# A Review of Some Exact Solutions to the Planar Equations of Motion of a Thrusting Spacecraft 

Anastassios E. Petropoulos* and Jon A. Sims ${ }^{\dagger}$<br>Jet Propulsion Laboratory<br>California Institute of Technology<br>Mail Stop 301-140L<br>4800 Oak Grove Drive<br>Pasadena, CA 91109-8099<br>USA


#### Abstract

With the complexities in computing optimal low thrust trajectories, easily-computed, good sub-optimal trajectories provide both a practical alternative for mission designers and a starting point for optimisation. The present paper collects in one place for easy reference and comparison several exact solutions that have been obtained in the literature over the last few decades: The logarithmic spiral, Pinkham's variant thereof, Forbes' spiral, the exponential sinusoid, the case of constant radial thrust, Markopoulos's Keplerian thrust arcs, Lawden's spiral, and the analogous Bishop and Azimov spiral. For most of these, the shape of the trajectory, the velocity at any point, and the requisite thrust magnitude and direction are available analytically in terms of the initial conditions and any remaining free trajectory parameters. The thrust and delta-vee characteristics are explored for some of the solutions, with applicability to both spiralling transfers and interplanetary trajectories.


## Nomenclature

| $a$ | Thrust acceleration normalised by local gravitational acceleration |
| :---: | :---: |
| $c$ | 1) $\cos \left(k_{2} \theta+\phi\right)$ (exponential sinusoid), or 2) $\cos \alpha$ (Lawden's spiral) |
| $f$ | Thrust acceleration, $f=F / m\left(\mathrm{~ms}^{-2}\right)$ |
| $g_{\oplus}$ | Acceleration due to gravity at the Earth's surface ( $\mathrm{ms}^{-2}$ ) |
| $h$ | Orbital angular momentum per unit mass ( $\mathrm{m}^{2} / \mathrm{s}$ ) |
| $k$ | Shape parameter for Pinkham's spiral |
| $k_{0}$ | Scale parameter for the exponential sinusoid (m) |
| $k_{1}$ | Dynamic range parameter for the exponential sinusoid |
| $k_{2}$ | Winding parameter for the exponential sinusoid |
| $m$ | Spacecraft mass (kg) |
| $n$ | Mean motion of a conic orbit (1/s) |
| $p$ | Semi-latus rectum of a conic orbit (m) |
| $q$ | Parameter for the logarithmic spiral ( $q=\tan \gamma$ ) and related curves |
| $r$ | Radial distance from the central body (m) |
| $s$ | 1) $\sin \left(k_{2} \theta+\phi\right)$ (exponential sinusoid), or 2) $\sin \alpha$ (Lawden's spiral) |
| $t$ | Physical time (s) |
| $v$ | Spacecraft velocity (m/s) |
| $v_{c}$ | Circular orbit speed at the current radius ( $\mathrm{m} / \mathrm{s}$ ) |
| $\boldsymbol{v}_{\infty}$ | Hyperbolic excess velocity ( $\mathrm{m} / \mathrm{s}$ ) |
| $w$ | Speed perpendicular to the thrust vector (in same sense as $\hat{\theta}$ ) (m/s) |
| $\dot{A}$ | Rate at which the radius vector sweeps out area ( $\mathrm{m}^{2} / \mathrm{s}$ ) |
| $B$ | Arbitrary constant for trajectories using Keplerian thrust programs |
| $E$ | Generalised eccentric anomaly (rad.) |
| F | Thrust (kgms ${ }^{-2}$ ) |
| $N$ | Number of revolutions |
| $Q$ | Throttling function for Keplerian thrust ( $\mathrm{m}^{2} \mathrm{~s}^{-2}$ ) |

[^0]| $\alpha$ | Thrust angle (rad.) |
| :--- | :--- |
| $\gamma$ | Flight path angle (rad.) |
| $\gamma_{q}$ | Pseudo flight path angle for Pinkham's spiral (rad.) |
| $\theta$ | Polar angle (rad.) |
| $\mu$ | Gravitational parameter of the central body $\left(\mathrm{m}^{3} \mathrm{~s}^{-2}\right)$ |
| $\phi$ | Phase angle for exponential sinusoid (rad.) |
| $\omega$ | Phase parameter for Pinkham's spiral (rad.) |
| $\Delta V$ | Velocity change or characteristic velocity, $\int f \mathrm{~d} t(\mathrm{~m} / \mathrm{s})$ |
| ()$_{0}$ or () | Initial value |
| ()$_{f}$ or () | Final value |
| ()$_{e}$ | Value at escape from the central body |
| ()$_{r}$ | Radial component |
| ()$_{s}$ | Scaling quantity |
| ()$_{\theta}$ | Circumferential component |
| $\dot{O}$ | Derivative with respect to time |
| ()$^{\prime}$ | Derivative with respect to a variable other than $t$ |

## 1 Introduction

In the past decade, a considerable body of research has addressed the problem of finding optimal lowthrust trajectories. The focus has been largely on computational aspects of the problem, as analytic solutions are available for only highly specialised cases. Notably, it has been found that in the cases where the thrusting occurs over many (e.g. hundreds) of revolutions, or is accompanied by numerous (e.g. three or more) gravity assists, the problem of optimisation becomes significantly more difficult. Given this difficulty, it seems prudent to investigate simple but good sub-optimal solutions as estimates in preliminary mission design and as guides and initial guesses in optimisation.

The present paper collects in one place some of the exact analytic solutions that have been found over the last five or so decades to the planar equations of motion for a thrusting spacecraft. As suggested by several examples, some of these solutions could find utility in preliminary design of orbit transfers and escape or capture trajectories, both for the multi-revolution spiralling case and for the case of few or no revolutions.

The analytic solutions presented are those of the logarithmic spiral [1-3], Pinkham's logarithmic spiral variant [4], Forbes' spiral [1], the exponential sinusoid [5], the case of constant radial thrust [6-9], Markopoulos's Keplerian thrust programs [10], Lawden's spiral [11,12], and the analogous spiral of Bishop and Azimov [13]. Apart from the constant radial thrust case, the solutions require a variable thrust magnitude. Some of the solutions, however, offer enough degrees of freedom, or are inherently so structured, as to permit nearly-constant thrust or thrust-acceleration.

Polar coordinates are used in the analyses, with various geometrical quantities defined in Fig. 1. With these definitions, the familiar equations of motion (EOMs) are

$$
\begin{align*}
\ddot{r}-r \dot{\theta}^{2}+\frac{\mu}{r^{2}} & =f \sin \alpha  \tag{1}\\
\frac{1}{r} \frac{d}{d t}\left(r^{2} \dot{\theta}\right) & =f \cos \alpha \tag{2}
\end{align*}
$$

Some of the analyses may be described as shape-based, that is, rather than trying to determine the trajectory followed due to an arbitrarily chosen thrust, the trajectory shape is itself directly assumed, with the requisite thrust computed a posteriori. By making a wise choice for the type of shape to use, only a few shape parameters will be needed to obtain a variety of realistic trajectory arcs. The thrust profile will typically not be unique, unless additional assumptions are made, such as constraining the thrust to lie along the velocity vector. The shape-based approach might be generalised to include any assumption regarding the orbit (other than thrust), not just the shape of the orbit, which allows the equations of motion to be simplified. The constant radial thrust case clearly does not fall in this category, and nor do the analyses of Lawden, Markopoulos and Bishop and Azimov.


Fig. 1 Geometry of a two-dimensional, thrusting arc in polar coordinates.

## 2 The shape-based approach

In the case where the shape of the orbit is known (or assumed), by substituting the shape function $r(\theta)$ into the EOMs, we can obtain an expression governing the angular rate, $\dot{\theta}$ :

$$
\begin{equation*}
\dot{\theta}^{2}=\left(\frac{\mu}{r^{3}}\right) \frac{a \sec \gamma \sin (\gamma-\alpha)+1}{2 \tan ^{2} \gamma-r^{\prime \prime} / r+1} \tag{3}
\end{equation*}
$$

where the prime, ('), here denotes differentiation with respect to $\theta$, the flight path angle, $\gamma$, is easily shown to be

$$
\begin{equation*}
\tan \gamma=\frac{1}{r} \frac{\mathrm{~d} r}{\mathrm{~d} \theta} \equiv \frac{r^{\prime}}{r} \tag{4}
\end{equation*}
$$

and $a$ is the ratio of the thrust acceleration to the local gravitational acceleration:

$$
\begin{equation*}
a \equiv \frac{f}{\mu / r^{2}} \tag{5}
\end{equation*}
$$

The quantity $a$ may be thought of as the thrust to local-weight ratio, or as the thrust acceleration expressed in units of local central-body "gee's." The differential equations governing the requisite $a$ and $\alpha$ may be derived from Eq. 3 and the EOMs, but these are rather lengthy and we therefore restrict ourselves to simplifications for particular shapes below. Typically, one would want $a$ to be a small number. For example, NASA's Deep Space 1 spacecraft had a maximum $a$ of around 0.04 just after launch from Earth. The normalisation by the local gravitational acceleration is particularly convenient for interplanetary studies involving solar electric propulsion, as the maximum thrust often depends almost linearly on the ion engine input power from the solar arrays (within the throttle range of the thruster), which, in turn, drops off roughly in proportion to $1 / r^{2}$. Thus, the maximum $a$ remains roughly constant, except for its gradual increase due to the reduction of spacecraft mass from propellant expenditure.

## 3 The logarithmic spiral

Perhaps the first instance in the literature of an assumed orbit shape is the logarithmic spiral, given in polar coordinates by

$$
\begin{equation*}
r=r_{0} \mathrm{e}^{q\left(\theta-\theta_{0}\right)} \tag{6}
\end{equation*}
$$

Low-thrust travel along the logarithmic spiral was mentioned as early as 1950 by Forbes [1] and 1959 by Tsu [2]. Unfortunately, while various analytic results are available for the logarithmic spiral, having essentially just one free parameter, $q$, the shape is rather restrictive and of limited practical utility. However,
it is interesting to note that in the case where both the thrust to local-weight ratio, $a$, and the thrust angle, $\alpha$, are constant, radius and polar angle are available as explicit functions of time:

$$
\begin{align*}
\theta & =\frac{2}{3 q} \ln \left(\tilde{n} t+\mathrm{e}^{\frac{3}{2} q \theta_{0}}\right)  \tag{7}\\
r & =r_{0}\left(\tilde{n} t+\mathrm{e}^{\frac{3}{2} q \theta_{0}}\right)^{\frac{2}{3}} \tag{8}
\end{align*}
$$

where, for convenience, we define what might be termed the logarithmic mean motion as

$$
\begin{equation*}
\tilde{n}= \pm \frac{3 q}{2} \sqrt{\frac{\mu}{r_{0}^{3}}} \sqrt{\frac{[(q \cos \alpha-\sin \alpha) a+1]}{\left(1+q^{2}\right)}} \tag{9}
\end{equation*}
$$

We note that $\tilde{n}$ is constant, since $a, \alpha$, and $q$ are all constant. In references [3,5], an expression is given for $a$ in terms of $r$, when only $\alpha$ and $q$ are assumed constant, only that in this case $r$ and $\theta$ are not available as explicit functions of time. However, even in the most general case, where also $\alpha$ is allowed to vary, an expression is supplied which shows that compared to a Hohmann transfer, a logarithmic spiral transfer will always have either a much longer time of flight, or a much larger launch $v_{\infty}$, or both. This drawback may also be seen geometrically: Due to the constant flight path angle, a considerable $\Delta V$ is required to transfer from a circular orbit onto the spiral, or the converse, unless the flight path angle is exceedingly small, which would mean very long flight times for any meaningful excursions in radius. One interesting observation for the logarithmic spiral is that the $\Delta V$ expended whilst thrusting on the spiral from radius $r_{1}$ to radius $r_{2}>r_{1}$ is exactly

$$
\begin{equation*}
\Delta V=v_{c 1}-v_{c 2} \tag{10}
\end{equation*}
$$

for the case where tangential thrust acceleration, proportional to $1 / r^{2}$, is assumed. This $\Delta V$ does not include any $\Delta V$ required for enjoining the spiral or departing from it.

## 4 Pinkham's logarithmic spiral variant

A remarkable variant on the logarithmic spiral is given by Pinkham [4], who finds that if the thrustacceleration components are given by

$$
\begin{align*}
f_{r} & =\frac{q \sqrt{\mu p}}{2 r^{2}} v_{r}  \tag{11}\\
f_{\theta} & =\frac{q \sqrt{\mu p}}{2 r^{2}} v_{\theta} \tag{12}
\end{align*}
$$

where $q$ is a constant and $p$ is the semi-latus rectum of the osculating orbit, then the equations of motion have the solution

$$
\begin{align*}
p & =p_{s} \mathrm{e}^{q \theta}  \tag{13}\\
r & =\frac{p\left(1+q^{2}\right)}{1+\mathrm{e}^{q \theta}\left(1+q^{2}\right) k \cos (\theta-\omega)} \tag{14}
\end{align*}
$$

where $p_{s}, k$, and $\omega$ are arbitrary constants. Unlike the logarithmic spiral, Pinkham's spiral has three constants (apart from scaling and phasing factors) available for matching particular boudary conditions or for optimisation. Also unlike the logarithmic spiral, hyperbolic speeds can be attained on this shape after an initial spiralling period. The thrust is seen to be tangential, and vary approximately as $1 / r^{2}$, making this shape suitable for solar electric propulsion on interplanetary legs where available power is less than the engine's maximum input power. The flight path angle is easily obtained from Eq. 14 as

$$
\begin{equation*}
\tan \gamma=q+\frac{r}{p_{s}} k \sqrt{1+q^{2}} \sin \left(\theta-\omega-\gamma_{q}\right) \tag{15}
\end{equation*}
$$

where

$$
\begin{equation*}
\tan \gamma_{q} \equiv q \tag{16}
\end{equation*}
$$

The polar angle rate is found from Eqs. 3 and 14 to be given by

$$
\begin{equation*}
\dot{\theta}^{2}=\left(\frac{\mu}{r^{3}}\right) \frac{1}{\left(1+q^{2}\right)\left[1-\frac{r}{p_{s}} k \cos (\theta-\omega)\right]} \tag{17}
\end{equation*}
$$

Pinkham's spiral can be used, for example, to escape from an initially circular orbit, or from any point on an elliptic orbit. In this case, only one of the three parameters remains free, most conveniently taken as $q$. A polar plot of departure from a circular orbit is shown in Fig. 2 for $q=0.1$. The corresponding speed, normalised by the local circular speed, $\sqrt{\mu / r}$, is shown as a function of polar angle in Fig. 3. The thrust magnitude for this trajectory is shown in Figs. 4 and 5 as a function of polar angle and radius respectively. While thrust is not a single-valued function of radius, it does indeed drop-off roughly as $1 / r^{2}$. As evident from the shape and thrust equations, a higher value for the parameter $q$ would translate to higher thrust levels, and escape in fewer revolutions. The freedom to choose the parameter $q$, given initial position and velocity vectors, makes this shape much more useful than the logarithmic spiral. Apart from this noteworthy solution, Pinkham also gives simple equations for using his spiral as a reference trajectory in cases where the thrust does not obey precisely the requisite form of Eqs. 11 and 12.


Fig. 2 Pinkham's spiral, with $q=0.1$ and starting in circular orbit.

## 5 Forbes' spiral

Turning now to assumptions other than trajectory shape, we note Forbes' [1] observation that practical and analytic results can be obtained if the thrust is assumed collinear with the velocity $(\sin (\gamma-\alpha)=0)$, and the circumferential speed, $v_{\theta}$, is assumed equal to the local circular speed, $v_{c},\left(\dot{\theta}^{2}=\mu / r^{3}\right)$. His results can be reproduced from within the present analytical framework by noting from Eq. 3 that the assumptions result in

$$
\begin{equation*}
r r^{\prime \prime}-2{r^{\prime}}^{2}=0 \tag{18}
\end{equation*}
$$

which may be solved to give:

$$
\begin{equation*}
r=\frac{r_{0}}{1-\left(\theta-\theta_{0}\right) \tan \gamma_{0}} \tag{19}
\end{equation*}
$$

Forbes gives the requisite thrust acceleration as

$$
\begin{equation*}
f=\frac{\mu}{r_{0}^{2}} \cdot \frac{\tan ^{2} \gamma_{0}}{2 \sin \gamma} \tag{20}
\end{equation*}
$$

Exact analytic expressions are also available for the time of flight and the speed. These results are applicable to spiralling outwards after starting with an arbitrarily small initial flight path angle and $a_{0} \approx \gamma_{0} / 2$.


Fig. 3 Speed on Pinkham's spiral of Fig. 2, normalised by local circular speed.


Fig. 4 Normalised thrust acceleration as a function of polar angle for the Pinkham spiral of Fig. 2.


Fig. 5 Normalised thrust acceleration as a function of radius for the Pinkham spiral of Fig. 2.

Moeckel [14] also studies the $v_{\theta}=v_{c}$ assumption, but without the tangential thrust requirement. With the simplifications afforded to the equations of motion, he is able to assume $r$ to be an arbitrary function of time, and determine the thrust profile therefrom. He gives the example of a circle to circle orbit transfer, and the selection of a simple function $r(t)$ which satisfies the boundary conditions on the transfer arc. Presumably, arbitrary constants could be retained in the chosen function, so as to permit some sort of optimisation.

## 6 Exponential sinusoid

The exponential sinusoid was developed [5] in an attempt to correct the shortcomings of the logarithmic spiral - namely its constant flight path angle. The shape is given in polar coordinates by

$$
\begin{equation*}
r=k_{0} \mathrm{e}^{k_{1} \sin \left(k_{2} \theta+\phi\right)} \tag{21}
\end{equation*}
$$

Having two parameters, $k_{1}$ and $k_{2}$, apart from the scaling and phase parameters, gives the shape more flexibility than the logarithmic spiral, but not quite as much as Pinkham's spiral. The flight path angle on the shape is given by:

$$
\begin{equation*}
\tan \gamma=k_{1} k_{2} c \tag{22}
\end{equation*}
$$

An example of the shape is shown in Fig. 6. By adjusting the parameter $k_{2}$, a smaller or greater number of revolutions may be obtained betweeen periapsis and apoapsis. The fewer (or no) revolution spiral is suitable for interplanetary transfers, and the many-revolution spiral is suitable for orbit transfer, in particular between circular orbits. One of the simplest cases studied is that of tangential thrust, which results in the following expressions for angular rate and normalised thrust acceleration:

$$
\begin{align*}
\dot{\theta}^{2} & =\left(\frac{\mu}{r^{3}}\right) \frac{1}{\tan ^{2} \gamma+k_{1} k_{2}^{2} s+1}  \tag{23}\\
a & =\frac{(-1)^{n} \tan \gamma}{2 \cos \gamma}\left[\frac{1}{\tan ^{2} \gamma+k_{1} k_{2}^{2} s+1}-\frac{k_{2}^{2}\left(1-2 k_{1} s\right)}{\left(\tan ^{2} \gamma+k_{1} k_{2}^{2} s+1\right)^{2}}\right] \tag{24}
\end{align*}
$$

where $n$ is chosen as 0 or 1 so as to make $a$ positive; the thrust is then directed either along ( $\alpha=\gamma$ ) or against ( $\alpha=\gamma+\pi$ ) the velocity according to

$$
\begin{equation*}
\alpha=\gamma+n \pi \tag{25}
\end{equation*}
$$

For small values of $k_{1}$ and $k_{2}$, the thrust will typically be directed along the velocity vector for outbound spacecraft, and against the velocity for inbound spacecraft.


Fig. 6 Part of an exponential sinusoid arc.
To use the exponential sinusoid shape for transferring between two circular orbits, it is most advantageous to enjoin and leave the exponential sinusoid at its apses, since the required velocity at these points on the exponential sinusoid is parallel and almost equal to the local circular velocity. In particular, supposing the initial orbit to be of radius $r_{1}$ and the final orbit of radius $r_{2}>r_{1}$, we find from Eq. 23 the following $\Delta V$ s:

$$
\begin{align*}
& \Delta V_{1}=v_{c 1}\left(\frac{1}{\sqrt{1-k_{1} k_{2}^{2}}}-1\right) \approx v_{c 1} \frac{\ln \left(r_{2} / r_{1}\right)}{16 N^{2}}  \tag{26}\\
& \Delta V_{2}=v_{c 2}\left(1-\frac{1}{\sqrt{1+k_{1} k_{2}^{2}}}\right) \approx v_{c 2} \frac{\ln \left(r_{2} / r_{1}\right)}{16 N^{2}} \tag{27}
\end{align*}
$$

where $\Delta V_{1}$ is the impulsive velocity increment needed to enjoin the exponential sinusoid at its periapsis from the circular orbit at $r_{1}, \Delta V_{2}$ is the corresponding velocity increment needed to enter circular orbit at $r_{2}$, and $N$ is the number of revolutions around the central body required for the transfer. The $\Delta V$ expended whilst thrusting on the exponential sinusoid arc itself, denoted $\Delta V_{a}$, is found by integrating Eq. 24:

$$
\begin{equation*}
\Delta V_{a} \approx v_{c 1}-v_{c 2} \tag{28}
\end{equation*}
$$

It should be noted that the following constraint must be met in order to travel along the exponential sinusoid using tangential thrust:

$$
\begin{equation*}
N \geq \sqrt{\frac{1}{8} \ln \left(r_{2} / r_{1}\right)} \tag{29}
\end{equation*}
$$

This requirement is most easily seen by noting the singularity that occurs in Eq. 26 when $k_{1} k_{2}^{2} \geq 1$.
For illustrative purposes, the variation of thrust acceleration over polar angle is shown in Fig. 7 for a sample 200 -revolution transfer with a five-fold increase in radius. The thrust acceleration is normalised by the acceleration due to gravity at $r_{1}$, rather than by the local gravity, since, for an orbit transfer, one might expect constant power and thrust levels to be available for the duration of the transfer, unlike the interplanetary case. For comparison, the normalised acceleration is about $10^{-4}$ for a 100 kg spacecraft using an NSTAR engine at maximum thrust (about 95 mN ) in low Earth orbit. For the same spacecraft on the surface of the asteroid Vesta, the normalised acceleration would be about $3.4 \times 10^{-3}$. Greater insight can be gained from Fig. 8, which shows how the maximum thrust required on the transfer varies with the number of revolutions and the ratio of final to initial radius. Also, Fig. 9 shows that the total $\Delta V$ required for transfer (i.e., $\Delta V_{1}+\Delta V_{a}+\Delta V_{2}$ ) is slightly less than $v_{c 1}-v_{c 2}$.


Fig. 7 Thrust profile for a 200 revolution exponential-sinusoid transfer.

## 7 Radial thrust

Tsien [6] shows that exact analytic solutions, in terms of elliptic integrals, are available for the case of constant radial thrust (see also Battin [7,8]). Unfortunately, from an initially circular orbit, this thrust scheme requires very high thrust levels to obtain any significant excursions in radial distance (see also the more recent analysis by Prussing and Coverstone-Carroll [9]). Perhaps the equations could find utility in the case of near-radial motion (such as near the asymptotes of hyperbolas, or along highly elongated ellipses), since radial thrust would nearly maximise the rate of change of energy of such motion. The related case of radial thrust acceleration inversely proportional to the square of the radius, while obviously tractable analytically, is also of little practical utility. Whittaker [15] notes that all central forces depending only on radial distance permit solutions to the EOMs in terms of quadratures.

## 8 Markopoulos's Keplerian thrust programs

While studying the problem of optimal, powered spacecraft motion, Markopoulos [10] happened upon a class of analytic solutions, separate from the optimisation problem, which have close analogues to pure


Fig. 8 Maximum thrust acceleration as a function of the number of revolutions required for exponentialsinusoid transfer for various ratios of final to initial radius.


Fig. 9 The total exponential-sinusoid transfer $\Delta V$ compared with $\left(v_{c 1}-v_{c 2}\right)$ as a function of number of revolutions, for various ratios of final to initial radius.

Keplerian motion. Markopoulos defines the term "Keplerian thrust" to mean a thrust program wherein

$$
\begin{align*}
f_{\theta} & =\frac{Q}{r}  \tag{30}\\
f_{r} & =\frac{Q \dot{r}+2 \dot{Q} r}{h} \tag{31}
\end{align*}
$$

and $Q$ is an arbitrary, explicit, differentiable function of time. It is worth interjecting here that Forbes [1] makes a more general observation regarding an extension to Kepler's second law of planetary motion. From the equations of motion he finds that

$$
\begin{equation*}
\frac{f}{\dot{r} \sin \gamma}=\frac{\ddot{A}}{\dot{A}} \tag{32}
\end{equation*}
$$

This clearly reduces to "equal areas in equal time" when $f=0$. Eq. 32 is certainly very elegant and worthy of further study. Forbes himself obtains solutions for $f$ in the special cases noted in previous subsections, and states that "undoubtedly, further relations can be found which would lead to solutions." Markopoulos finds such solutions, apparently unaware of Forbes' remarks. He terms his thrust profile Keplerian because he is able to reduce the shape of the orbit, $r(\theta)$, to functional forms akin to Kepler's conic relations, both elliptic and hyperbolic. Using the normalisations $\tilde{r}=r / r_{s}$ and $\tilde{h}=h / \sqrt{\mu r_{s}}$, the shape of the elliptic-type trajectory is given by

$$
\begin{align*}
\tilde{r} & =\frac{\tilde{h}^{2}}{1-B^{2}} \cdot(1-B \cos E)  \tag{33}\\
\tan \frac{\theta}{2} & =\sqrt{\frac{1+B}{1-B}} \tan \frac{E}{2} \tag{34}
\end{align*}
$$

where, $B \in(0,1)$ is an arbitrary constant, $E$ is what Markopoulos terms the generalised eccentric anomaly, and the angular momentum $h$ is given by

$$
\begin{equation*}
h=h_{0}+\int_{0}^{t} Q \mathrm{~d} t \tag{35}
\end{equation*}
$$

The angular momentum is also related to $E$ by

$$
\begin{equation*}
\int_{0}^{\bar{t}} \frac{1}{\tilde{h}^{3}} \mathrm{~d} \tilde{t}=\frac{1}{\left(1-B^{2}\right)^{3 / 2}} \cdot\left[E-E_{0}-B\left(\sin E-\sin E_{0}\right)\right] \tag{36}
\end{equation*}
$$

where the normalisation $\tilde{t}=t \sqrt{\mu / r_{s}^{3}}$ has been used. Analogous results are available for the case of $B>1$, describing hyperbolic-type trajectories, and simple results are available for the special cases $B=0$ and $B=1$.

Markopoulos notes that when $Q=Q_{0}$ is constant, the thrust is tangential. The thrust acceleration is then found from Eqs. 30 and 31 to be:

$$
\begin{equation*}
f=\frac{Q_{0}}{r \cos \gamma} \tag{37}
\end{equation*}
$$

The angular momentum for the tangential thrust case may also be expressed as a function of $E$, using Eqs. 35 and 36:

$$
\begin{equation*}
\tilde{h}^{2}=\left\{\frac{2 \tilde{Q}_{0}}{\left(1-B^{2}\right)^{3 / 2}} \cdot\left[E-E_{0}-B\left(\sin E-\sin E_{0}\right)\right]+1\right\}^{-1} \tag{38}
\end{equation*}
$$

where $\tilde{Q}_{0}=Q_{0} /\left(\mu / r_{s}\right)$.
Other functional forms for $Q$ will typically only permit the $\tilde{h}$ of Eq. 33 to be defined implicitly in terms of $E$ from Eqs. 35 and 36 (each of which involve a quadrature). While the thrust profile cannot be chosen explicitly as a function of time, by choosing a suitable form for $Q$, a desired thrust profile could be approximated. This drawback aside, there are two main advantages to this formulation, apart from the clear functional similarities to the conic orbits. The first is the analytic nature of the equations, which makes them suitable for preliminary mission design - for example, by assuming $Q$ is a polynomial in time, its coefficients can be used to meet a large variety of boundary conditions, such as would arise in problems of escape, transfer, and rendezvous. The second advantage is that, according to Markopoulos, Keplerian thrust programs are "near optimal" with regard to propellant, for power-limited systems.

## 9 Lawden's spiral

For the case of constant $I_{s p}$, but variable, limited thrust, Lawden [11,12] has shown that arcs of intermediate thrust can satisfy necessary condtions for optimality with regard to propellant mass for time-free transfers. In particular, Lawden found a specific, two-dimensional spiral which satisfied the necessary conditions. Although later research (see for example Refs. [16-19]) has shown Lawden's spiral to be non-optimal on account of various additional necessary conditons not being met, it is worth examining its characteristics. Lawden's spiral is given by:

$$
\begin{align*}
r & =\frac{r_{s} s^{6}}{1-3 s^{2}}  \tag{39}\\
w & =-\sqrt{\frac{\mu}{r_{s}}} \cdot \frac{1}{s^{2}}  \tag{40}\\
\theta & =\theta_{0}-4 \alpha-3 \cot \alpha  \tag{41}\\
v_{r} & =\sqrt{\frac{\mu}{r_{s}}} \cdot \frac{6\left(1-2 s^{2}\right) c}{s^{2}\left(3-5 s^{2}\right)}  \tag{42}\\
v_{\theta} & =\sqrt{\frac{\mu}{r_{s}}} \cdot \frac{\left(3-4 s^{2}\right)\left(1-3 s^{2}\right)}{s^{3}\left(3-5 s^{2}\right)}  \tag{43}\\
f & =\sqrt{\frac{\mu}{r_{s}^{2}}}\left(\frac{1-3 s^{2}}{3-5 s^{2}}\right)^{3} \frac{1}{s^{1} 1} \cdot\left(27-75 s^{2}+60 s^{4}\right)  \tag{44}\\
\Delta V & =\text { const }-\sqrt{\frac{\mu}{r_{s}}} \cdot \frac{3\left(1-2 s^{2}\right)\left(1-5 s^{2}\right) c}{s^{3}\left(3-5 s^{2}\right)} \tag{45}
\end{align*}
$$

Although Lawden's spiral was developed with transfer between two arbitrary states in mind, the spiral does not offer enough degrees of freedom to accomplish this. For example, if an initial radius and flight path angle are specified, then all other quantities on the spiral are determined. Fig. 10 shows a portion of the spiral, where the initial radius is taken as unity, and the initial $\alpha$ is taken as $5^{\circ}$, which corresponds to $\gamma \approx 10^{\circ}$. The required thrust to local-weight ratio at this initial flight path angle is very high, however.


Fig. 10 A portion of Lawden's spiral.
More realistic thrust to local-weight ratios require much smaller initial flight-path angles. The required thrust acceleration does decrease, however, as radial distance increases. These two characteristics are indicated in Fig. 11 which shows the variation of the thrust acceleration (normalised by the initial gravitational acceleration) as a function of revolutions made around the central body, for various initial fight-path angles. Fig. 12 shows the same curves, but as a function of time on the spiral (normalised by the period of a circular orbit at the initial radius). Normalised thrust above $10^{-4}$ is only required for a relatively short initial period,
compared to the time needed for escape. The corresponding $\Delta V$ that is expended whilst travelling along the spiral is shown in Fig. 13 as a function of the number of revolutions. The vertical axis in the figure shows the fraction by which the $\Delta V$ is less than $\left(v_{c 1}-v_{c}\right)$, that is, the difference between the initial circular orbit speed and the circular orbit speed at the current radius. Contours are shown for various initial flight-path angles, and also for various excursions in radial distance as well as the point of escape. For example, a hundred-fold increase in initial radius requires about 590 turns if the initial flight path angle is $0.05^{\circ}$, whereas it requires about 420 turns for $\gamma_{1}=0.07^{\circ}$. The $r / r_{1}$ and escape contours are shown as linear fits between data points on the $\gamma_{1}$ contours.


Fig. 11 Thrust acceleration for the Lawden spiral, normalised by initial gravitational acceleration, as a function of revolutions completed around the central body, for various initial flight-path angles.


Fig. 12 Thrust acceleration for the Lawden spiral, normalised by initial gravitational acceleration, as a function of time travelled on the spiral, for various initial flight-path angles.


Fig. $13 \Delta V$ expended on Lawden's spiral, compared to ( $v_{c 1}-v_{c}$ ) as a function of revolutions completed around the central body for various initial flight-path angles and including contours for various increases in radial distance.

We see graphically in Fig. 13 that

$$
\begin{equation*}
\Delta V \approx v_{c 1}-v_{c} \tag{46}
\end{equation*}
$$

for shallow initial flight-path angles. This same result can be found by examining Eq. 45. It should be noted, however, that if we include the $\Delta V$ required to enjoin Lawden's spiral from a circular orbit and to circularise once a certain radius is reached, then the total $\Delta V$ will exceed $\left(v_{c 1}-v_{c}\right)$ by much more than $1 \%$ for revolutions below about 100, and by about $0.1-0.2 \%$ for revolutions above about 400 , for the initial flight-path angles shown in Fig. 13.

## 10 Bishop and Azimov spiral

The case of a power-limited, variable- $I_{s p}$ engine operating at maximum power and inside (not on) the $I_{s p}$ bounds is studied by Bishop and Azimov [13]. Based on the assumptions, the thrust is also between an upper and a lower bound, paralleling the intermediate-thrust case of Lawden, except that $c$ rather than $\beta$ is now the control providing the variable thrust. Bishop and Azimov obtain an explicit analytic description of planar spiral trajectories that are optimal in terms of propellant mass. The equations are parametrised using the thrust angle $\alpha$, just as in the case of the Lawden spiral. They state that their spiral can be used to transfer optimally between ellipses, but not between perfectly circular orbits. We note, however, that just as in the case of Lawden's spiral, there is no general analytic criterion which specifies when the use of intermediate thrust actually satisfies the necessary conditions for optimality.

## 11 Conclusions

A number of special solutions to the planar equations of motion of a thrusting spacecraft have been collected from the literature. Some of their characteristics are described in detail, others left for the interested reader to pursue further. The solutions cover a spectrum of characteristics, permitting the mission designer to choose the most appropriate one for the problem at hand, whether that be a rough estimate of key parameters for a specific trajectory type, an initial guess for use in optimisation, or even an automated search for low-thrust trajectories over varied boundary conditions.

## Acknowledgements

This research was carried out at the Jet Propulsion Laboratory, California Institute of Technology, under a contract with the National Aeronautics and Space Administration

## References

${ }^{1}$ Forbes, G. F., "The Trajectory of a Powered Rocket in Space," Journal of the British Interplanetary Society, Vol.9, No.2, 1950, pp.75-79, (abridgement of Master's Thesis submitted to the Massachusetts Institute of Technology).
${ }^{2}$ Tsu, T. C., "Interplanetary Travel by Solar Sail," J. American Rocket Society, Vol.29, 1959, pp.422-427.
${ }^{3}$ Petropoulos, A.E., Longuski, J.M., and Vinh, N.X., "Shape-Based Analytic Representations of Low-Thrust Trajectories for Gravity-Assist Applications," AAS/AIAA Astrodynamics Specialist Conference, AAS Paper 99-337, Girdwood, Alaska, Aug. 1999.
${ }^{4}$ Pinkham, G., "Reference Solution for Low Thrust Trajectories," J. American Rocket Society, Vol.32, No.5, May 1962, pp.775-776.
${ }^{5}$ Petropoulos, A. E., "A Shape-Based Approach to Automated, Low-Thrust, Gravity-Assist Trajectory Design," Ph.D. Thesis, School of Aeronautics and Astronautics, Purdue University, West Lafayette, IN, May 2001.
${ }^{6}$ Tsien, H. S., "Take-off from Satellite Orbit," J. American Rocket Society, Vol.23, July-Aug. 1953, pp.233-236.
${ }^{7}$ Battin, R. H., Astronautical Guidance, McGraw-Hill, New York, 1964.
${ }^{8}$ Battin, R. H., An Introduction to the Mathematics and Methods of Astrodynamics, 1st ed. 4th printing, AIAA, New York, 1987.
${ }^{9}$ Prussing, J. E. and Coverstone-Carroll, V., "Constant Radial Thrust Acceleration Redux," J. Guidance, Control, and Dynamics, Vol.21, No.3, May-June 1998, pp.516-518.
${ }^{10}$ Markopoulos, N., "Non-Keplerian Manifestations of the Keplerian Trajectory Equation and a Theory of Orbital Motion Under Continuous Thrust," AAS/AIAA Space Flight Mechanics Meeting, AAS Paper 95-217, Albuquerque, New Mexico, Feb. 1995.
${ }^{11}$ Lawden, D. F., "Optimal Intermediate-Thrust Arcs in a Gravitational Field," Astronautica Acta, Vol.8, 1962, pp.106-123.
${ }^{12}$ Lawden, D. F., Optimal Trajectories for Space Navigation, Butterworths, London, 1963.
${ }^{13}$ Bishop, R. H. and Azimov, D. M., "New Analytic Solutions to the Fuel-Optimal Orbital Transfer Problem Using LowThrust Exhaust-Modulated Propulsion," AAS/AIAA Space Flight Mechanics Meeting, AAS Paper 00-131, Clearwater, Florida, Jan. 2000.
${ }^{14}$ Moeckel, W. E., "Interplanetary Trajectories for Electrically Propelled Vehicles," Astronautica Acta, Vol.7, 1961, pp.430444.
${ }^{15}$ Whittaker, E. T., A Treatise on the Analytical Dynamics of Particles and Rigid Bodies, $4^{t h}$ ed., 1999 reprint, Cambridge University Press, Cambridge, 1947, pp.80-81.
${ }^{16}$ Robbins, H. M., "Optimality of Intermediate-Thrust Arcs of Rocket Trajectories," AIAA Journal, Vol.3, No.6, June 1965, pp.1094-1098.
${ }^{17}$ Keller, J. L., "On Minimum Propellant Paths for Thrust Limited Rockets," Astronautica Acta, Vol.10, 1964, pp.262-269.
${ }^{18}$ Bell, D. J., "The Non-Optimality of Lawden's Spiral," Astronautica Acta, Vol.16, 1971, pp.317-324.
${ }^{19}$ Marec, J., Optimal Space Trajectories, Elsevier, Amsterdam, 1979.


[^0]:    *Senior Member of the Engineering Staff;
    Email: Anastassios.E.Petropoulos@jpl.nasa.gov, Tel.: (1)(818)354-1509, Fax: (1)(818)393-9900.
    ${ }^{\dagger}$ Senior Member of the Engineering Staff;
    Email: Jon.A.Sims@jpl.nasa.gov, Tel.: (1)(818)354-0313, Fax: (1)(818)393-9900.

