gives us sufficiently large extra $\eta$

$$\eta = \int \frac{cdt}{a(t)} = \int \frac{cdt}{a_0(t/t_0)^n} = \frac{ct_0}{a_0}\int (\frac{t}{t_0})^{-n} d\frac{t}{t_0} = \frac{ct_0}{a_0}(-n+1)\left(\frac{t}{t_o}\right)^{1-n}$$

Once again, if $n > 1$ (accelerating universe) one can find negative values of $\eta$ which are arbitrarily large as one approaches zero ($t \rightarrow 0$).

In terms of equation of state

$$\eta = \frac{ct_0}{a_0}(1+3w)\left(\frac{t}{t_o}\right)^{(1+3w)}$$

Thus when $(1+3w) < 0$, $\eta$ can be negative with large magnitude.

## Solution to Pollution is Dilution

Another problem solved by Inflation (a sustained period of accelerating universe) in addition to moving (postponing) curvature problems outside the horizon is reducing unwanted relics of initial conditions and phase transitions.

Unified gauge theories predict that each spontaneous symmetry breaking must produce relics. E.g. magnetic monopoles (one per room), topological defects, superheavy particles, etc. **We see no evidence for these or other previously unknown relics from the early Universe.**

Assume that the GUT symmetry breaking scale is $10^{16}$ GeV: Correponds to a time of about $t_{GUT} \sim 10^{-38}$ seconds and thus a horizon size of $d_{GUT} \sim ct_{GUT} \sim 3 \times 10^{-28}$ cm. The Universe has expanded about a factor of $10^{29}$ since and this corresponds to a present day size of 30 cm. There should be one monopole per such volume size left from the GUT symmetry breaking.

If we have an accelerating phase that expands the Universe by a factor of $10^{28} = e^{64}$ or greater then, then the present size of the GUT symmetry breaking causal volume is greater than the horizon distance ($R_{Horizon} = c/H_0 \sim 10^{28}$ cm).

## Perturbations from Inflation

Consider the case where Inflation (accelerating universe) occurs at times between $10^{-38}$ to $10^{-34}$ seconds.

Very short time so can be considered like a sharp hammer blow or bremsstrahlung event. All modes from essentially zero frequency to $1/\tau = 10^{34}$ Hz are produced with nearly equal amplitude.

The Universe has expanded by about a factor $(1+z)$

$$\frac{a_0}{a_{inflation}} = \left(\frac{t_0}{t_{inflation}}\right)^{1/2} \cong \left(\frac{10^{10} \times 3 \times 10^7}{10^{-34}}\right)^{1/2} \sim 10^{25}$$

so that the frequencies have been downshifted by that same factor.

Predict then that to first order there will be scalar, vector, and tensor perturbations to the RW metric. Vector perturbations are surpressed by conservation of angular momentum. The scalar and tensor perturbations to the curvature will be essentially scale invariant and would cutoff at a frequency of $10^9$

Tensor modes become gravitational waves when they enter the causal horizon and their energy density decreases as $(1+z)^{-1}$.

Scalar modes undergo acoustic oscillations inside the horizon and high frequencies are damped out.

## Perturbations from Inflation

In simplest version of Inflation

$$R(t) \propto e^{Ht} \qquad H^2 = \frac{8\pi G}{3} V = \frac{8\pi}{3} \frac{V}{M_{Planck}^2} \cong \frac{M^4}{M_{Planck}^2}$$

$$H \sim \frac{V^{1/2}}{M_{Planck}} = \frac{M^2}{M_{Planck}}$$

Horizon size (separation distance at which things move apart a speed $c$)

$$v = Hd \qquad d_H = \frac{c}{H} = \frac{M_{Planck}c}{V^{1/2}} = \frac{M_{Planck}c}{M^2}$$

Uncertainty Principle

$$\Delta E \Delta t \geq \hbar \qquad \Delta p \Delta x \geq \hbar$$

$$\Delta E / H \sim \hbar \qquad \Delta pc \sim h \frac{H}{2\pi}$$

$$\Delta E \sim \hbar H \qquad \Delta \phi = \frac{H}{2\pi}$$

Thus the metric fluctuations will be of order $\frac{M^2}{M_{Planck}^2}$

If the potential corresponds to an energy of about $3 \times 10^{16}$ GeV, the metric (curvature) fluctuations will be of order
$h \sim \frac{(3 \times 10^{16})^2}{(1.22 \times 10^{19})^2} = 10^{-5}$

## Perturbations from Inflation: 2

A more careful look at the perturbations from Inflation

$$\delta lna = \frac{dlna}{d\phi}\delta\phi = \frac{dlna}{dt}\frac{dt}{d\phi}\delta\phi = H/\frac{d\phi}{dt}\delta\phi; \qquad H \equiv \frac{dlna}{dt}$$

In a de Sitter background, the rms fluctuations are $\delta\phi = H/2\pi$.

$$kT_{de\ Sitter} = kT_{\mathrm{Gibbons-Hawking}} = \frac{\hbar H}{2\pi} \quad T_{deSitter} = T_{\mathrm{GH}} = \frac{H}{2\pi}$$

For tensor modes (Gravity waves) there is a direct correlation

$$h_{GW} = \sqrt{16\pi G}\delta\phi = \sqrt{16\pi}\delta\phi_{GW}/M_{Planck} \sim H/M_{Planck}$$

For scalar modes the curvature perturbations can be related to the density fluctuations by the relativistic continuity equation:

$$\frac{d\rho}{dt} + 3H(\rho + p/c^2) = 0; \qquad H \equiv \frac{dlna}{dt}$$

Multiply through by $\delta t$ and we have

$$\delta\rho + 3(\rho + p/c^2)\delta lna = 0 \quad \text{or} \quad \delta lna = -\frac{1}{3}\frac{\delta\rho}{(\rho + p/c^2)}.$$

Note that the pressure is negative in inflation so that the density fluctuations are enhanced compared to the expected curvature fluctuations.

Since $H^2 = 8\pi V(\phi)/3M^2_{Planck}$

$$\delta lna = -\frac{3H^2}{dV/d\phi}\delta\phi = -\frac{8\pi V}{V' M^2_{Planck}}\delta\phi$$

The ratio of rms gravity wave fluctuations to density fluctuations is

$$R = \frac{\left\langle h_{GW}^2 \right\rangle^{1/2}}{\left\langle h_\phi^2 \right\rangle^{1/2}} = \sqrt{\frac{2}{\pi}} \frac{V' M_{Planck}}{V}$$

Thus the ratio of tensor to scalar modes is roughly the ratio of the slope of the inflaton potential (times a Planck energy) divided by the potential. Slow roll (flat slope) inflation can have a significant enhancement of scalar to tensor modes.

## Dynamics with Changing Equation of State

The consequences of dark energy follow from its effect on the expansion rate for a flat universe (negligible curvature):

$$\begin{aligned}
H^2 &= \frac{8\pi G}{3}\{\rho_M + \rho_\gamma + \rho_X\} \\
H(z)^2 &= H_0^2\left[\Omega_M(1+z)^2 + \Omega_\gamma(1+z) + \Omega_X e^{3\int_0^z (1+w(y))dln(1+y)}\right] \\
&= H_0^2\left[\Omega_M(1+z)^2 + \Omega_\gamma(1+z) + \Omega_X(1+z)^{3(1+w)}\right] \qquad (6)
\end{aligned}$$

where $\Omega_M$, $\Omega_\gamma$, and $\Omega_X$ are the fractions of the critical density contributed by matter, radiation, and dark energy, respectively. The last equality is only for constant $w$.

This is derived by integration its equation of motion

$$d\left(\rho_X a^3\right) = -p_X da^3$$

where $a$ is the cosmic scale factor.

Then one has

$$\begin{aligned}
t_0 &= \int_0^{t_0} dt = \int_0^{\infty} \frac{dz}{(1+z)H(z)} \\
r(z) &= \int_0^z \frac{dy}{H(y)} \\
d_L(z) &= (1+z)r(z) \\
\frac{dV}{dzd\Omega} &= \frac{r^2(z)}{H(z)} \qquad (7)
\end{aligned}$$