

Approximation to the Probability Density at the Output of a Photomultiplier Tube

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In this paper, the probability density of the integrated output of a PMT is approximated by the Gaussian, Rayleigh, and Gamma probability densities. The accuracy of the approximations depends on the signal energy α : the Gamma distribution is accurate for all α , the Rayleigh distribution is accurate for small α (≤ 1 photon) and the Gaussian distribution is accurate for large α (≥ 10 photons).

I. Introduction

H. H. Tan (Ref. 1) used a Markov diffusion model to determine an approximate probability density at the output of a photomultiplier tube (PMT). The resulting expression (Eq. 66, Ref. 1) is difficult to evaluate. Even by computer, the computational error can become serious since the number of significant terms in the summation of Bessel functions in Eq. (66) increases as α increases. Furthermore, in order to obtain the probabilities of detection and false alarm for a direct detection optical communications system (Ref. 2), this density must be convolved with a Gaussian noise density and integrated. As a result of these complications, three simply computed probability densities (Gaussian, Rayleigh, Gamma) will be considered as approximations to the exact density function.¹

II. Statistical Model

The input to the PMT (Fig. 1) is a pulse with power λ photons/sec of duration T_s seconds. The number of photo-

electrons emitted at the PMT cathode² during the time slot $[0, T_s]$ is assumed to be a poisson random variable with a mean value of $\alpha = \lambda T_s$.

The PMT is characterized by its average gain A and the number of dynode states ν . The variance of the PMT output is $2AB$ where $B = 1/2 (A - 1)/(A^{1/\nu} - 1)$. It is assumed that the bandwidth of the PMT (W_p) is greater than the integration bandwidth ($W_s = 1/T_s$); i.e., $W_p \gg W_s$, and consequently all the signal energy appears at the integrated output. The sampled voltage at the PMT output is $(eR/T_s)y$ where R is the anode resistance and $e = 1.6 \times 10^{-19}$ coulombs. The value ν ($Y = y$) is a dimensionless "gain" and Y is the statistic of interest. The probability density of Y (Eq. 66, Ref. 1) is the combination of an impulse at $Y = 0$ and a non-impulse density, $P_X(y)$:

$$P_Y(y) = c\delta(y) + (1 - c)P_X(y) \quad ; \quad y \geq 0 \quad (1)$$

where

$$c = e^{\alpha(e^{-A/B} - 1)} \quad (2)$$

¹In this article, the density of Eq. (66), Ref. 1, will be called the exact density even though it is a Markov diffusion approximation of a Galton-Watson branching process.

²For the sake of simplicity in this paper, we assume that the PMT quantum efficiency is unity.

III. Approximations to the Non-Impulse Density $P_X(y)$

The mean and variance of the densities $P_Y(y)$ and $P_X(y)$ are

$$m_y = \alpha A \quad ; \quad m_x = m_y / (1 - c) \quad (3)$$

$$\sigma_y^2 = \alpha(A^2 + 2AB) \quad ; \quad \sigma_x^2 = \frac{\sigma_y^2}{1 - c} - cm_x^2 \quad (4)$$

A Rayleigh approximation to the probability density $P_X(y)$ is obtained by equating the mean values. The Gamma and Gaussian approximations are obtained by equating the first and second moments.

The three approximations to $P_X(y)$ and consequently $P_Y(y)$ (Eqs. 1, 2) are:

1. Rayleigh Approximation

$$P_X(y) \approx \frac{y}{\sigma^2} \exp[-y^2/2\sigma^2] \quad ; \quad y \geq 0 \quad (5)$$

$$\sigma = \sqrt{\frac{2}{\pi}} m_x$$

2. Gamma Approximation

$$P_X(y) \approx \frac{a}{\Gamma(b)} (ay)^{b-1} e^{-ay} \quad ; \quad y \geq 0$$

$$a = \frac{1}{k_1 \left[\frac{k_2}{k_1^2} - 1 \right]} \quad ; \quad b = ak_1 \quad (6)$$

$$k_1 = m_x \quad ; \quad k_2 = \frac{m_y^2 + \sigma_y^2}{1 - c}$$

3. Truncated Gaussian Approximation

$$P_X(y) \approx \frac{e^{-\frac{(y - m_x)^2}{2\sigma_x^2}}}{\sqrt{2\pi}\sigma_x \left[1 - Q\left(\frac{m_x}{\sigma_x}\right) \right]} \quad ; \quad y \geq 0 \quad (7)$$

$$Q(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\beta^2/2} d\beta$$

IV. Discussion

The 11-dynode ($\nu = 11$) RCA 31034 A PMT has a usable average gain in a range from 10^5 to 10^7 . The curves of Figs. 2-6 are calculated with an average gain of $A = 10^6$. Several different values of α are considered. The case of $\alpha = 5.2 \times 10^{-6}$ photons/slot (Fig. 2) corresponds to a typical dark current ($T_s = 100$ ns, $\lambda = 52$ counts/sec) in the optical channel (Ref. 2). For this case of small α (Fig. 2), the Rayleigh and Gamma distributions reproduce the shape of the exact distribution with the Gamma distribution being more accurate in the tail regions. Figures 3-5 demonstrate some representative light current values for the optical channel. The fidelity of the Gamma approximation varies as a function of α and is most accurate for $\alpha \approx 1$ and becomes asymptotically close as α increases. Figure 6 demonstrates the validity of the Gaussian approximation for large α . For the Rayleigh distribution, the ratio of the mean and standard deviation is equal to a constant since the density function depends on one parameter. Consequently, the Rayleigh distribution becomes a poor approximation as the mean value increases. Figures 2-6 demonstrate the accuracy of the Gamma distribution for all values of α . The Rayleigh and Gaussian distributions are easily computed functions which can be used for small and large values of α respectively.

References

1. Tan, H. H., "A Statistical Model of the Photomultiplier Gain Process with Applications to Optical Pulse Detection," *TDA Progress Report 42-68*, Jet Propulsion Laboratory, Pasadena, Calif., pp. 55-67, Feb. 15, 1982.
2. Lesh, J., Katz, J., Tan, H., and Zwillinger, D., "The 2.5 Bit/Detected Photon Demonstration Program: Description, Analysis and Phase I. Results," *TDA Progress Report 42-66*, Jet Propulsion Laboratory, Pasadena, Calif., pp. 115-132, Dec. 15, 1981.

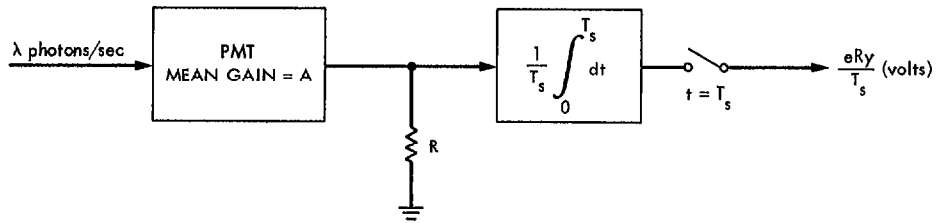


Fig. 1. Model of the integrated output of a random gain PMT

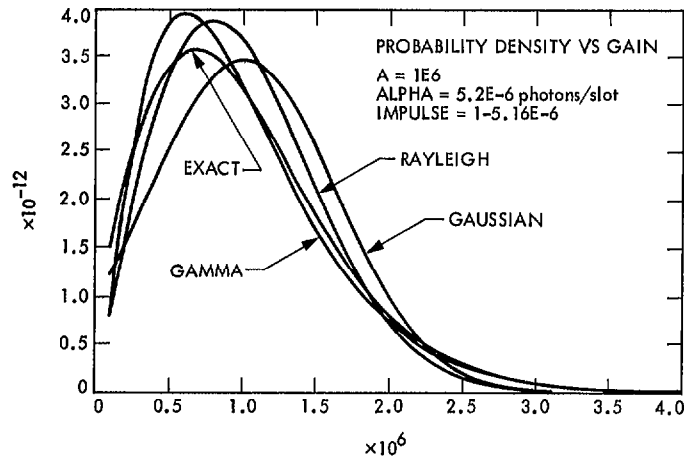


Fig. 2. Small α case: A typical dark current of $\lambda = 52$ cts/s will result in $\alpha = 5.2 \times 10^{-6}$ photons/slot when $T_s = 100$ ns. For this case, the probability that the output is nonzero is 5.16×10^{-6}

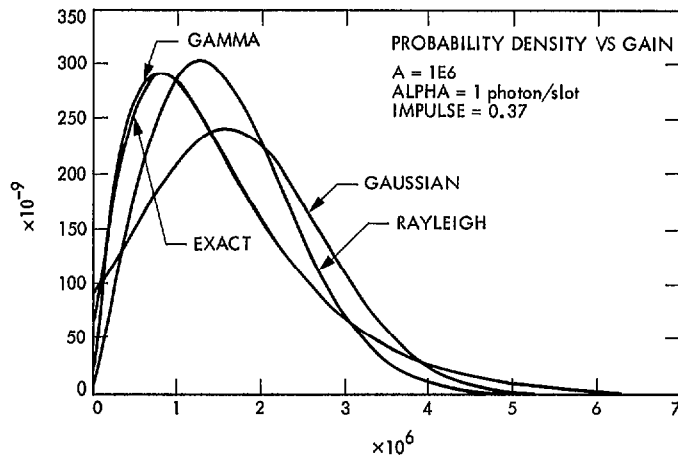


Fig. 3. Moderate α case: An optical channel (256-ary PPM) light current of $\alpha = 1$ photon/slot corresponds to 8 bits/photon; i.e., $\rho = 8/\alpha$ (bits/photon)

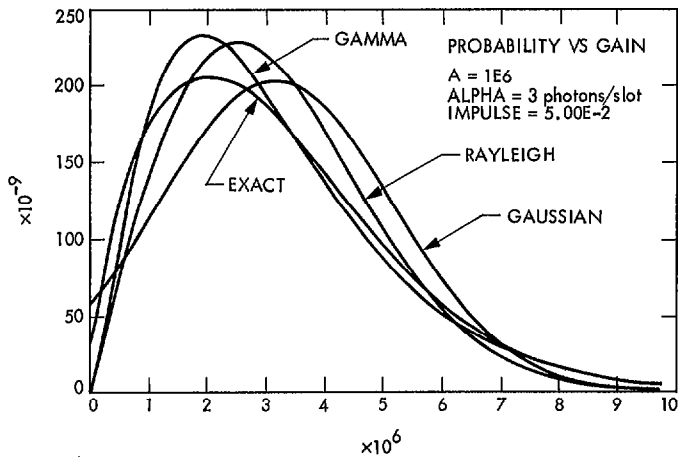


Fig. 4. Moderate α case: An optical channel (256-ary PPM) light current of $\alpha = 3$ photons/slot corresponds to 2.67 bits/photons; i.e., $\rho = 8/\alpha$ (bits/photon)

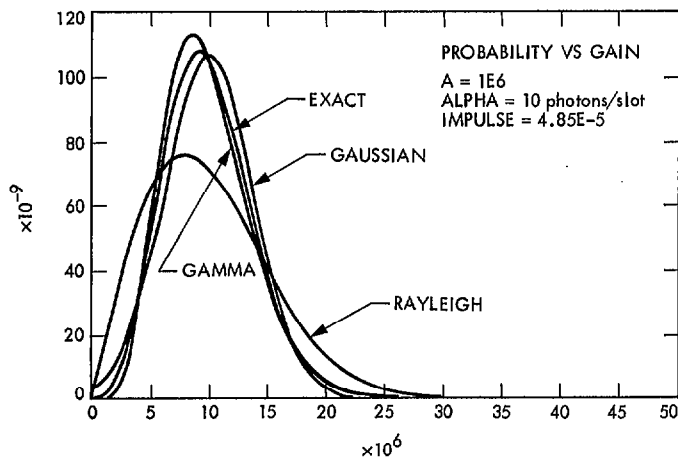


Fig. 5. Moderate α case: An optical channel (256-ary PPM) light current of $\alpha = 10$ photons/slot corresponds to .8 bits/photons; i.e., $\rho = 8/\alpha$ (bits/photon)

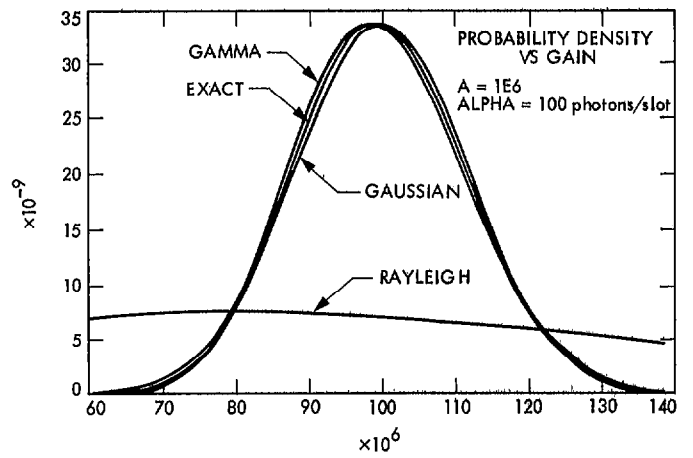


Fig. 6. Large α case: An optical channel (256-ary PPM) light channel of $\alpha = 100$ photons/slot corresponds to $\rho = 12.5$ photons/bit