Chapter 5

Generating Function Formalism and Robust Observables

In order to determine the distribution of the neutral fraction (f) of pions in the MiniMax detector, we would like to count the number of charged and neutral pions from a collision entering our acceptance. (Even if only a piece of a DCC domain enters the acceptance, we would expect on average to observe the same f as for the entire domain.) This approach is complicated by many things, including the fact that π^0's decay almost immediately into two photons which we identify only if they convert in the lead or scintillator within the MWPC telescope. To precisely determine the number of π^0's, we would need to identify both γ's from a decay and reconstruct the pion mass using the electromagnetic calorimeter.

We are unable to do this due to the fact that the probability of both photons from a decay entering the small acceptance of the calorimeter is only about 15%. However, we have found a set of observables which sample the charged-neutral distribution and are independent of many of the detector-related complications and (uncorrelated) efficiencies, and take very different values for pure DCC and for generic particle production. We make many bold assumptions about the production and detection of particles going into the detector in order to establish the robustness of these observables; however, simulations indicate that the assumptions are not unreasonable (see Sec. 6.1.1). We assume that particles other than pions can be ignored, that charged particles and photons are not misidentified, that the production process can be modeled as a two-step process, with the total number of pions given by a parent distribution and the fraction of pions that are neutral given by, e.g., a binomial or DCC distribution function, and that detection efficiencies for finding a charged particle or photon do not depend on the nature of the rest of the event.

These observables and their properties are best understood using generating functions and their factorial moments to describe probability distributions for the production of some species of particle. The generating function formalism has been widely used in multiparticle analysis [51]-[55] and was extended to two variables [56]-[59], [32], e.g., charged and neutral pions by the MiniMax Collaboration in Ref. [60]. The ideas in this section are described in detail in that paper.

Some of the difficulties in measuring the charged-neutral distribution are listed below:

1. The MiniMax acceptance is small, so that it is improbable that both γ's from a π^0 enter the detector acceptance;

2. the conversion efficiency per γ is only about 50%.;

3. not all γ's come from π^0's;

4. not all charged tracks come from $\pi^{\pm}$'s;

5. because of the small acceptance, the multiplicities are rather low, so that statistical fluctuations are very important;

6. detection efficiencies are not the same for charged tracks and γ's and are momentum-dependent;

7. efficiency functions may depend on the observed multiplicity or other parameters;

8. the efficiency for triggering when no charged track or converted γ is produced within the acceptance is relatively low and different from that for events in which at least one charged particle or converted γ is detected.

5.1 Generating functions for charged-pion–neutral-pion distributions

If the set of (normalized) probabilities for producing N particles in a given region of phase space is $\{P(N)\}$, then the generating function can be defined as

$$G(z) = \sum_{N=0}^{\infty} z^N P(N), \tag{5.1}$$

and contains all the information of the $\{P(N)\}$:

$$P(N) = \frac{1}{N!}\left(\frac{d^N G}{dz^N}\right)_{z=0}. \tag{5.2}$$

Information can likewise be extracted from the factorial moments, defined as

$$f_i \equiv \left(\frac{d^i G(z)}{dz^i}\right)_{z=1} = \langle N(N-1)\cdots(N-i+1)\rangle, \tag{5.3}$$

where $\langle \mathcal{O}\rangle = \sum_N \mathcal{O} P(N)$.

Now if $p(n_{ch}, n_0)$ is the probability distribution for producing n_{ch} charged and n_0 neutral pions, the generating function is

$$G(z_{ch}, z_0) = \sum_{n_{ch}=0}^{\infty}\sum_{n_0=0}^{\infty} p(n_{ch}, n_0) z_{ch}^{n_{ch}} z_0^{n_0}, \tag{5.4}$$

and the factorial moments for charged (ch) and neutral (0) pions are

$$f_{i,j}(ch,0) \equiv \left(\frac{\partial^{i+j} G(z_{ch}, z_0)}{\partial z_{ch}{}^i \partial z_0{}^j} \right)_{z_{ch}=z_0=1} . \tag{5.5}$$

We assume that $p(n_{ch}, n_0)$ can be written as the product of a parent distribution $P(N)$ for producing N total pions, and $\hat{p}(n_{ch}, n_0; N)$, which gives the charged-neutral distribution of pions for a given $N = n_{ch} + n_0$:

$$p(n_{ch}, n_0) = P(N)\hat{p}(n_{ch}, n_0; N), \tag{5.6}$$

where

$$\sum_{N=0}^{\infty} P(N) = 1, \tag{5.7}$$

$$\sum_{n_{ch}=0}^{\infty} \sum_{n_0=0}^{\infty} \delta_{N, n_{ch}+n_0}\, \hat{p}(n_{ch}, n_0; N) = 1. \tag{5.8}$$

For generic production of pions, the charge is distributed according to a binomial (bin) with mean neutral fraction $\hat{f} = 1/3$, so that

$$\hat{p}_{bin}(n_{ch}, n_0; N) = \frac{N!}{n_0!(N-n_0)!} \hat{f}^{n_0} (1-\hat{f})^{n_{ch}}. \tag{5.9}$$

The corresponding generating function [from Eqs. (5.9), (5.6) and (5.4)] is

$$G_{bin}(z_{ch}, z_0; \hat{f}) = \sum_N P(N) [\hat{f} z_0 + (1-\hat{f}) z_{ch}]^N. \tag{5.10}$$

Note that G_{bin} depends only on the linear combination

$$\zeta \equiv \hat{f} z_0 + (1-\hat{f}) z_{ch}. \tag{5.11}$$

Much of the simplicity of the generic case is also realized for what can be called the binomial transform,

$$\hat{p}(n_{ch}, n_0; N) = \frac{N!}{n_0!(N-n_0)!} \int_0^1 df p(f) f^{n_0} (1-f)^{n_{ch}}, \tag{5.12}$$

of the arbitrary normalized distribution $p(f)$, such as the DCC distribution $p(f) = 1/(2\sqrt{f})$. This leads to a wide class of possible pion factorial-moment generating functions, namely

$$G(z_{ch}, z_0) = \int_0^1 df p(f) G_{bin}(z_{ch}, z_0; f), \tag{5.13}$$

where $G_{bin}(z_{ch}, z_0; f)$ is given by (5.10) with $\hat{f}$ replaced by an arbitrary f, $0 \leq f \leq 1$.

If $P(N)$ is a Poisson distribution, $[P(N) = \frac{\mu^N}{N!} e^{-\mu}$ with $\mu = \langle N \rangle]$ then $G_{bin}(z_{ch}, z_0; \hat{f}) = e^{-\mu + \mu\zeta}$, so that $\ln G_{bin}(z_{ch}, z_0; \hat{f})$ is linear in ζ. The PYTHIA simulations yield generating functions that, to good approximation, depend only on a fixed linear combination of z_{ch} and z_0 (Fig. 5.1); the full detector simulation with GEANT is found to alter this linear behavior slightly. Compare this to the generating function for the DCC distribution which depends on both z_{ch} and z_0 (Fig. 5.2).

5.2 Generating functions for charged-pion–photon distributions

Next we take into account the probability ϵ_{ch} for observing a given primary charged pion in the detector and a probability $(1-\epsilon_{ch})$ for not observing it, and the probabilities ϵ_m, $m = 0, 1, 2$, for observing m photons from a π^0 decay, with $\epsilon_0 + \epsilon_1 + \epsilon_2 = 1$. We assume that these efficiencies are uncorrelated. Then the generating function for the distribution of observed particles, including efficiencies, is obtained from $G(z_{ch}, z_0)$ [47] by replacing z_{ch} by the generating function

$$g_{ch}(z_{ch}) = (1-\epsilon_{ch}) + \epsilon_{ch} z_{ch}, \tag{5.14}$$

and z_0 by the generating function

$$g_0(z_\gamma) = \epsilon_0 + \epsilon_1 z_\gamma + \epsilon_2 {z_\gamma}^2. \tag{5.15}$$

For the class of production models characterized by (5.13), this leads to the following factorial-moment generating function for the distribution of observed charged pions and photons:

$$G_{obs}(z_{ch}, z_\gamma) = \int_0^1 df\, p(f) G_{bin}(g_{ch}(z_{ch}), g_0(z_\gamma); f). \tag{5.16}$$

The charged-pion–photon factorial moments are

$$f_{i,j} \equiv \left(\frac{\partial^{i,j} G(z_{ch}, z_\gamma)}{\partial z_{ch}{}^i \partial z_\gamma{}^j} \right)_{z_{ch}=z_\gamma=1}, \tag{5.17}$$

which introduces the indexing (i, j) with respect to charged particles and photons which will be used in the remainder of this section. The two lowest orders of factorial moments are

$$\begin{aligned}
f_{1,0} = &\ \langle n_{ch} \rangle &&= \langle 1 - f \rangle\, \epsilon_{ch}\, \langle N \rangle, \\
f_{0,1} = &\ \langle n_\gamma \rangle &&= \langle f \rangle\, (\epsilon_1 + 2\epsilon_2)\, \langle N \rangle, \\
f_{2,0} = &\ \langle n_{ch}(n_{ch} - 1) \rangle &&= \langle (1 - f)^2 \rangle\, \epsilon_{ch}{}^2\, \langle N(N-1) \rangle, \\
f_{1,1} = &\ \langle n_{ch} n_\gamma \rangle &&= \langle f(1 - f) \rangle\, \epsilon_{ch}(\epsilon_1 + 2\epsilon_2)\, \langle N(N-1) \rangle, \\
f_{0,2} = &\ \langle n_\gamma(n_\gamma - 1) \rangle &&= \langle f^2 \rangle\, (\epsilon_1 + 2\epsilon_2)^2\, \langle N(N-1) \rangle + \langle f \rangle\, 2\epsilon_2\, \langle N \rangle,
\end{aligned} \tag{5.18}$$

where the overall statistical averages for the charged, photon, and charged-photon factorial moments are expressed, in an obvious notation, in terms of the independent moments taken with respect to the $P(N)$ and $p(f)$ distributions. The second-order factorial moments represent the lowest-order correlative effects among charged pions and photons.

5.3 Robust observables

We would like to construct a measure from the moments in the form of a ratio in order to cancel out as many effects as possible, apart from the $p(f)$

averages which give information about the charged-neutral distribution. The gamma-gamma correlation ($f_{0,2}$) involves a term proportional to $\epsilon_2 \langle N \rangle$ that cannot be cancelled by any other moments of this order. However, the ratio

$$r_{1,1} = \frac{f_{1,1} f_{1,0}}{f_{2,0} f_{0,1}} = \frac{\langle n_{ch} n_\gamma \rangle \langle n_{ch} \rangle}{\langle n_{ch}(n_{ch} - 1) \rangle \langle n_\gamma \rangle} \ . \tag{5.19}$$

involving the other four moments has complete cancellation of all reference to the background distribution $P(N)$ and the efficiencies ϵ_1, ϵ_2, and ϵ_{ch} for generating functions of the form (5.16):

$$r_{1,1} = \frac{\langle f(1-f) \rangle \langle (1-f) \rangle}{\langle (1-f)^2 \rangle \langle f \rangle} \ . \tag{5.20}$$

For generic pion production, $p(f) = \delta(f - \hat{f})$ and

$$r_{1,1}(gen) = 1. \tag{5.21}$$

For a DCC distribution, $p(f) = 1/(2\sqrt{f})$, the ratio is

$$r_{1,1}(DCC) = \frac{1}{2} \ . \tag{5.22}$$

Therefore, the pure DCC and generic distributions should be easily distinguishable if the statistical uncertainties are not too large.

We can simplify the formulas slightly by introducing normalized factorial moments. For the production of N particles, these are given by

$$F_i \equiv \frac{\langle N(N-1)\ldots(N-i+1)\rangle}{\langle N\rangle^i} \ . \tag{5.23}$$

A generalization of the F_i's to normalized moments for charged track and photon production is

$$F_{i,j} = \frac{\langle n_{ch}(n_{ch}-1)\ldots(n_{ch}-i+1)\ n_\gamma(n_\gamma-1)\ldots(n_\gamma-j+1)\rangle}{\langle n_{ch}\rangle^i \langle n_\gamma\rangle^j} \ . \tag{5.24}$$

In particular,

$$F_{i,0} = \frac{F_i \left\langle (1-f)^i \right\rangle}{\left\langle (1-f)\right\rangle^i} \tag{5.25}$$

and

$$F_{i,1} = \frac{F_{i+1} \left\langle f(1-f)^i \right\rangle}{\langle f\rangle \left\langle (1-f)\right\rangle^i} \ , \tag{5.26}$$

where F_i refers to the ith normalized factorial moment (5.23) of the $P(N)$ distribution for the total multiplicity.

We see that $r_{1,1} = F_{1,1}/F_{2,0}$. A generalization of $r_{1,1}$ to a family of robust observables is

$$r_{i,1} = \frac{F_{i,1}}{F_{i+1,0}} = \frac{\langle (1-f)\rangle \left\langle f(1-f)^i \right\rangle}{\langle f\rangle \left\langle (1-f)^{i+1}\right\rangle} \ , \tag{5.27}$$

where again the dependence on the parent distribution and efficiencies has dropped out. For all $i \geq 1$, generic particle production yields

$$r_{i,1}(gen) = 1, \tag{5.28}$$

while for DCC,

$$r_{i,1}(DCC) = \frac{1}{i+1} \ . \tag{5.29}$$

Thus, $r_{i,1}$ becomes more sensitive to the difference between DCC and generic production mechanisms with increasing order of the moments. This reflects the broadness of the DCC neutral-fraction distribution relative to the binomial distribution of the generic case. If only a fraction of particle production is due to DCC, the signal will be easier to see in the higher-order ratios, which are sensitive to the tail of the charged-photon distribution, where the ratio of DCC to generic production is relatively high. The ratios

$$r_{i,j} = \frac{F_{i,j}}{F_{i+j,0}} \tag{5.30}$$

involving higher-order gamma correlations ($F_{i,j}$, $j > 1$) are not robust because the moments $F_{i,j}$ depend on the photon detection efficiencies. However, the moments can be expressed in terms of only one combination of these efficiencies along with the mean number of photons, namely

$$\xi = \frac{2\epsilon_2}{(\epsilon_1 + 2\epsilon_2)\langle n_\gamma \rangle} \ , \tag{5.31}$$

as

$$F_{i,j} = \sum_{m=0}^{[j/2]} c_{j,m} \xi^m F_{i+j-m} \frac{\langle (1-f)^i f^{j-m} \rangle}{\langle (1-f) \rangle^i \langle f \rangle^{j-m}} \ . \tag{5.32}$$

The coefficients $c_{j,m}$ are obtained from the identity [61], true for any differentiable function $D(z^2)$,

$$\frac{d^j D(z^2)}{(dz)^j} = \sum_{m=0}^{[j/2]} c_{j,m} 2^m (2z)^{j-2m} \frac{d^{j-m} D(z^2)}{(dz^2)^{j-m}} \ . \tag{5.33}$$

The first few $c_{j,m}$ are

$$\begin{aligned} c_{j,0} &= 1, \\ c_{j,1} &= j(j-1)/2, \\ c_{j,2} &= 3\ j!/4!(j-4)!\,. \end{aligned} \tag{5.34}$$

The ratios $r_{i,j}$ can be used in the analysis of experimental distributions, with the understanding that the parameter ξ is to be determined from the data.

Generally, we have the bounds and limiting values

$$r_{i,j}(gen) \geq 1, \tag{5.35}$$

$$r_{i,j}(gen)|_{\xi=0} = 1, \tag{5.36}$$

$$r_{i,j}(DCC)|_{\xi=0} = \frac{i!(2j-1)!!}{(i+j)!} \ . \tag{5.37}$$

Finally, we turn to the effect of the MiniMax trigger on the moments and ratios. As discussed in Sec. 2.2, events in which no charged particle or converted γ goes through the acceptance of the detector are triggered with lower efficiency, ϵ, than other events. Take $p^{obs}(n_{ch}, n_\gamma)$ to be the (normalized) probability for observing an event with n_{ch} charged particles and n_γ converted γ's in the acceptance assuming perfect triggering, and $p^{trig}(n_{ch}, n_\gamma)$ to be the probability including the effects of both the trigger and the particle detection efficiencies. An effective model for the effect of the trigger on the probability is given by

$$p^{trig}(0,0) = \epsilon\alpha p^{obs}(0,0), \qquad n_{ch} = n_\gamma = 0, \tag{5.38}$$

$$p^{trig}(n_{ch}, n_\gamma) = \alpha p^{obs}(n_{ch}, n_\gamma), \quad n_{ch} + n_\gamma > 0, \tag{5.39}$$

where α is a normalization factor,

$$\alpha = \left[1 + (1-\epsilon)\, p^{obs}(0,0)\right]^{-1}. \tag{5.40}$$

When we incorporate these trigger efficiencies,

$$f_{i,j} \to \alpha f_{i,j}, \tag{5.41}$$

and

$$F_{i,j} \to \alpha^{1-i-j} F_{i,j}, \tag{5.42}$$

leaving the $r_{i,j}$ robust in the sense that they are also independent of ϵ.

In summary,

1. the $r_{i,j}$ do not depend upon the form of the parent pion multiplicity distribution;

2. the $r_{i,j}$ are independent of the detection efficiencies for finding charged tracks, provided these efficiencies are not correlated with each other or with other variables such as total multiplicity or background level;

3. the $r_{i,1}$ are also independent of the γ efficiencies in the same sense as above; the $r_{i,(j>1)}$ depend only upon one parameter, ξ, which reflects the relative probability of both photons from a π^0 being detected in the same event;

4. in all cases the $r_{i,j}$ are independent of the magnitude of the null trigger efficiency;

5. the ratios $r_{i,j}$ possess definite and very different values for pure generic and pure DCC pion production.

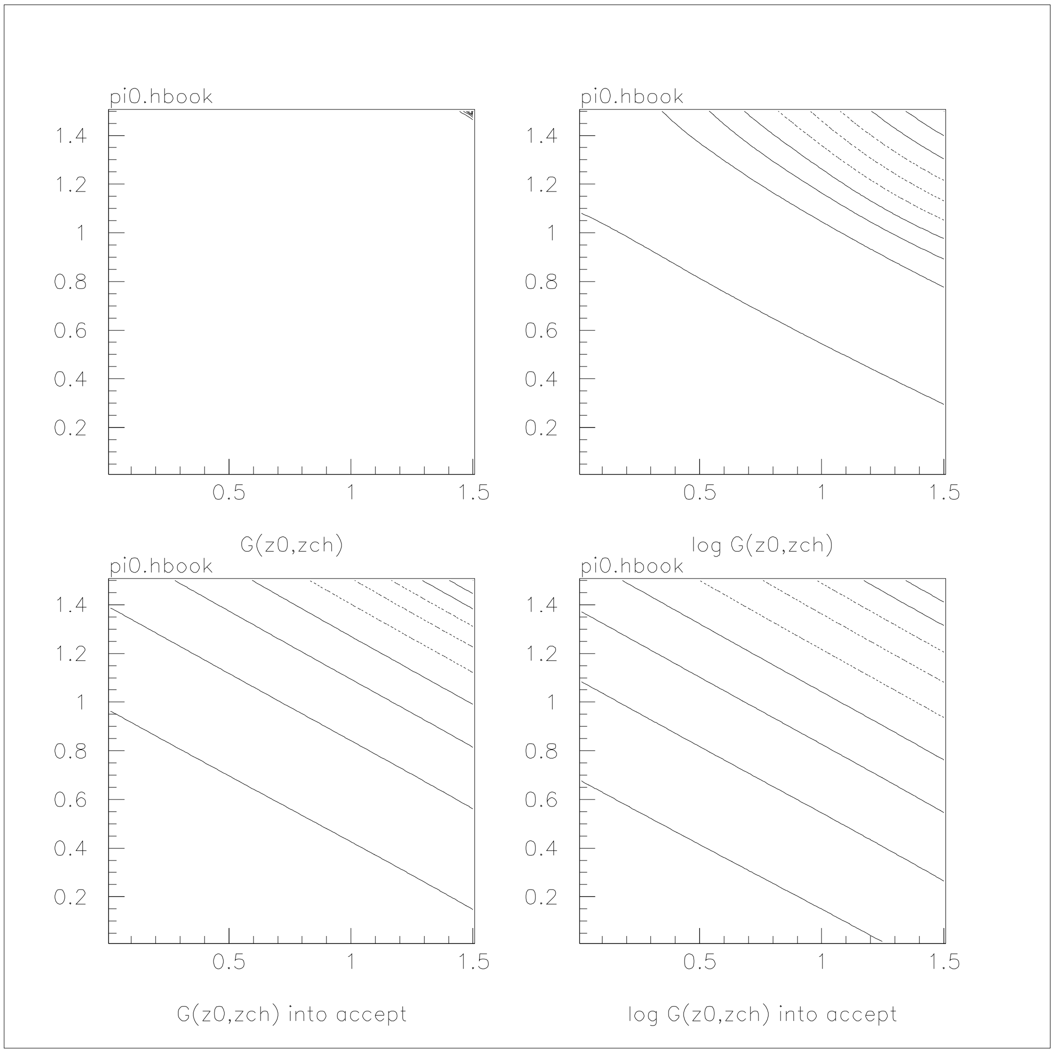

Figure 5.1: Contour plots of the generating function $G(z_{ch}, z_0)$ and $\log G(z_{ch}, z_0)$ from all PYTHIA charged and neutral pions, and from those entering the MiniMax acceptance (the vertical axis is z_{ch} and the horizontal is z_0).

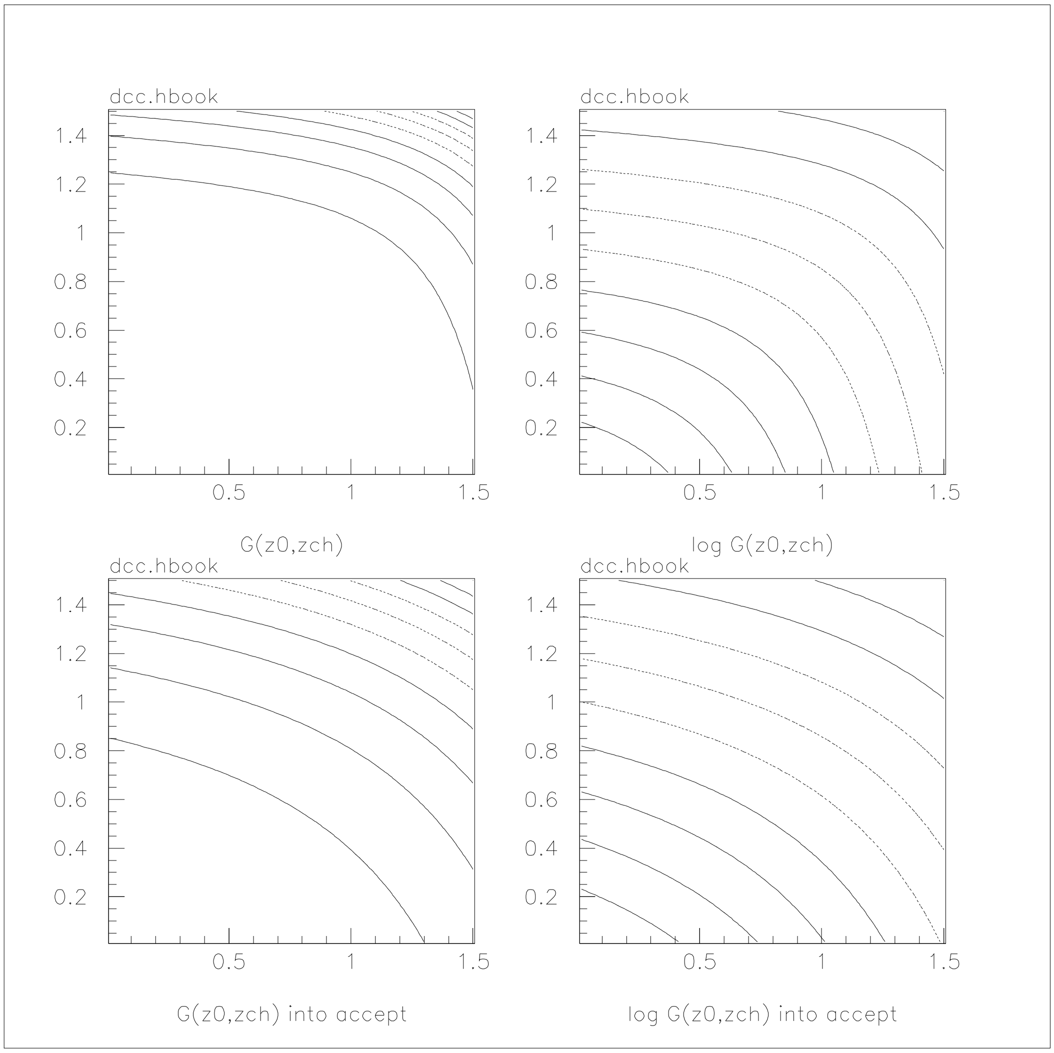

Figure 5.2: Contour plots of the generating function $G(z_{ch}, z_0)$ and $\log G(z_{ch}, z_0)$ from all charged and neutral pions from the DCC generator, and from those entering the MiniMax acceptance (the vertical axis is z_{ch} and the horizontal is z_0).