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OPTIMUM CONTROLLERS FOR LINEAR CLOSED-LOOP SYSTEMS

By Aaron S. Boksenbom, David Novik
and Herbert Heppler

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Cleveland, Ohio

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SUMMARY

An analysis for optimum controllers of general, linear, time-invariant multiloop systems is presented. Optimization is based on minimizing mean-square or integral-square errors for either stationary statistical or transient inputs, with limits and constraints of mean- or integral-square form. General representations of controller characteristics, stable process characteristics, and error relations are used. A method is shown of assuring stability of the multiloop system during the optimization process and casting the multiloop controlled system into an equivalent open loop so that the methods of optimum filter theory can be used. General solutions are obtained for four special cases. Two examples for speed control of a turbojet engine illustrate the methods developed in this report. The data for a controlled turbojet engine were in substantial agreement with the theoretical results.

INTRODUCTION

One of the fundamental functions of controllers is the reduction of certain variables, called errors, to small values. This must be accomplished with the dynamical nature of the system, the difficulty of accurate measurement, the power requirements to manipulate valves, and so forth, the random effects at the inputs, and the general effects of noise taken into consideration. At various stages in the design of an over-all controlled system, the question arises as to what computations the controller should make to minimize the errors.

In this report, the problem of optimizing this computational aspect of the controller is analyzed. Linear systems are assumed throughout. Optimization is considered under either transient or statistical inputs and disturbances and is based on the minimization of mean-square errors or integral-square errors. Constraints and limits are included as mean-square or integral-square values.

Most investigations in this field optimize the controlled system by adjusting certain parameters of an otherwise fixed system. In this report the more general approach of optimizing the system by adjusting the entire frequency-response characteristics of the controllers is used. This general approach was first developed in reference 1 for the design of filters, and a corresponding development is presented herein for a closed-multiloop controlled system. New problems of structural stability and physical realizability arise in the closed-loop cases. In addition, necessary constraints and limits are included in this analysis.

The problem of optimizing the computational aspect of the controller is first reduced to a standard form by generalizing the frequency-response characteristics of both the process to be controlled and the controller, the conditions of stability, and the specifications on errors, constraints, and limits. The general solution for the optimum controllers giving an absolute minimization of errors is derived. In addition, the expressions for the minimum errors, and the additional errors suffered when nonoptimum controllers are used, are shown.

It is expected that the results of this analysis, conducted at the NACA Lewis laboratory, can be used as a basis for controller design, as a standard under which controllers can be evaluated, or as a possible basis of specifications on controller dynamics. Applications of the results of this analysis to several examples of controlling gas turbine engines are shown.

SYMBOLS

The following symbols are used in this report:

a	engine or fuel servo dead time
b,c	constants for describing spectral densities of inputs
C	general linear operators representing controller
E	stable linear operators representing engine or process
e	errors to be minimized by control and constrained variables
e*	errors resulting from nonoptimum controller
F,G	stable linear operators representing parts of controller
H	general linear operators giving response of errors

K	loop gain
R	arbitrary stable linear operators
t	time
w	inputs and disturbances affecting errors
x	manipulated variables
y	inputs to controller
z	inputs and disturbances affecting y's
δF	arbitrary stable linear operators
λ	Lagrangian multiplier
σ,τ	engine time constants
τ_1	integral time constant
ω	frequency

Subscripts:

j,n,r,s,v	summation indices
opt	optimum

Superscripts:

$\overline{}$	complex conjugate
$\widetilde{}$	expected value or integral

Correlation function notation:

For statistical inputs,

$$(w_j\bar{z}_v) = (\bar{z}_v w_j) = \frac{1}{2\pi}\int_{u=-\infty}^{\infty} \widetilde{z_v(t)w_j(t + u)}\, \varepsilon^{-i\omega u}\, du$$

For transient inputs,

$$(w_j\bar{z}_v) = \frac{1}{2\pi}\int_{-\infty}^{\infty} w_j(t)\ \varepsilon^{-i\omega t}\ dt \int_{-\infty}^{\infty} z_v(u) \quad {}^{+i\omega u}\ du$$

$$(\bar{w}_j z_v) = \overline{(w_j\bar{z}_v)}$$

Matrix notation:

E_{jk} indicates element of j^{th} row, k^{th} column of matrix E

ANALYSIS

Scope of Analysis

The scope of the analysis is summarized as follows:

1. Time-invariant linear systems are assumed throughout.

2. The process to be controlled is inherently stable, with one variable to be manipulated by the controller.

3. Optimization is to be realized for either transient or stationary random inputs.

4. Optimization is based on minimizing mean-square or integral-square errors. The constraints and limits of the system are included by constraining the mean-square or integral-square values of the variables involved.

5. Complete freedom in computation is assumed for the controller.

The same formal equations in the time domain are obtained for the nonstationary, time-varying cases as for the stationary, time-invariant cases (ref. 2); but, even for the open-loop filter problem, the equations for the time-varying cases can be solved only by numerical means. Extension of the analysis of this report to the time-varying cases would require such techniques as those in reference 2. The linear assumptions do not completely preclude application of the methods of this report to nonlinear systems. Many nonlinearities can be handled so as to allow linear techniques without omitting the essential nature of the nonlinearity (refs. 3 and 4).

When the process to be controlled is inherently unstable, the techniques of preserving stability of the closed-loop system are more involved and these rather uncommon cases are not treated herein. The extension of this analysis to cases having several manipulated variables follows in a straightforward manner, but the final equations to be solved are more complex. The fundamental development of these cases is shown in the appendix.

The optimization problem for transient inputs is the same as that for statistical inputs (ref. 5) if mean squares are replaced by integral squares and if spectral densities are replaced by products of Fourier transforms, as shown in the list of symbols. The analysis which follows was guided by the desire to keep the systems studied and the problems resolved as general as possible so that wide applications may be made of the results. In order to utilize available open-loop theory, it was necessary to determine an open loop equivalent to the actual closed loops which would still assure structural stability of the actual system during the optimization procedure. Techniques of accomplishing this end were found for those cases of stable process characteristics.

Characterization of Process

The general form of the systems considered is shown in figure 1. The time-invariant linear process to be controlled has certain inputs x (in this case only one, such as a fuel-valve position signal) which can be and are chosen to be manipulated by the controller. There are a certain number of outputs y (such as measured engine speed, measured engine speed error, throttle position, altitude pressure, etc.) which can be and are chosen to be used by the controller. Each output is affected by transient and statistical disturbances and inputs z of altitude, air speed, throttle position, and so forth, and by the manipulated variable through a linear operator E. In general, E would be a rectangular array of operators, but for the case of one manipulated variable E is a column of operators. The general form of the process to be controlled can be written operationally as

$$y_j = z_j + E_j \cdot x \tag{1}$$

This form follows directly from one method of obtaining these characteristics. If x is held constant, measured y gives the nature of z. If x is varied sinusoidally, the harmonic analysis over a large number of cycles of measured y filters out any effects of z because the harmonic content at any one frequency of either a transient or statistical z is relatively small. This can be seen from the following analysis:

If

$$x(t) = \mathrm{Re}\ (\varepsilon^{i\omega t}) \qquad t > 0$$

$$= 0 \qquad t < 0$$

then

$$y(t) = z(t) + \mathrm{Re}\ \left[E(i\omega)\ \varepsilon^{i\omega t}\right] \quad \text{as} \quad t \to \infty$$

and

$$\frac{1}{T}\int_{L}^{L+T} \varepsilon^{-i\omega t}\, y(t)dt = \frac{1}{T}\int_{L}^{L+T} \varepsilon^{-i\omega t}\, z(t)dt + E(i\omega)$$

where

$$T = \frac{2\pi k}{\omega} \qquad (k \text{ is an integer})$$

and

$$L \rightarrow \infty$$

The first term on the right side of the preceding equation approaches zero as $k \rightarrow \infty$ for either a transient or statistical z, and

$$\frac{1}{T}\int_{L}^{L+T} \varepsilon^{-i\omega t}\, y(t)dt = E(i\omega) \quad \text{for } L, k \rightarrow \infty$$

Characterization of Controllers

The controller to be designed is to operate on the y's to give x according to the operational equation

$$x = \sum_{j} C_j \cdot y_j \tag{2}$$

As the optimization of only the computational aspect of the controller will be considered, the y's and the x are outputs and input, respectively, which isolate the engine or process to be controlled and which can be freely used in computation.

The C's of equation (2) are arbitrary linear operators which need only to keep the closed-loop system stable. But, in order to assure stability, the responses of all variables in the system to all possible disturbances must be considered. These include variables and disturbances internal to the C and E boxes. Because arbitrary linear operators can be built up from stable structures and the small internal disturbances of such structures can be considered as applied to the input or output as small disturbances, the proper discussion of stability should deal only with stable structures.

The controller operational equation should therefore be written

$$x = \sum_j F_j \cdot y_j + G \cdot x \tag{3}$$

where F_j and G represent stable structures. The only new modes of oscillation in the system in addition to the stable modes generated by E, F, and G are those represented by the response $\dfrac{1}{1 - G - \sum_j F_j E_j}$. This criterion of stability can be seen by deriving the responses of all variables in the closed-loop system to all possible disturbances. Each response will have the preceding factor multiplied by E, F, or G, or by the sum of the products of these operators. It can also be shown that no loss of generality is caused by allowing

$$G = -\sum_j F_j E_j \tag{4}$$

in which case stability of the entire closed-loop system is assured. The number of functions F still to be determined is the same as the original number of unknown functions C of equation (2).

There are alternative ways of obtaining the same over-all operation expressed by equations (2) or (3), some of which may be unstable. One general method, always stable, is shown by figure 2, in which the F_j's are arbitrary stable structures.

Combining equations (1), (3), and (4) gives

$$x = \sum_j F_j \cdot z_j \tag{5}$$

The original closed-loop system of figure 1 is now equivalent to the open-loop system of figure 3, and this open-loop system is now comparable with the systems of reference 1.

An expression for C_j of equation (2) is

$$C_j = \frac{F_j}{1 + \sum_j E_j F_j} \tag{5a}$$

Specifications on Controller

The errors to be minimized by the controller are characterized in the same way as the y's as they have a controlled and an uncontrolled part. The specifications are written

$$\sum_j \lambda_j \widetilde{e_j^2} = \text{a minimum} \tag{6}$$

where

$$e_j = w_j + H_j \cdot x \tag{7}$$

and H is any general linear operator.

The λ multiplier technique is used to allow additional degrees of freedom whenever the optimum values of the errors are not independent or when constraints of mean-square form are to be imposed. The λ's may then appear as parameters of the controller and are adjusted to give a compromise among the various dependent errors or to set the mean-square values of the constrained variables.

Minimization of Errors

From equations (7) and (5), the expressions for the quantities to be minimized or constrained are

$$e_j = w_j + H_j \sum_n F_n \cdot z_n$$

and

$$\sum_j \lambda_j \widetilde{e_j^2} = \sum_j \lambda_j \int_{-\infty}^{\infty} d\omega \Bigg[\sum_v H_j F_v (z_v \overline{w}_j) + \sum_v \overline{H}_j \overline{F}_v (\overline{z}_v w_j) +$$

$$\sum_v \sum_n |H_j|^2 F_v \overline{F}_n (z_v \overline{z}_n) + (w_j \overline{w}_j) \Bigg] \tag{8}$$

If the symbol F is used for the optimum system resulting in error e and the symbol $F + \delta F$ is used for any other system resulting in error e^*, then

$$\sum_j \lambda_j \widetilde{e^{*}_j{}^2} - \left(\sum_j \lambda_j \widetilde{e_j^2} \right)_{opt} = \sum_j \lambda_j \int_{-\infty}^{\infty} d\omega \sum_v \overline{\delta F}_v \Bigg[\overline{H}_j (\overline{z}_v w_j) +$$

$$\sum_n |H_j|^2 F_n (z_n \overline{z}_v) \Bigg] + \sum_j \lambda_j \int_{-\infty}^{\infty} d\omega \sum_v \delta F_v \Bigg[H_j (z_v \overline{w}_j) + \sum_n |H_j|^2 \overline{F}_n (\overline{z}_n z_v) \Bigg] +$$

$$\sum_j \lambda_j \int_{-\infty}^{\infty} d\omega \sum_v \sum_n |H_j|^2 (z_v \overline{z}_n) \delta F_v \overline{\delta F}_n$$

The first two terms of the right-hand side of the preceding equation are equal because the integrands are merely conjugate. The third term is nonnegative as it is equal to the mean-square error resulting when F is δF and $w = 0$. Thus the necessary and sufficient condition for an absolute minimum is that

$$\sum_j \lambda_j \int_{-\infty}^{\infty} d\omega \sum_v \overline{\delta F}_v \Bigg[\overline{H}_j (\overline{z}_v w_j) + \sum_n |H_j|^2 F_n (z_n \overline{z}_v) \Bigg] = 0 \tag{9}$$

and then

$$\sum_j \lambda_j \widetilde{e^*_j{}^2} - \left(\sum_j \lambda_j \widetilde{e_j{}^2}\right)_{opt} = \int_{-\infty}^{\infty} d\omega \left(\sum_j \lambda_j |H_j|^2\right) \sum_v \sum_n (z_v \bar{z}_n) \delta F_v \overline{\delta F}_n \geqq 0 \tag{10}$$

Because δF_v is any arbitrary but structurally stable system, equation (9) reduces to

$$\sum_j \lambda_j \bar{H}_j (\bar{z}_v w_j) + \left(\sum_j \lambda_j |H_j|^2\right) \sum_n F_n (z_n \bar{z}_v) = \bar{R}_v \tag{11}$$

for all values of v for which there are a corresponding F_v and C_v, and where R_v represents any arbitrary stable structure.

Cases in Which z's Are Independent

A general solution of equation (11) has been found only for the case in which the z's are independent ($(z_n \bar{z}_v)$ is a diagonal matrix). In this case the general solution is

$$F_v = -\frac{A_{v,1}}{MZ_v} \tag{12}$$

where

$$A_v = \frac{\sum_j \lambda_j \bar{H}_j (\bar{z}_v w_j)}{\overline{MZ}_v}$$

$$\sum_j \lambda_j |H_j|^2 = |M|^2$$

$$(z_v \bar{z}_v) = |Z_v|^2$$

The operators M and Z_v represent minimum phase structures, and $A_v = A_{v,1} + A_{v,2}$ where $A_{v,1}$ and $\bar{A}_{v,2}$ represent stable structures. This solution follows the general pattern of reference 1 in the frequency domain. The determination of M and Z_v is the factorization problem of reference 1. The function $A_{v,1}$ can be obtained formally by the following equation:

$$A_{v,1}(i\omega) = \frac{1}{2\pi}\int_{t=0}^{\infty} \varepsilon^{-i\omega t}\, dt \int_{u=-\infty}^{\infty} A_v(iu)\, \varepsilon^{iut}\, du$$

From equation (8), a general expression for the minimum error gives

$$\left(\sum_j \lambda_j \widetilde{e_j^2}\right)_{opt} = \sum_j \lambda_j \int_{-\infty}^{\infty} d\omega (w_j \bar{w}_j) - \int_{-\infty}^{\infty} d\omega \sum_v |A_{v,1}|^2 \qquad (13)$$

From equation (10), the difference between nonoptimum and optimum errors is

$$\sum_j \lambda_j \widetilde{e^*{}_j^2} - \left(\sum_j \lambda_j \widetilde{e_j^2}\right)_{opt} = \sum_v \int_{-\infty}^{\infty} d\omega |M\delta F_v Z_v|^2 \qquad (14)$$

EXAMPLES

Example 1

The first example taken from a problem in the control of a turbojet engine is illustrated in figure 4. The engine-speed error is fed into the controller and a fuel-valve position signal is varied by the controller. Linearized engine characteristics are represented by a lag (ref. 6), and the fuel servo is considered as having only dead time (the output reproduces the input a seconds later).

Then

$$e_1 = y_1 = \text{engine speed error}$$

$$w_1 = z_1 = \text{negative of speed setting}$$

$$E_1 = H_1 = \frac{\varepsilon^{-ai\omega}}{1 + \tau i\omega}$$

$$x = \text{signal to fuel servo}$$

From equation (12), since $|M|^2 = \dfrac{\lambda_1}{|1 + \tau i\omega|^2}$,

$$M = \frac{\lambda_1^{1/2}}{1 + \tau i\omega}$$

then

$$A_1 = \lambda_1^{1/2}\varepsilon^{ai\omega}z_1$$

Letting

$$(z_1\bar{z}_1) = \frac{c^2}{|b + i\omega|^2}$$

then

$$Z_1 = \frac{c}{b + i\omega}$$

$$A_1 = \lambda_1^{1/2}\,\frac{\varepsilon^{ai\omega}c}{b + i\omega}$$

$$A_{1,1} = \frac{1}{2\pi}\int_0^{\infty} \varepsilon^{-i\omega t}\, dt \int_{u=-\infty}^{\infty} \varepsilon^{iut}\,\frac{\lambda_1^{1/2}\, c\, \varepsilon^{aiu}}{b + iu}\, du$$

$$= \int_0^{\infty} \varepsilon^{-i\omega t}\, dt\, \lambda_1^{1/2}\, c\, \varepsilon^{-b(t+a)}$$

$$= \frac{\lambda_1^{1/2}\, c\, \varepsilon^{-ab}}{b + i\omega}$$

and

$$F_1 = -\varepsilon^{-ab}(1 + \tau i\omega)$$

From equation (5a)

$$C_1 = \frac{-\varepsilon^{-ab}(1 + \tau i\omega)}{1 - \varepsilon^{-ab}\varepsilon^{-ai\omega}}$$

From equation (13)

$$\left(\widetilde{e_1^2}\right)_{opt} = \frac{c^2}{2b}\left(1 - \varepsilon^{-2ab}\right)$$

For the controller action derived, the responses of speed and x to speed setting are

$$\text{Speed} = \varepsilon^{-ab}\varepsilon^{-ai\omega} \cdot (\text{speed setting})$$

and

$$x = \varepsilon^{-ab}(1 + \tau i\omega) \cdot (\text{speed setting})$$

The order property and thus the derivative action of the preceding equation indicates the large fuel flow and temperature variations encountered with the derived controller because no limit or constraint was placed on these variables.

Example 2

The second example in which the same control problem as in example 1 is analyzed with a constraint on the mean- or integral-square turbine temperature variation and in which the servo lag has been omitted is illustrated in figure 5. For this example,

$$e_1 = y_1 = \text{engine speed error}$$

$$w_1 = z_1 = \text{negative of speed set}$$

$$e_2 = \text{turbine temperature}$$

$$w_2 = 0$$

$$H_1 = E_1 = \frac{1}{1 + \tau i\omega}$$

$$H_2 = \frac{1 + \sigma i\omega}{1 + \tau i\omega}$$

$$x = \text{signal to fuel servo}$$

From equation (12), since $|M|^2 = \dfrac{\lambda_1 + \lambda_2 |1 + \sigma i\omega|^2}{|1 + \tau i\omega|^2}$,

$$M = \frac{(\lambda_1 + \lambda_2)^{1/2} + \lambda_2^{1/2}\sigma i\omega}{1 + \tau i\omega}$$

then

$$A_1 = \frac{\lambda_1 z_1}{(\lambda_1 + \lambda_2)^{1/2} - \lambda_2^{1/2}\sigma i\omega}$$

Letting

$$(z_1\bar{z}_1) = \frac{c^2}{|b + i\omega|^2}$$

then

$$z_1 = \frac{c}{b + i\omega}$$

$$A_1 = \frac{\lambda_1 c}{(b + i\omega)\left[(\lambda_1 + \lambda_2)^{1/2} - \lambda_2^{1/2}\sigma i\omega\right]}$$

$$A_{1,1} = \frac{\lambda_1 c}{(b + i\omega)\left[(\lambda_1 + \lambda_2)^{1/2} + \lambda_2^{1/2}\sigma b\right]}$$

and

$$F_1 = -\frac{\lambda_1(1 + \tau i\omega)}{\left[(\lambda_1 + \lambda_2)^{1/2} + \lambda_2^{1/2}\sigma b\right]\left[(\lambda_1 + \lambda_2)^{1/2} + \lambda_2^{1/2}\sigma i\omega\right]}$$

From equation (5a)

$$C_1 = \frac{-\lambda_1(1 + \tau i\omega)}{\left[(\lambda_1 + \lambda_2)^{1/2} + \lambda_2^{1/2}\sigma b\right]\left[(\lambda_1 + \lambda_2)^{1/2} + \lambda_2^{1/2}\sigma i\omega\right]} - \lambda_1$$

From equation (13)

$$\left(\sum_j \lambda_j \widetilde{e_j^2}\right)_{opt} = \frac{\lambda_1 c^2}{2b} \frac{\left[\lambda_2(1 + \sigma^2 b^2) + 2(\lambda_1 + \lambda_2)^{1/2}\lambda_2^{1/2}\sigma b\right]}{\left[(\lambda_1 + \lambda_2)^{1/2} + \lambda_2^{1/2}\sigma b\right]^2} \tag{15}$$

The responses of speed and x to speed setting are

$$\text{Speed} = \frac{\lambda_1}{\left[(\lambda_1 + \lambda_2)^{1/2} + \lambda_2^{1/2}\sigma b\right]\left[(\lambda_1 + \lambda_2)^{1/2} + \lambda_2^{1/2}\sigma i\omega\right]} \cdot (\text{speed setting})$$

$$x = -F_1 \cdot (\text{speed setting})$$

In all the preceding equations, only the ratio of the two λ's is effective. This ratio can now be set by evaluating the individual errors that make up equation (15). Choosing the ratio of the λ's involves a compromise between temperature and speed errors, as these quantities vary oppositely to the ratio of λ's.

EXPERIMENTAL RESULTS

An axial-flow turbojet engine was operated on a sea-level static test stand to obtain some verification of the analysis. Figure 4 of example 1 describes the system. The frequency response of engine speed to fuel pressure obtained by harmonic analysis of transient data is shown in figure 6. An approximation to this data gave $\tau = 1.6$ seconds and $a = 0.163$ second.

A proportional-plus-integral controller of the form $C = -K\left[1 + \frac{1}{\tau_1 i\omega}\right]$ was used over a range of K's (loop gain) from 3.5 to 11 and a range of τ_1's from 1 to 2.5 seconds. The integral-square percentage error was obtained for each controller setting from the response to a step in speed setting, and the results are shown in figure 7.

The smallest error (of magnitude 0.41 sec) was obtained at $K = 9.3$ and $\tau_1 = 1.6$ seconds.

The results of example 1 with $b = 0$ (step input) gave

$$C_{opt} = \frac{-(1 + \tau i\omega)}{1 - \varepsilon^{-ai\omega}}$$

With $\varepsilon^{-ai\omega} \doteq \dfrac{1 - \dfrac{ai\omega}{2}}{1 + \dfrac{ai\omega}{2}}$, $C_{opt} \doteq -\dfrac{\tau}{a}\left(1 + \dfrac{1}{\tau i\omega}\right)\left(1 + \dfrac{ai\omega}{2}\right)$. The small derivative action $\left(1 + \dfrac{ai\omega}{2}\right)$ indicated was not used. For $\tau = 1.6$ seconds and $a = 0.163$ second, the preceding equation indicates

$$K_{opt} = \frac{\tau}{a} = \frac{1.6}{0.163} = 9.82$$

$$\tau_{1,opt} \doteq \tau = 1.6 \text{ seconds}$$

which is in substantial agreement with the data of figure 7.

SUMMARY OF RESULTS

An analysis was developed for the optimum controllers of general linear closed-multiloop systems under either stationary statistical or transient inputs to minimize mean-square or integral-square errors. General solutions were obtained for the case of independent inputs and one manipulated variable. Several examples of this case were shown for the speed control of a turbojet engine. Experimental data from a controlled axial-flow turbojet engine were in substantial agreement with the theoretical results.

For the general multiloop systems of any number of manipulated variables, general solutions, which are shown in the appendix, were obtained for four special cases.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, February 16, 1953

APPENDIX - GENERAL MULTILOOP SYSTEMS

The scope of this analysis is the same as for the preceding analysis except that it includes any number of manipulated variables.

Characterization of system. - The general form of the system to be considered is shown in figure 8. The general form of the process to be controlled is written as follows in matrix notation:

$$y = z + E \cdot x \tag{16}$$

The controller to be designed is to operate on the y's in order to give the x's according to the operational equation

$$x = C \cdot y \tag{17}$$

or

$$x = F \cdot y + G \cdot x \tag{18}$$

where every element of the F and G matrices represents stable structures. The new modes of oscillation generated by the closed loops are represented by the responses

$$[1 - G - FE]^{-1}$$

No loss in generality is caused by allowing

$$G = - FE \tag{19}$$

and stability is now assured. There are alternate ways of building the same over-all operation expressed by equation (17) or (18), some of which may be unstable. One definitely stable method is shown in figure 9.

Combining equations (16), (18), and (19) gives

$$x = F \cdot z \tag{20}$$

which represents an equivalent open-loop system. An expression for the C of equation (17) is

$$C = (1 + FE)^{-1}F \tag{20a}$$

Specifications on controller. - The errors and constraints are characterized in the same way as y. The specifications are thus written

$$\sum \lambda_j \widetilde{e_j^2} = \text{a minimum} \tag{21}$$

where

$$e = w + H \cdot x \tag{22}$$

and the elements of H are general linear operators.

<u>Minimization of errors</u>. - From equations (22) and (20), the expression for the quantities to be minimized or constrained is

$$e = w + HF \cdot z$$

and

$$\sum_j \lambda_j \widetilde{e_j^2} = \sum_j \lambda_j \int_{-\infty}^{\infty} d\omega \left[\sum_v (HF)_{jv}(z_v \bar{w}_j) + \sum_v (\overline{HF})_{jv}(\bar{z}_v w_j) + \right.$$

$$\left. \sum_v \sum_n (HF)_{jv}(\overline{HF})_{jn}(z_v \bar{z}_n) + (w_j \bar{w}_j) \right]$$

If the symbol F is used for the optimum system resulting in error e and the symbol F + δF is used for any other system resulting in error e*, then

$$\sum_j \lambda_j \widetilde{e^*{}_j^2} - \left(\sum_j \lambda_j \widetilde{e_j^2} \right)_{opt} = \sum_j \lambda_j \int_{-\infty}^{\infty} d\omega \sum_v (\overline{H\delta F})_{jv} \left[(\bar{z}_v w_j) + \sum_n (HF)_{jn}(z_n \bar{z}_v) \right] +$$

$$\sum_j \lambda_j \int_{-\infty}^{\infty} d\omega \sum_v (H\delta F)_{jv} \left[(z_v \bar{w}_j) + \sum_n (\overline{HF})_{jn}(\bar{z}_n z_v) \right] +$$

$$\sum_j \lambda_j \int_{-\infty}^{\infty} d\omega \sum_v \sum_n (H\delta F)_{jv}(\overline{H\delta F})_{jn}(z_v \bar{z}_n)$$

The first two terms of the right side of the preceding equation are equal since the integrands are merely conjugate. The third term is nonnegative as it is equal to the mean-square error resulting when F is δF and $w = 0$. Thus the necessary and sufficient condition for an absolute minimum is that

$$\sum_j \lambda_j \int_{-\infty}^{\infty} d\omega \sum_v \left(\overline{H\delta F}\right)_{jv} \left[\left(\bar{z}_v w_j\right) + \sum_n (HF)_{jn} \left(z_n \bar{z}_v\right) \right] = 0 \tag{24}$$

and then

$$\sum_j \lambda_j \widetilde{e^*_j{}^2} - \left(\sum_j \lambda_j \widetilde{e_j{}^2} \right)_{opt} = \sum_j \lambda_j \int_{-\infty}^{\infty} d\omega \sum_v \sum_n (H\delta F)_{jv} \left(\overline{H\delta F}\right)_{jn} \left(z_v \bar{z}_n\right) \geqq 0 \tag{25}$$

Because each element of the δF matrix is any arbitrary but structurally stable system, equation (24) reduces to

$$\sum_j \lambda_j \bar{H}_{jr} \left[\left(\bar{z}_v w_j\right) + \sum_n (HF)_{jn} \left(z_n \bar{z}_v\right) \right] = \bar{R}_{rv} \tag{26}$$

for all values of r and v running over the rows and columns of F, and where every element of R represents any arbitrary stable structure.

Equation (26) in matrix symbols becomes

$$\bar{H}'\lambda HFZ + \bar{H}'\lambda W = \bar{R} \tag{27}$$

where H' indicates the transpose of H and the matrices Z, W, and λ are

$$Z_{jv} = \left(z_j \bar{z}_v\right)$$

$$W_{jv} = \left(w_j \bar{z}_v\right)$$

$$\lambda = \text{diagonal } \left(\lambda_1, \lambda_2, \lambda_3, \ . \ . \ .\right)$$

The general solution of equation (27) or (26) has not been found. The factorization problem now seems to be that of factoring the Hermitian matrices $(\bar{H}'\lambda H)$ and Z. Equation (27) has been solved for certain special cases.

Case 1. - In the first case, H is a column matrix and the z's are independent. This is the case solved herein (eq. (12)) in which only the corresponding row of F (and C) is determined. It is noted that, in general, if any row of C vanishes, then the corresponding row of F likewise vanishes.

Case 2. - In the second case, H is a diagonal matrix and the z's are independent. The general solution for this case is

$$F_{rv} = -\frac{A_{vr,1}}{H_r Z_v} \tag{28}$$

where

$$A_{vr} = \frac{\overline{H}_{rr}(\overline{z}_v w_r)}{\overline{H}_r \overline{Z}_v}$$

$$|H_{rr}|^2 = |H_r|^2$$

$$(z_v \overline{z}_v) = |Z_v|^2$$

The functions H_r and Z_v are minimum phase structures and $A_{vr} = A_{vr,1} + A_{vr,2}$ where $A_{vr,1}$ and $\overline{A}_{vr,2}$ are stable structures.

Case 3. - Every element of H and H^{-1} is stable and the z's are independent in the third case. The solution for this case is

$$F_{rv} = -\sum_s \frac{(H^{-1})_{rs} A_{sv,1}}{Z_v}$$

where

$$A_{sv} = \frac{(w_s \overline{z}_v)}{\overline{Z}_v}$$

Z_v is defined in Case 2, and A_{sv} is to be separated into the two parts $A_{sv,1} + A_{sv,2}$ as in Case 2.

Case 4. - In the fourth case, $H_{jk} = BD_j M_{jk}$ and the z's are independent where $B\overline{B} = D_j \overline{D}_j = 1$ and every element of M and M^{-1} is stable. The solution for this case is

$$F_{rv} = - \sum_{s} \frac{(M^{-1})_{rs} A_{sv,1}}{Z_v}$$

where

$$A_{sv} = \frac{\overline{BD}_s(w_s \bar{z}_v)}{\bar{Z}_v}$$

Here Z_v is defined in Case 2, and A_{sv} is to be separated into the two parts $A_{sv,1} + A_{sv,2}$ as in Case 2.

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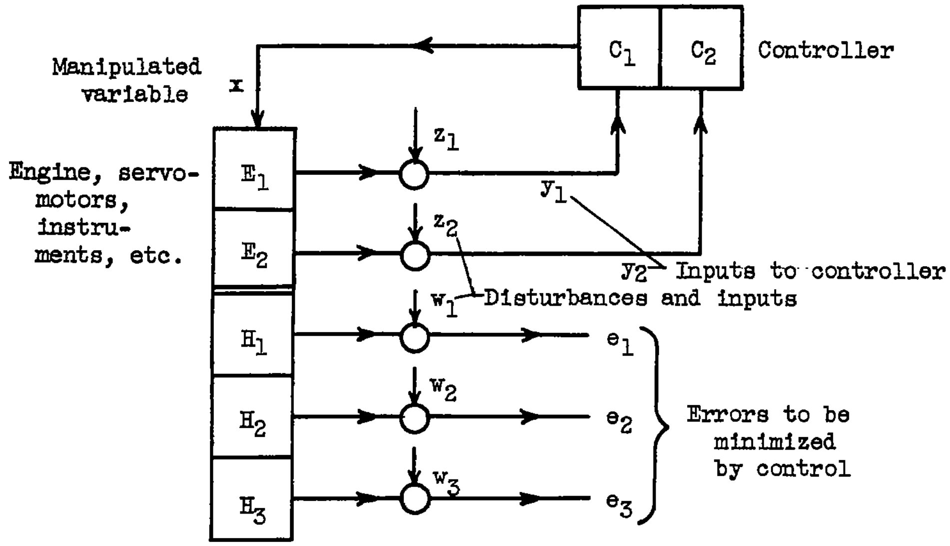

Figure 1. - General linear closed-loop controlled system; one manipulated variable. Circle indicates operation of addition.

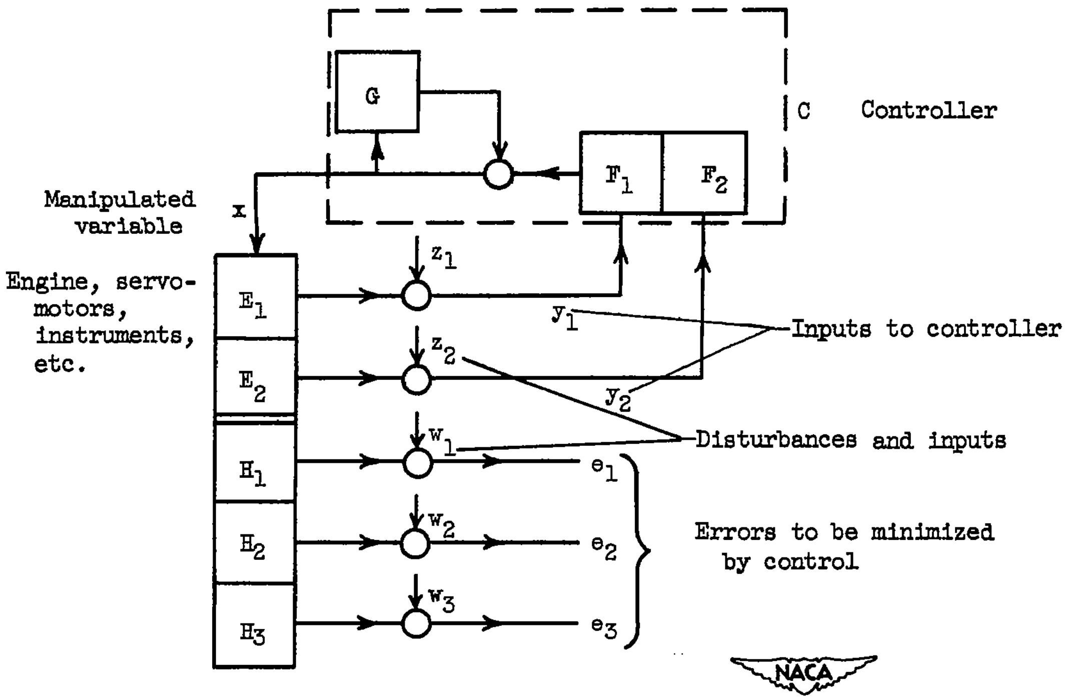

Figure 2. - Representation of general linear controlled system using only stable boxes E, F, and G; one manipulated variable; stable if $G = -\sum_j E_j F_j$.

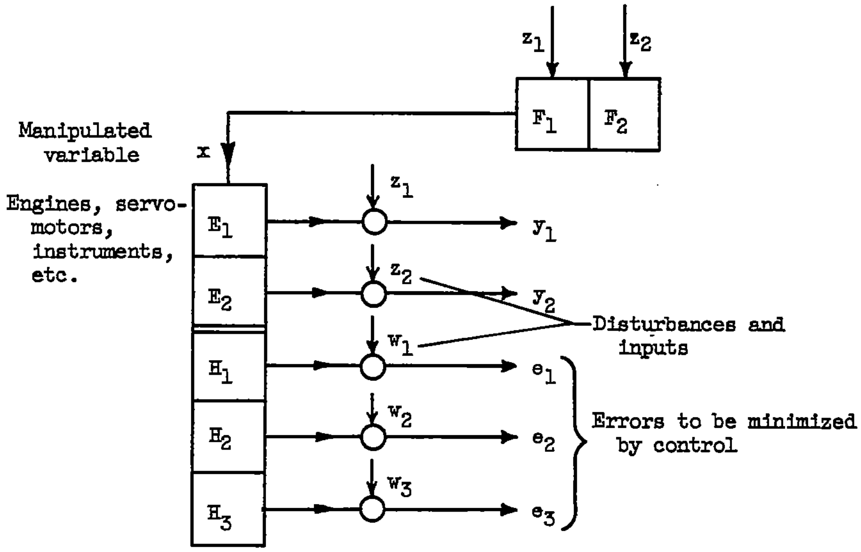

Figure 3. - Open-loop system equivalent to closed-loop system of figure 2 when $G = -\sum_j E_j F_j$.

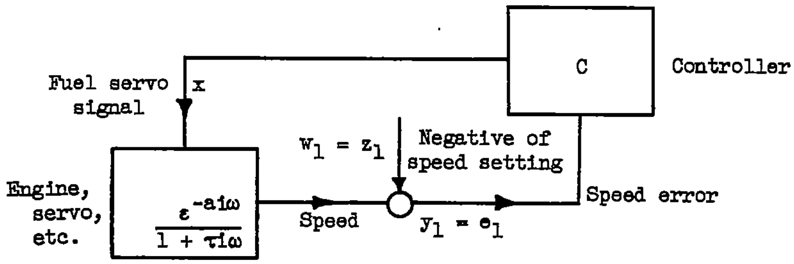

Figure 4. - Illustration of example 1, speed control of turbojet engine.

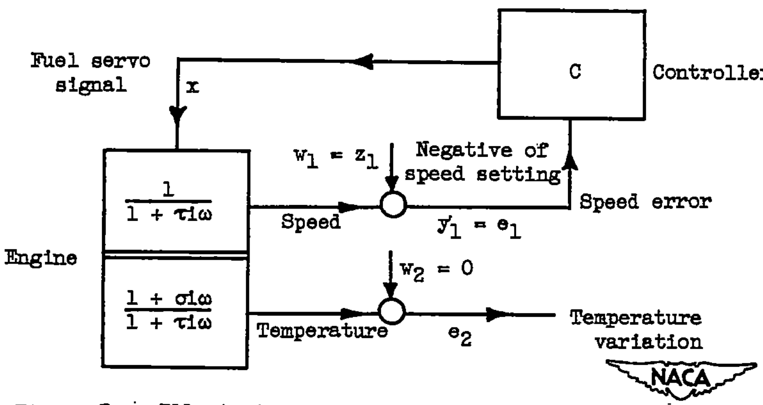

Figure 5. - Illustration of example 2, speed control of turbojet engine with temperature variation constrained.

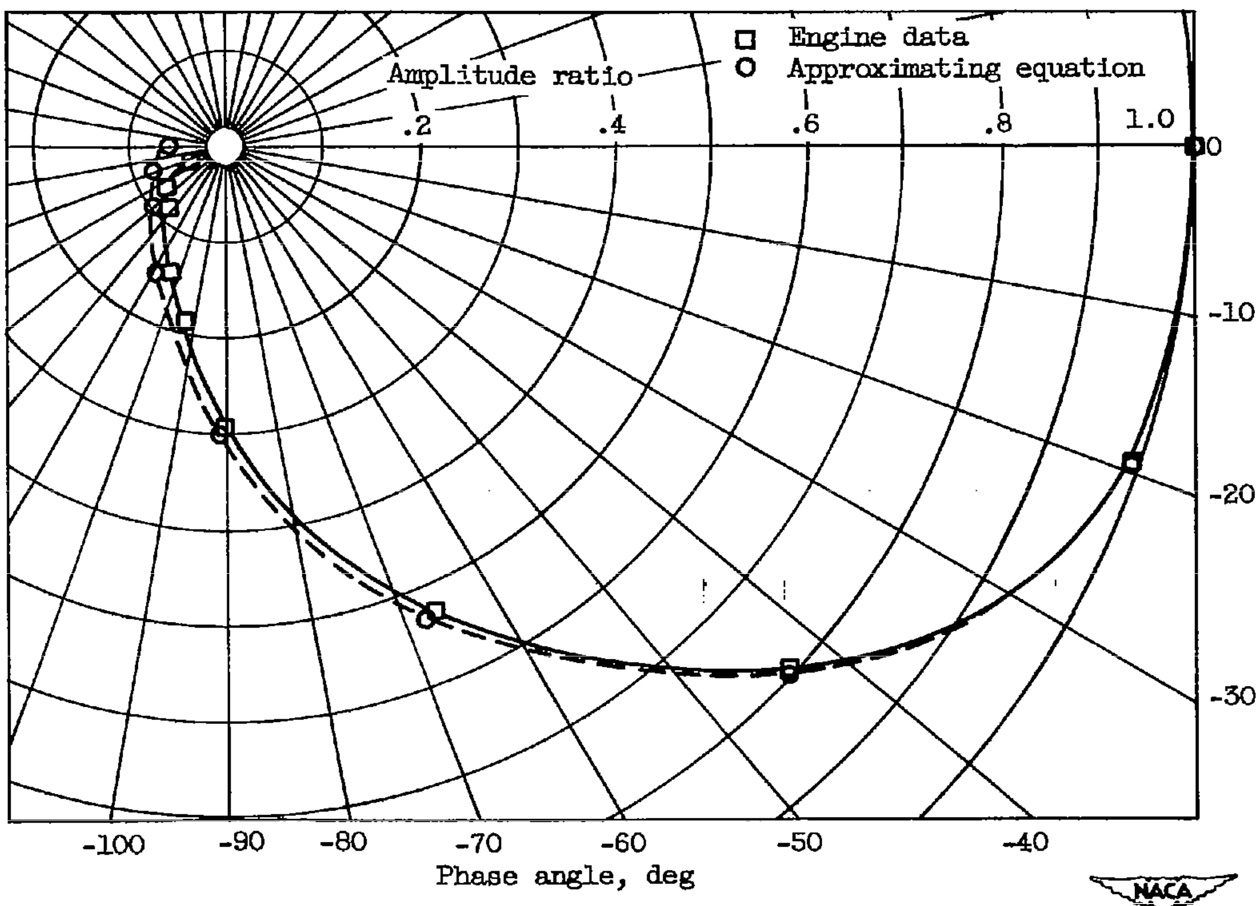

Figure 6. - Frequency response of engine speed to fuel pressure of axial-flow turbojet engine.

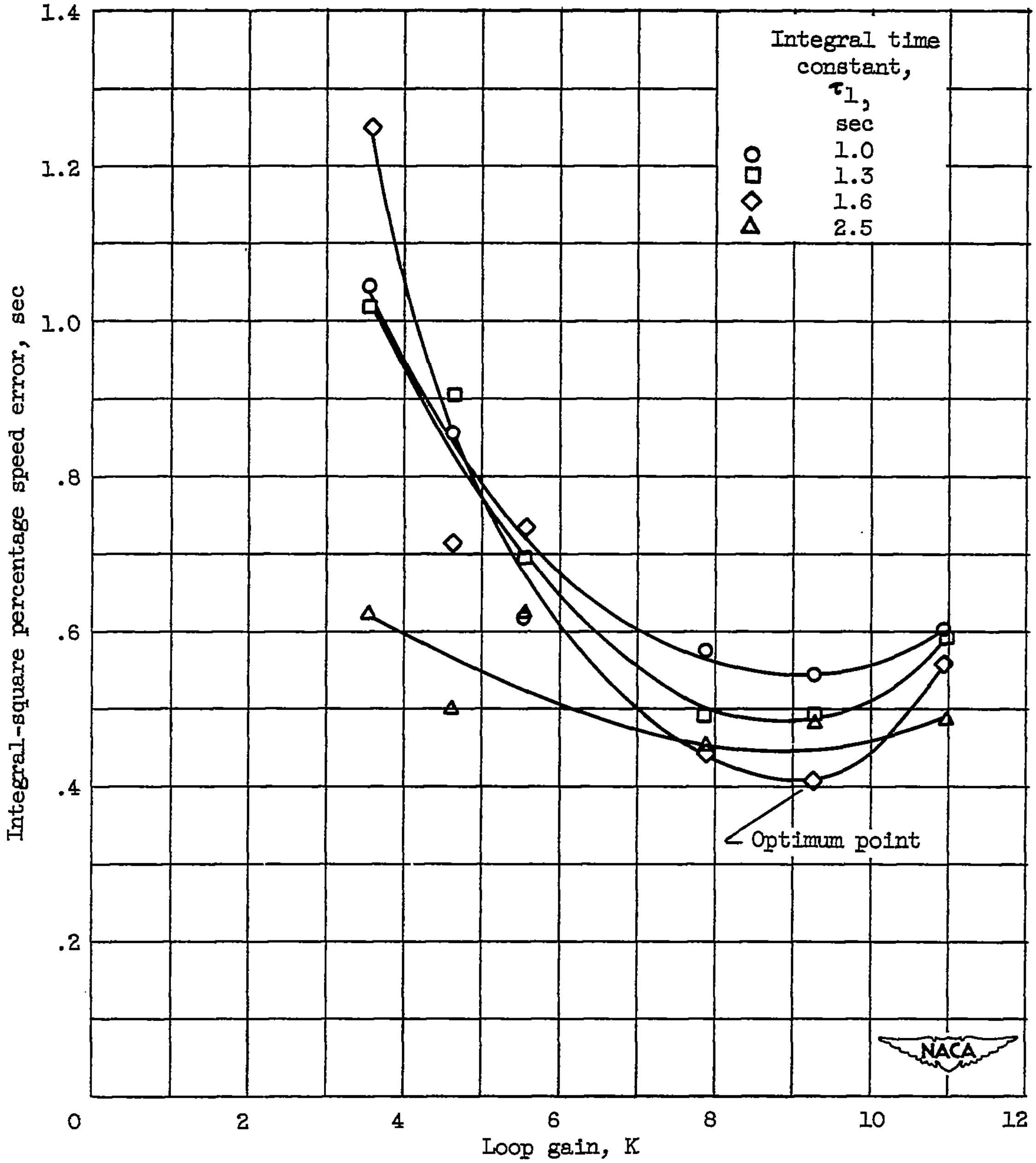

Figure 7. - Integral-square errors of controlled axial-flow turbojet engine for range of loop gains and integral time constants.

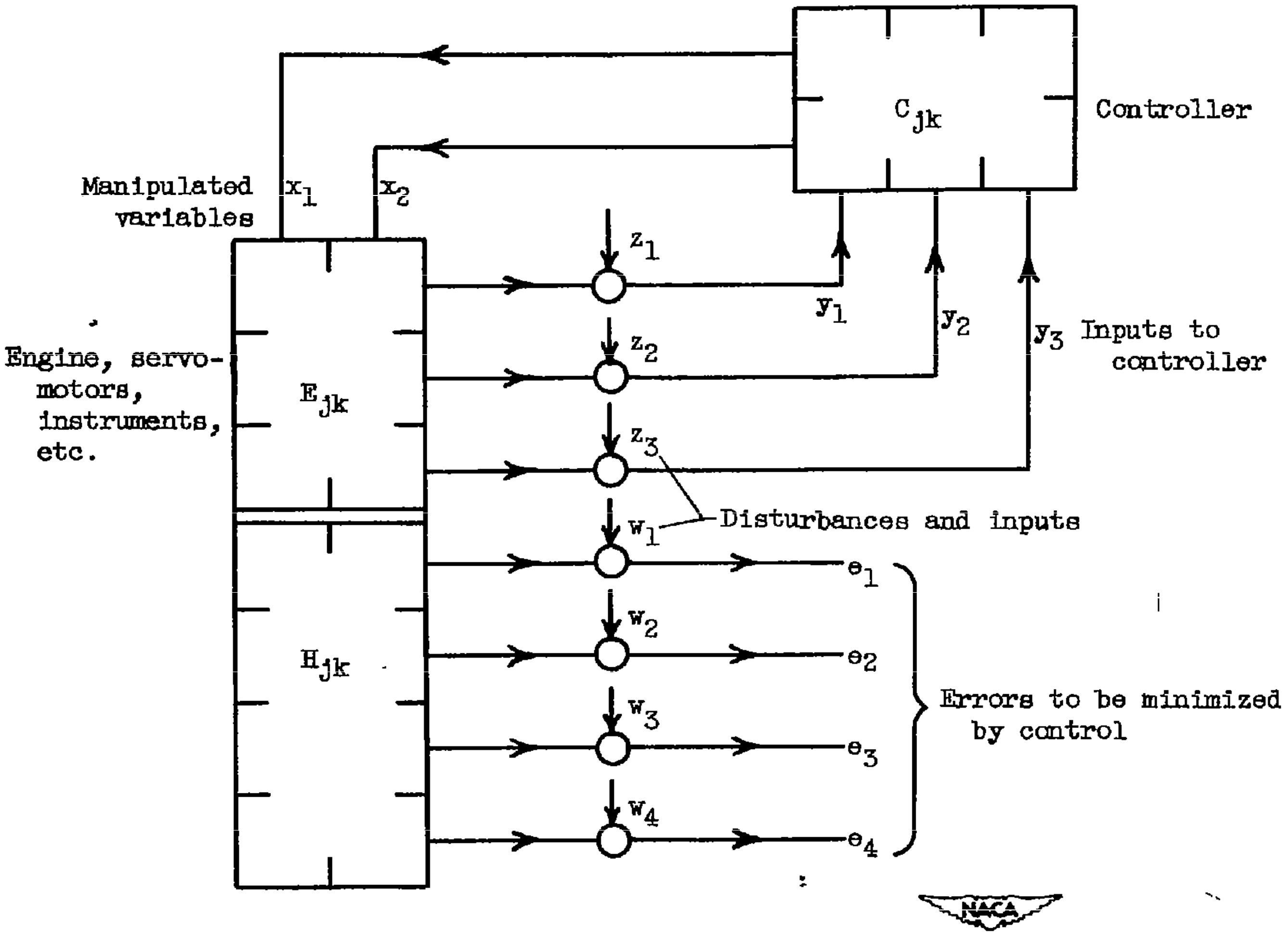

Figure 8. - General linear closed-loop controlled system.

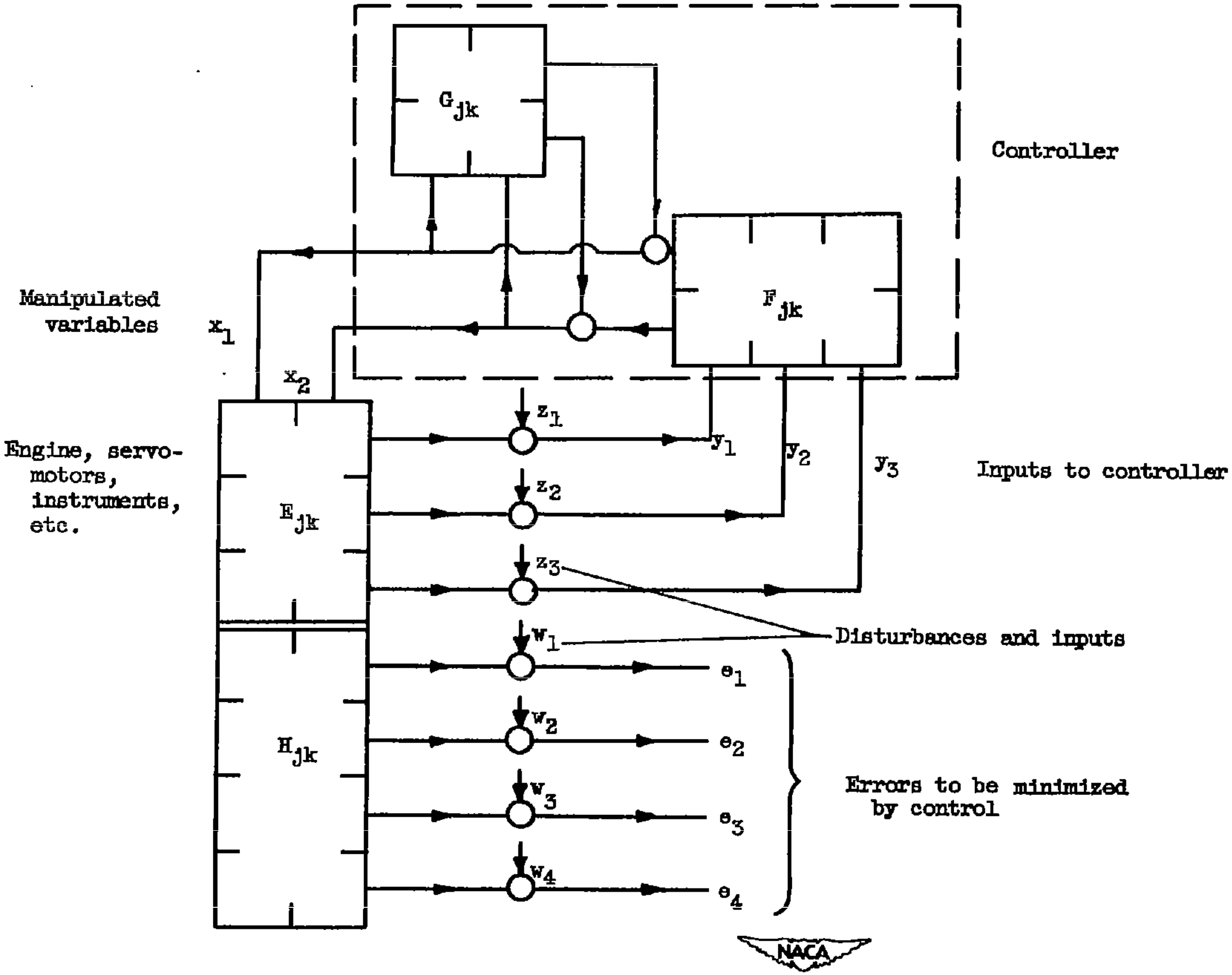

Figure 9. - Representation of general linear controlled system using only stable boxes E, F, and G; stable if $G = -FE$.

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