Coupled Discontinuous and Continuous Finite Element Methods for Shallow Water

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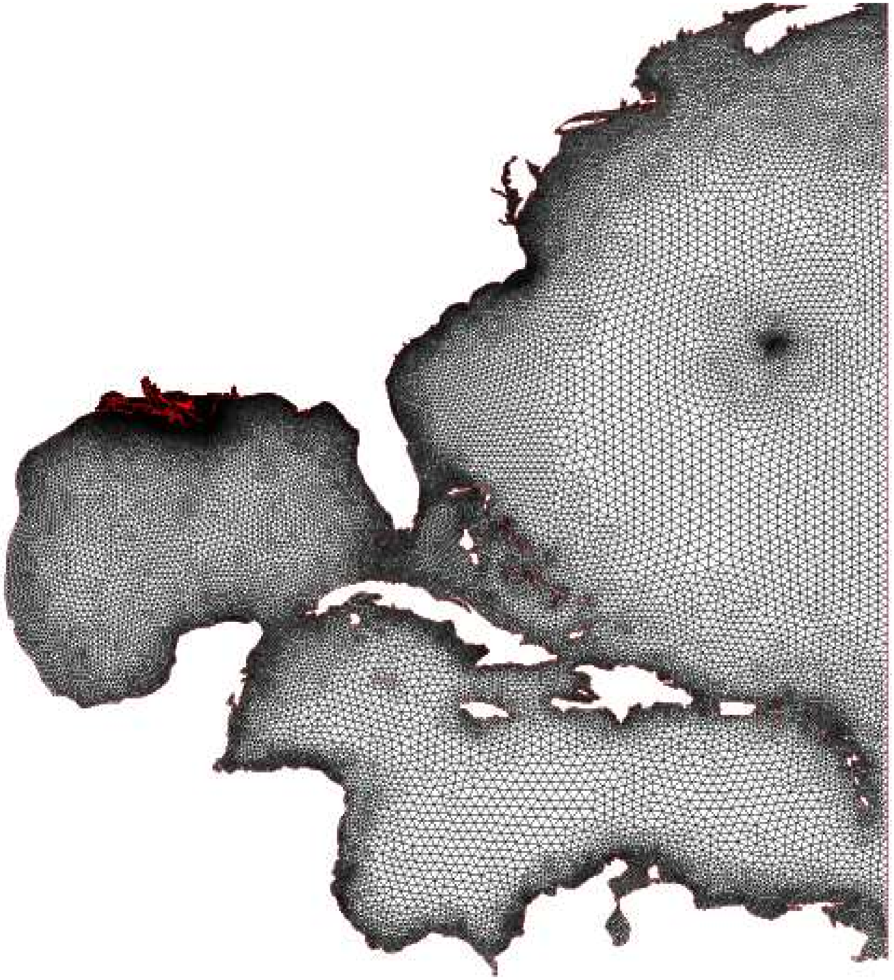
Outline

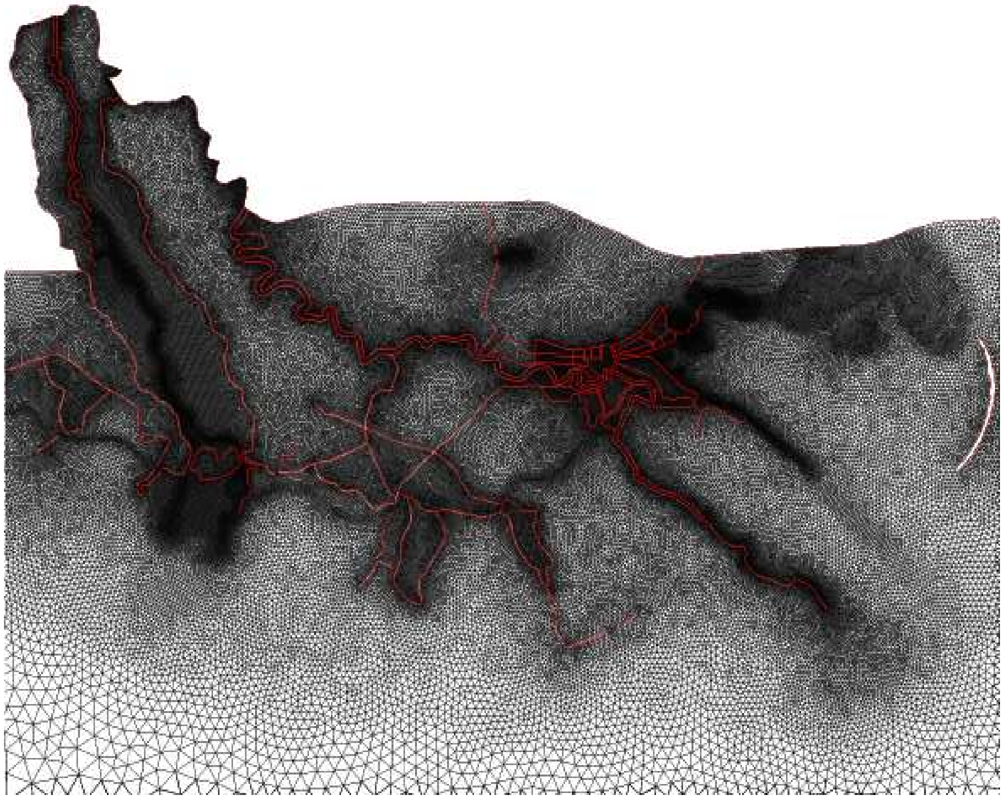
- Some motivation
- Model transport problem
- Coupling of streamline diffusion and discontinuous Galerkin methods
- The shallow water equations
- Coupling of DG and CG for shallow water
- Numerical results

Hurricane simulation



Western North Atlantic Ocean

Hurricane simulation



Lousiana Coast

Motivation for coupling methods

- different flow regimes require different numerical treatment
- completely different physics (fluid/structure, e.g.)
- reduce degrees of freedom
- use nonconforming grids without mortar spaces
- adaptivity in mesh/polynomial order

Model transport problem

$$c + \nabla \cdot (\mathbf{u}c - D\nabla c) = fc, \quad x \in \Omega, t > 0$$

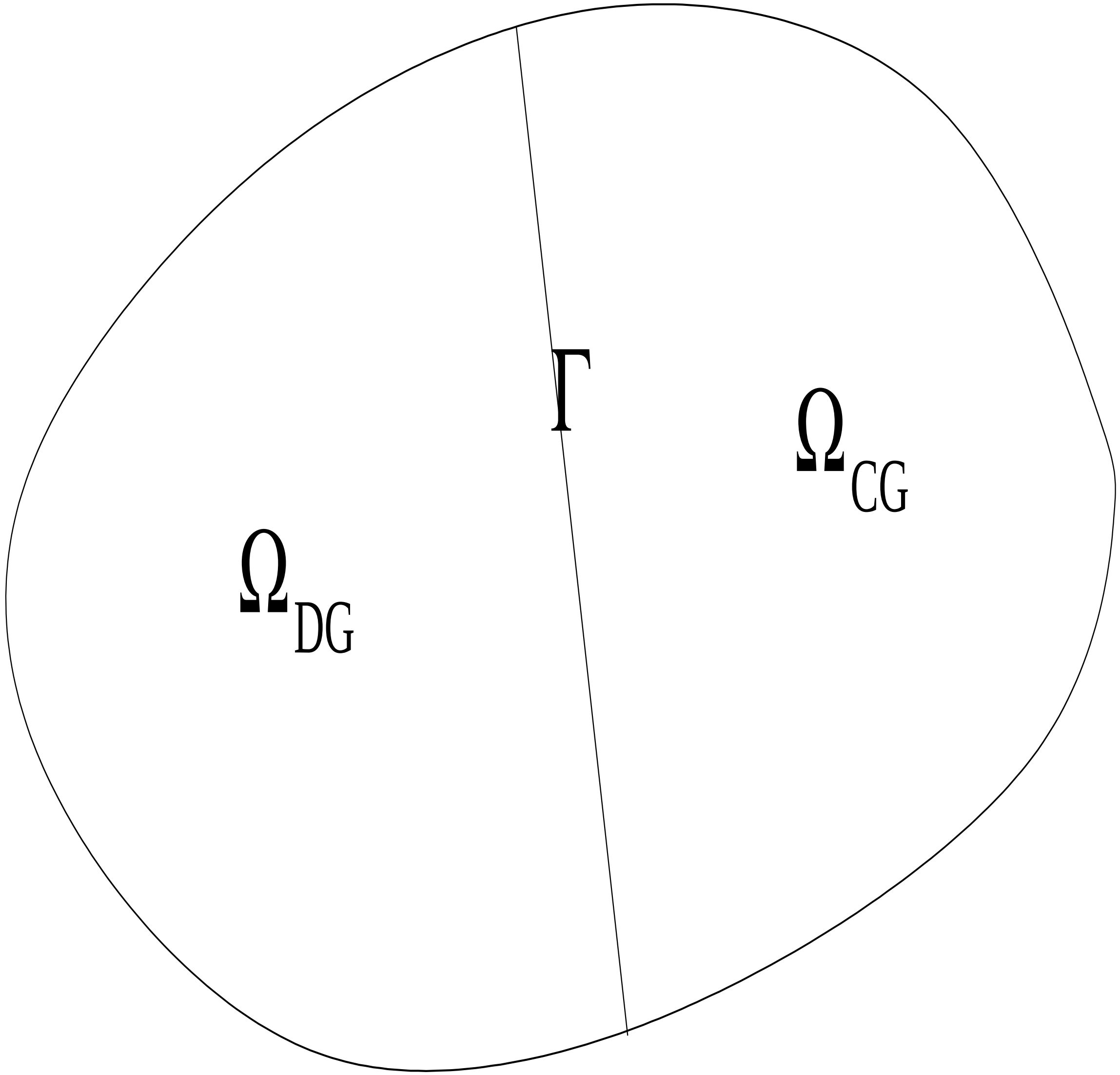
$$c(x,0) = c^0(x), \quad x \in \Omega.$$

$$\begin{aligned}(\mathbf{u}c - D\nabla c) \cdot \mathbf{n} = g_D \mathbf{u} \cdot \mathbf{n}, &\quad \Gamma_D \times (0,T], \\ (D\nabla c) \cdot \mathbf{n} = 0, &\quad \Gamma_N \times (0,T].\end{aligned}$$

Assume

$$\nabla \cdot \mathbf{u} = f$$

Coupling a DG and a CG method



Notation

- Let $\mathcal{T}_h$ denote regular partition of Ω into elements Ω_e with no elements overlapping Γ.
- Suppose Ω_e^- and Ω_e^+ are adjacent elements
- $(\mathbf{v}^\pm, w^\pm)$ denote the traces of $(\mathbf{v}, w)$ on the face e between Ω_e^+ and Ω_e^- from the interiors of the elements.
- Define the average $\{\cdot\}$ and jump $[\![\cdot]\!]$ for $\mathbf{x} \in e$ as follows:

$$\{\mathbf{v}\} = (\mathbf{v}^- + \mathbf{v}^+)/2, \qquad \{w\} = (w^- + w^+)/2,$$

$$[\![\mathbf{v}]\!] = \mathbf{v}^+ \cdot \mathbf{n}^+ + \mathbf{v}^- \cdot \mathbf{n}^-, \qquad [\![w]\!] = w^+\mathbf{n}^+ + w^-\mathbf{n}^-.$$

- $\mathcal{E}_i$ denotes set of interior element faces.

Coupling streamline diffusion and DG

The streamline diffusion method (Eriksson, Johnson, Hughes et al) uses the fact that

$$\nabla \cdot \mathbf{u} = f$$

and rewrites the transport problem in nonconservative form

$$c + \mathbf{u} \cdot \nabla c - \nabla \cdot (D \nabla c) = 0$$

Weak form on Ω:

$$\begin{aligned}(c + \mathbf{u} \cdot \nabla c, v + \delta(v + \mathbf{u} \cdot \nabla v))_{\Omega} \\ + (D \nabla c, \nabla v)_{\Omega} + \langle (g_D - c)\mathbf{u} \cdot \mathbf{n}, v \rangle_{\Gamma_D} \approx 0\end{aligned}$$

$\delta = \mathcal{O}(h)$.

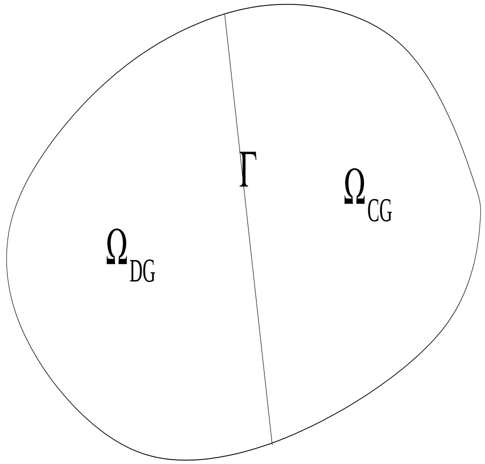
DG method

The DG method is based on the conservative form of the model. The nonsymmetric DG weak form on Ω (Oden, Baumann; Riviere, Wheeler):

$$
\begin{aligned}
&(c, w)_\Omega - (\mathbf{u}c, \nabla w)_\Omega + \langle \mathbf{u}\hat{c}, [\![w]\!]\rangle_{\mathcal{E}_i} + (D\nabla c, \nabla w)_\Omega \\
&\quad -\langle \{D\nabla c\}, [\![w]\!]\rangle_{\mathcal{E}_i} + \langle \{D\nabla w\}, [\![c]\!]\rangle_{\mathcal{E}_i} \\
&\quad = -\langle \mathbf{u}\cdot\mathbf{n} g_D, w\rangle_{\Gamma_D} + (fc, w)_\Omega
\end{aligned}
$$

The term $\hat{c}$ is usually taken to be the upwind value of c.

Coupling



On the coupling curve Γ, we want to "enforce" continuity of c and continuity of flux $D\nabla c$. We also want to make sure mass is conserved. Problem: The methods use different weak formulations, one integrates the advection term by parts and the other doesn't.

Coupled method

Conservation of mass $\Rightarrow$ couple through fluxes
DG method in Ω_{DG}:

$$(c,w)_{\Omega_{DG}} - (\mathbf{u}c,\nabla w)_{\Omega_{DG}} + \langle \mathbf{u}\hat{c}, [\![w]\!]\rangle_{\mathcal{E}_i}$$
$$+(D\nabla c,\nabla w)_{\Omega_{DG}} - \langle\{D\nabla c\},[\![w]\!]\rangle_{\mathcal{E}_i} + \langle\{D\nabla w\},[\![c]\!]\rangle_{\mathcal{E}_i}$$
$$\langle \mathbf{u}\cdot\mathbf{n}_\Gamma \hat{c}, w\rangle_\Gamma - \langle\{D\nabla c\}\cdot\mathbf{n}_\Gamma, w\rangle_\Gamma + \frac{1}{2}\langle D\nabla w\cdot\mathbf{n}_\Gamma, [\![c]\!]\rangle_\Gamma$$
$$= -\langle \mathbf{u}\cdot\mathbf{n} g_D, w\rangle_{\Gamma_D} + (fc,w)_\Omega$$

Coupled method

SD method in Ω_{CG}:

$$
\begin{aligned}
&(c + \mathbf{u}\cdot\nabla c, v + \delta(v + \mathbf{u}\cdot\nabla v))_{\Omega_{CG}} \\
&\quad + (D\nabla c, \nabla v)_{\Omega_{CG}} + \langle (g_D - c)\mathbf{u}\cdot\mathbf{n}, v\rangle_{\Gamma_D} \\
&\quad - \langle (\hat{c} - c^+)\mathbf{u}\cdot\mathbf{n}_\Gamma, v\rangle_\Gamma + \langle \{D\nabla c\}\cdot\mathbf{n}_\Gamma, v\rangle_\Gamma + \frac{1}{2}\langle D\nabla v\cdot\mathbf{n}_\Gamma, [\![c]\!]\rangle_\Gamma \\
&\quad = 0
\end{aligned}
$$

Approximate c by c_h, which is a discontinuous piecewise polynomial in Ω_{DG} and a continuous piecewise polynomial in Ω_{CG}. This coupled method is globally mass conservative and stable in L^2.

The shallow water equations-2D

Assuming vertical effects are negligible, and integrating over the water column, we obtain:

- Continuity Equation (CE):

$$\xi_t + \nabla \cdot (\mathbf{u}H) = 0$$

- Momentum Equation (ME):

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \tau_{bf}\mathbf{u} + g\nabla\xi - \mu\Delta\mathbf{u} = \mathbf{f}$$

Plus boundary and initial conditions

Definition of variables

- ξ = free surface elevation
- h_b = bathymetry
- $H = h_b + \xi$ = total water column
- $\mathbf{u}$ = depth averaged velocity
- μ = depth-averaged turbulent viscosity
- g =gravity
- $\mathbf{f}(H, \mathbf{u}, t, \mathbf{x})$ = bottom friction, Coriolis force, surface wind stress, atmospheric pressure, tidal potentials.

Other Forms of SWE

- Conservative momentum equation (CME):

$$\mathbf{q}_t + \nabla \cdot (\mathbf{q}\mathbf{q}/H) + \tau_{bf}\mathbf{q} + gH\nabla\xi - \nu H \Delta \mathbf{u} = \mathbf{f}H \qquad \text{(-11)}$$

- Generalized wave continuity equation (GWCE):

$$\text{CE}_t + \nabla \cdot \text{CME} + \tau_0 \text{CE} = 0 \qquad \text{(-11)}$$

- $\tau_0 =$ user defined parameter
- $\mathbf{q} = \mathbf{u}H$

Shallow water simulators

ADCIRC (Advanced Circulation Model; Luettich, Westerink, *et al*)

- Based on GWCE and ME
- Galerkin finite element code, piecewise linears on triangles.
- GWCE discretized in time using three time levels centered at t^n
- ME discretized in time using Crank-Nicholson except for nonlinear advection terms which are explicit.
- Careful selection of $\tau_0 \approx \tau_{bf}$

Shallow water simulators

UTBEST (University of Texas Bay and Estuary Simulator; Aizinger, Chippada, D.)

- Based on CE and CME
- Local discontinuous Galerkin (LDG) finite element code on triangles
- Roe numerical flux
- $p = 0, 1, 2$
- Runge-Kutta time stepping

GWCE vs. DG

GWCE formulation

- Stabilizes conventional finite element approaches for many tidal flows
- Not mass conservative
- No special technique for handling advection-dominated flow

DG formulation

- Mass conservative
- Can employ upwinding and stability post-processing
- More degrees of freedom than GWCE based on continuous Galerkin method

Coupling DG and CG for SWE

- Use CE/ME in Ω_{DG}
- Use GWCE/ME in Ω_{CG}
- Approximate ξ, $\mathbf{u}$ by ξ_h, $\mathbf{u}_h$, discontinuous in Ω_{DG}, continuous in Ω_{CG}, piecewise polynomials of degree k.
- Use Lesaint-Raviart upwinding technique and nonsymmetric interior penalty Galerkin (motivated by recent work of Girault, Riviere and Wheeler) in momentum equation
- Coupling is through fluxes

Wave continuity equation

$$\text{WCE} = \text{CE}_t + \nabla \cdot (H \ \text{ME})$$

resulting in

$$\xi_{tt} - \nabla \cdot \left[\tau_{bf}\mathbf{u}H + \mathbf{u}H \cdot \nabla \mathbf{u} - \mathbf{u}H_t + gH\nabla\xi - H\mathcal{F}\right] = 0$$

(Assuming $\mu = 0$)

Weak form

DG/PCE: $v \in L^2(\Omega_{DG}) \cap H^1(\Omega_e)$

$$(\xi_t, v)_{\Omega_{DG}} - (\mathbf{u}H, \nabla v)_{\Omega_{DG}} + \langle \mathbf{u}H, [\![v]\!] \rangle_{\mathcal{E}_i} + \langle \mathbf{u}H \cdot \mathbf{n}_\Gamma, v \rangle_\Gamma = 0$$

CG/WCE: $\nu \in H^1(\Omega_{CG})$

$$\begin{aligned}(\xi_{tt}, \nu)_{\Omega_{CG}} \\ &+ (\tau_{bf}\mathbf{u}H + Hg\nabla\xi + H\mathbf{u}\cdot\nabla\mathbf{u} - \mathbf{u}H_t - H\mathbf{f}, \nabla\nu)_{\Omega_{CG}} \\ &- \langle (\mathbf{u}H)_t \cdot \mathbf{n}_\Gamma, \nu \rangle_\Gamma + \langle (\mathbf{u}H)_t \cdot \mathbf{n}, \nu \rangle_{\partial\Omega_{CG}\setminus\Gamma} = 0\end{aligned}$$

The coupling is through $\mathbf{u}H$ on Γ.

Weak form

DG/CG for momentum equation:

$$
\begin{aligned}
&(\mathbf{u}_t, \mathbf{w})_\Omega + (\mathbf{u}\cdot\nabla\mathbf{u}, \mathbf{w})_\Omega \\
&\quad + \sum_{\partial\Omega_e^- \subset \Omega} \langle |\mathbf{u}\cdot\mathbf{n}_e|(\mathbf{u}^{int} - \mathbf{u}^{ext}), \mathbf{w}^{int}\rangle_{\partial\Omega_e^-} \\
&\quad + (\tau_{bf}\mathbf{u}, \mathbf{w})_\Omega + (g\nabla\xi, \mathbf{w})_\Omega \\
&\quad - \langle g[\![\xi]\!], \{\mathbf{w}\}\rangle_{\mathcal{E}_i} + \langle \sigma[\![\mathbf{u}]\!], [\![\mathbf{w}]\!]\rangle_{\mathcal{E}_i} \\
&\quad = (\mathbf{f}, \mathbf{w})_\Omega.
\end{aligned}
$$

Here

$$
\begin{aligned}
\partial\Omega_e^- &= \{x \in \partial\Omega_e : \mathbf{u}\cdot\mathbf{n}_e < 0\} \\
\sigma &= \mathcal{O}(h^{-1})
\end{aligned}
$$

Error estimate

Theorem (D. and Proft): Assume $\mathbf{u}$ and ξ and the initial data are sufficiently smooth, $\mu > 0$ and that $\mathcal{T}_h$ is quasiuniform. Then, there exists a constant $\bar{C}$ such that

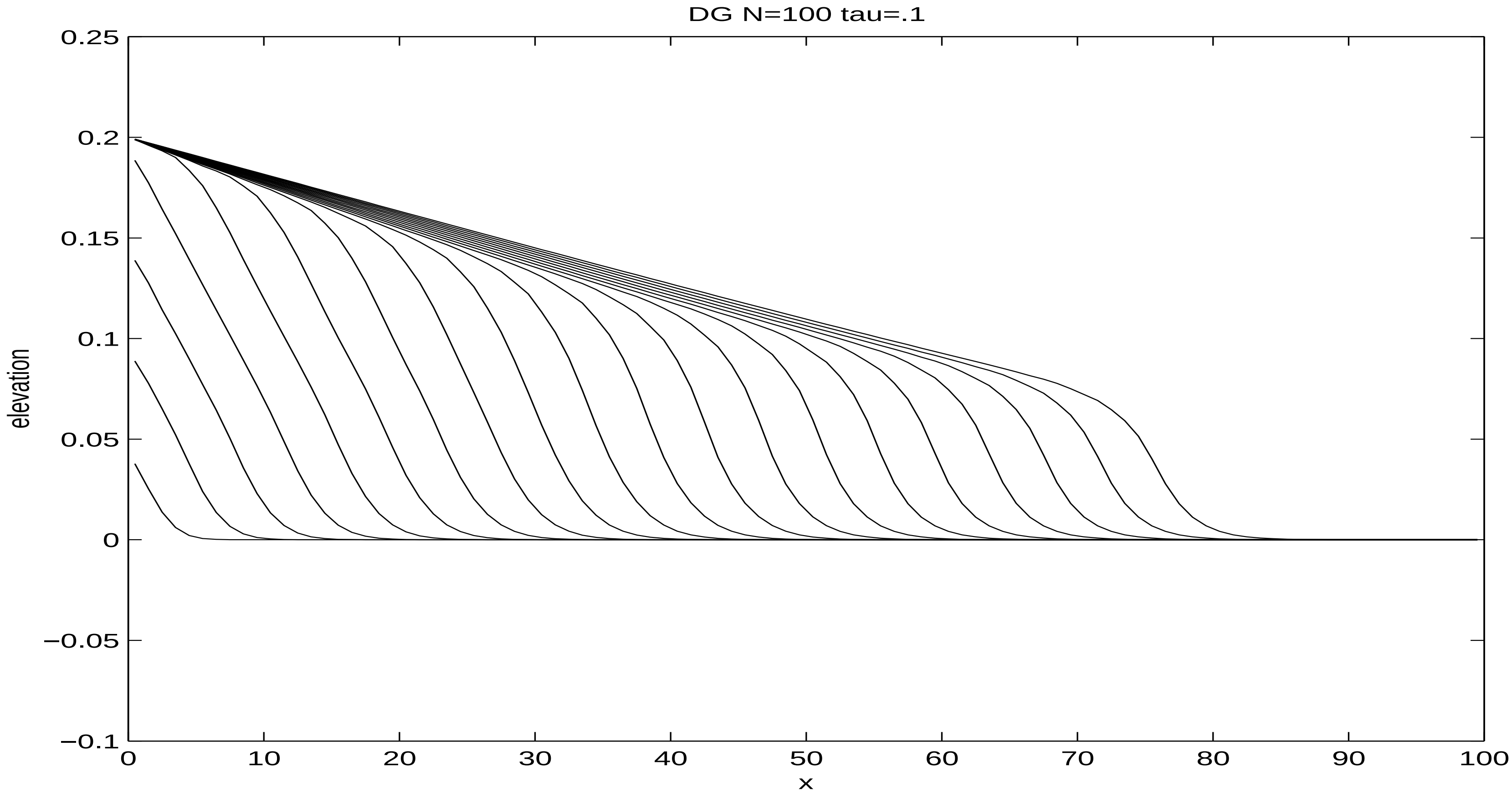
$$||\xi - \xi_h||_{L^\infty(0,T;L^2(\Omega))} + ||\mathbf{u} - \mathbf{u}_h||_{L^\infty(0,T;L^2(\Omega))}$$
$$+||\mathbf{u} - \mathbf{u}_h||_{L^2(0,T;H^1(\Omega))} \leq \bar{C}h^k.$$

Numerical results

Test case:

- 100 m channel, initially at rest, $H = 1$ m
- Ramp up elevation to $H = 1.2$ m at left boundary over 5 seconds
- $\mu = 0$
- Different values of τ_{bf}
- Piecewise linears in Ω_{DG}, continuous, piecewise linears in Ω_{CG}.
- Compare DG, coupled DG/CG, and CG
- Second order R-K time stepping in all methods

Test case 1: $\tau_{bf} = .1, h = 1$



DG

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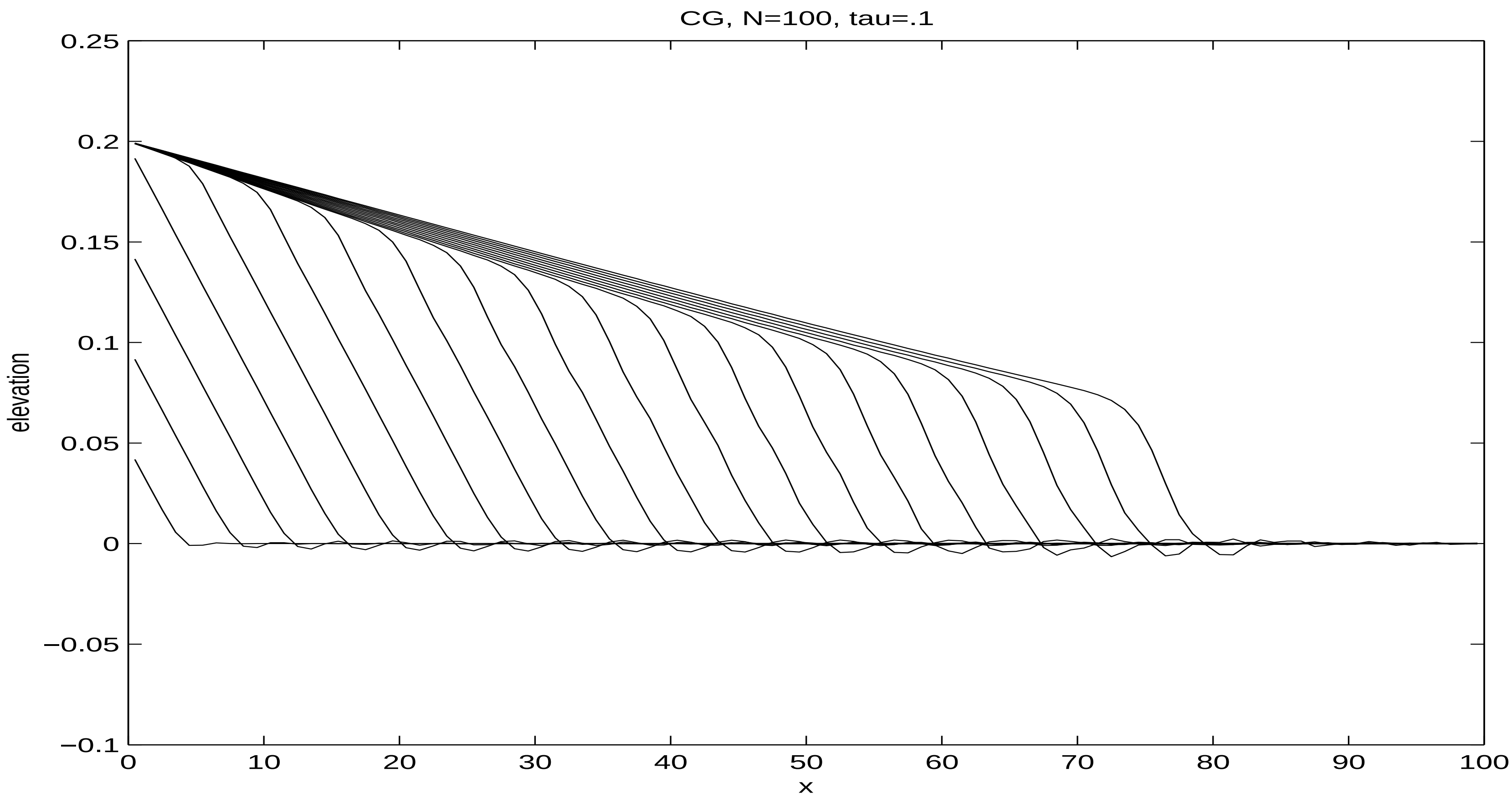

CG

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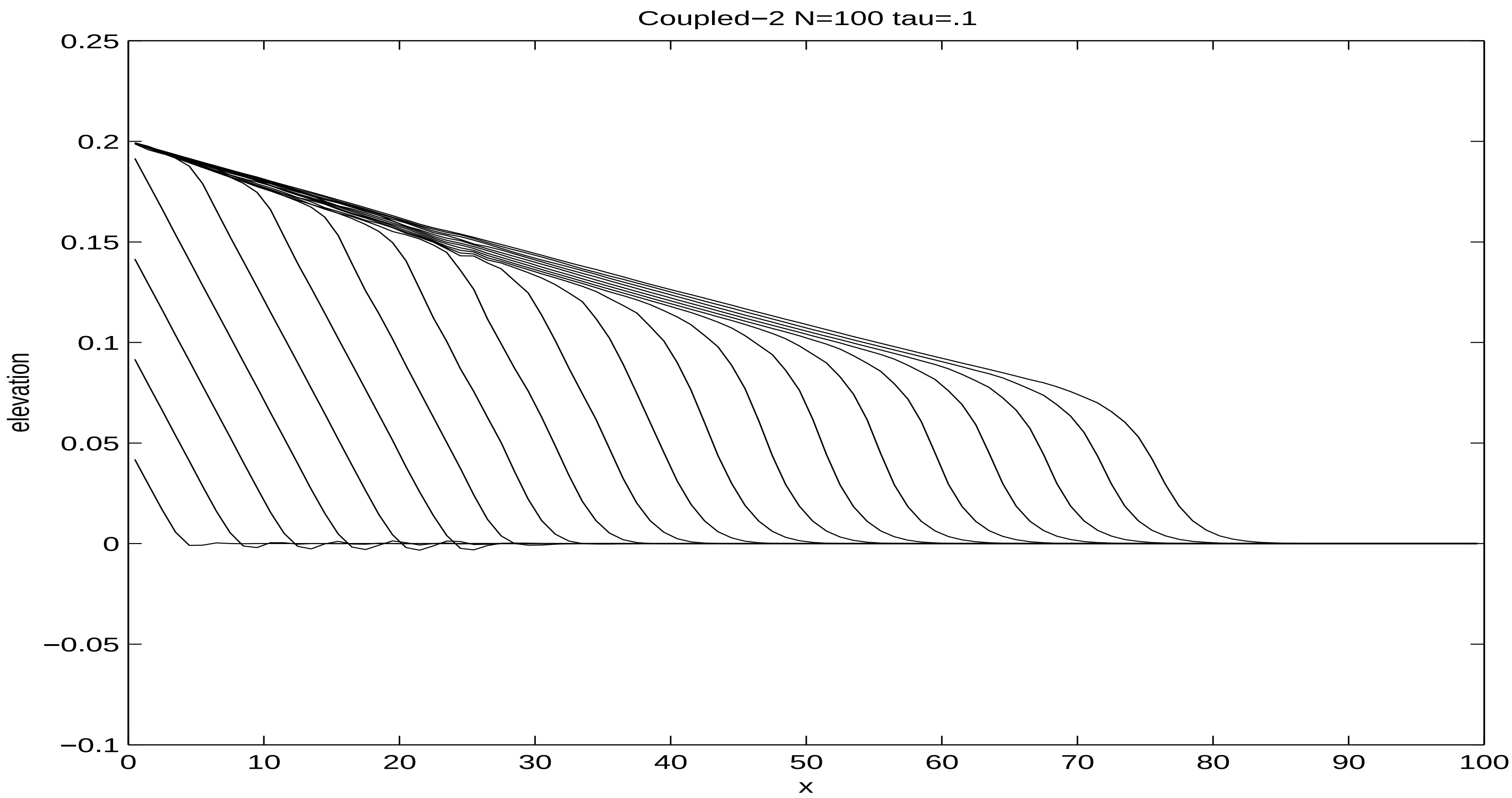
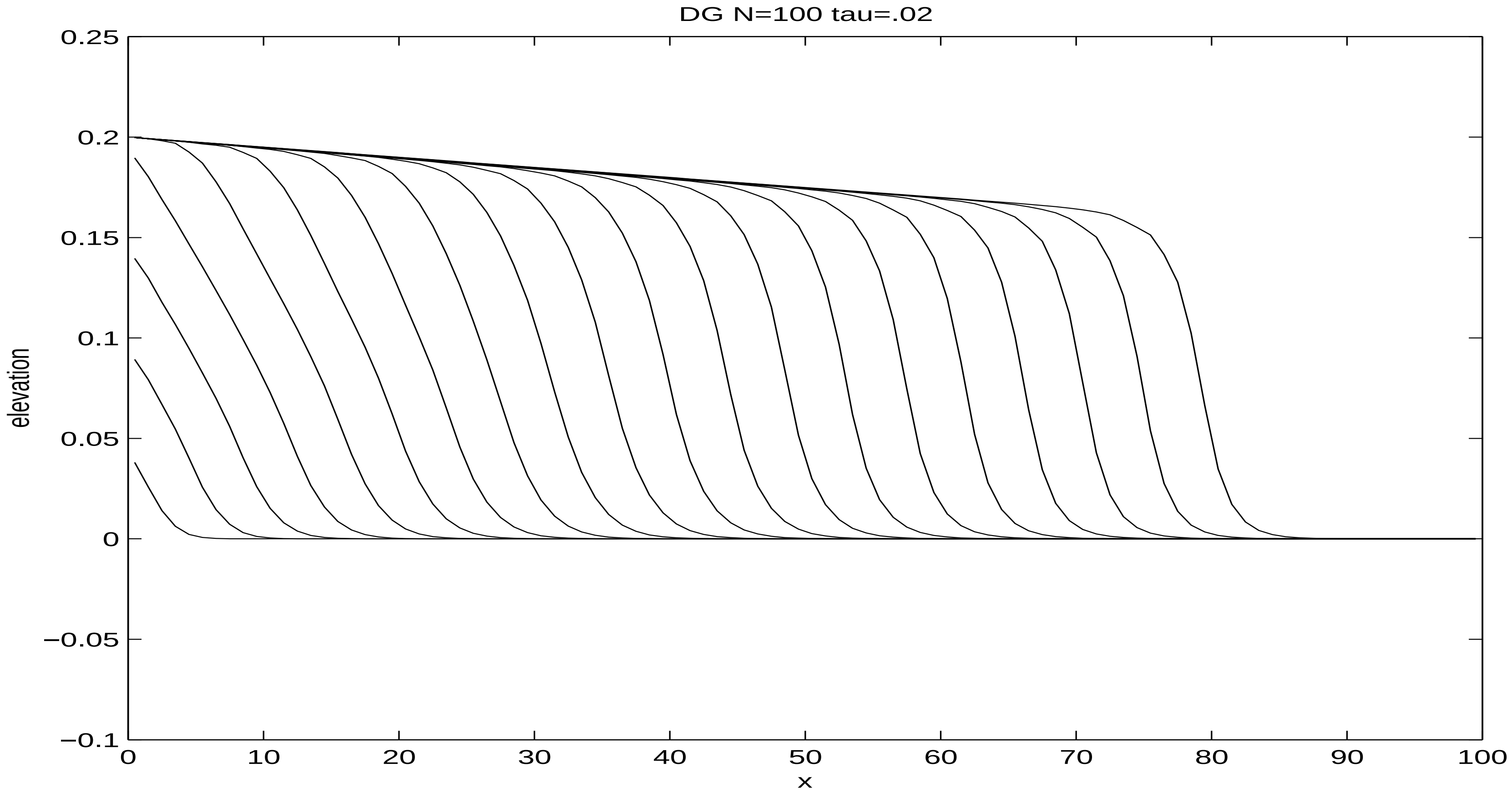


Coupled DG/CG: $\Omega_{CG} = (0, 25)$, $\Omega_{DG} = (25, 100)$.

Test case 2: $\tau_{bf} = .02$, $h = 1.$



DG

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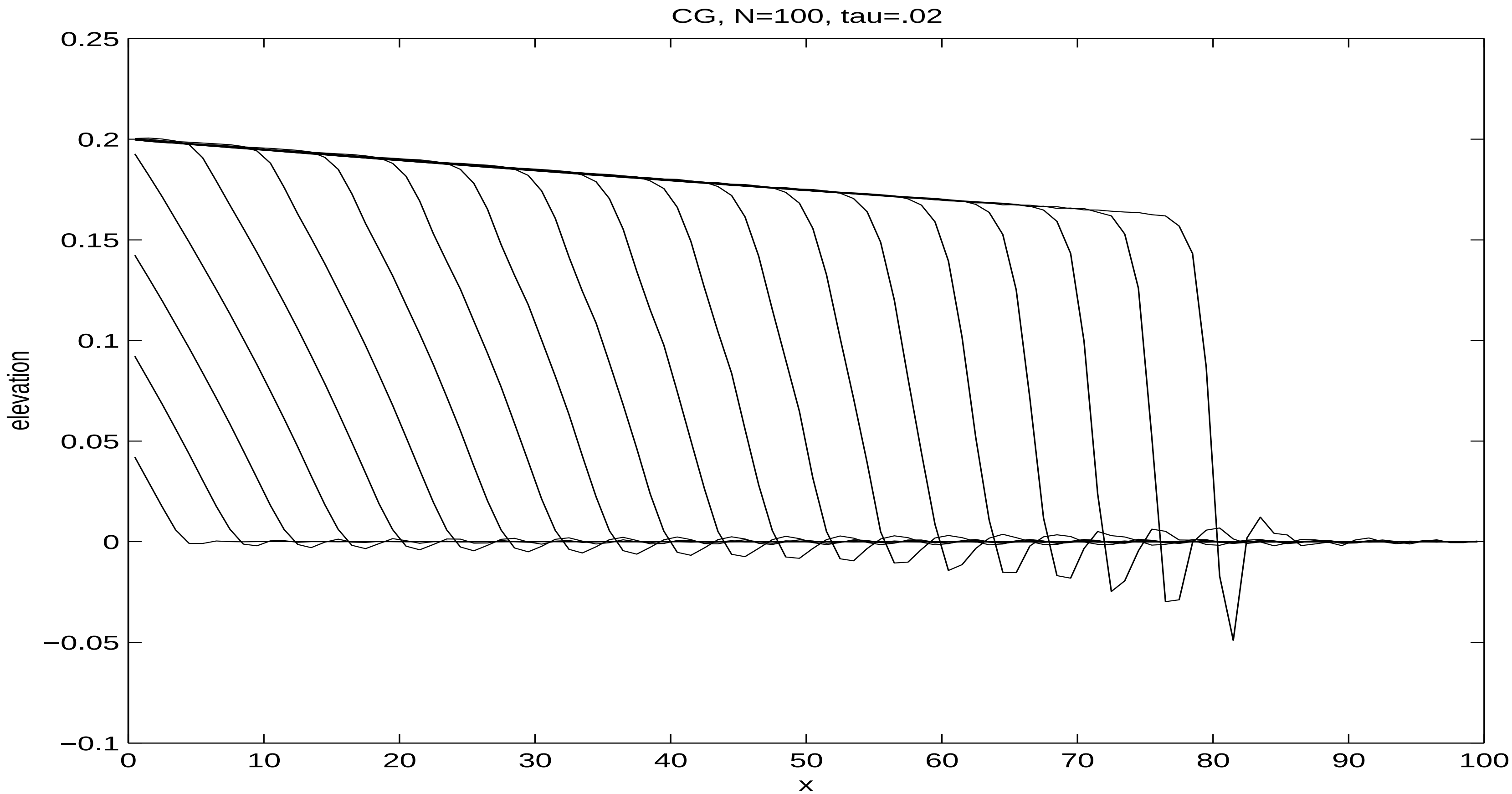

CG

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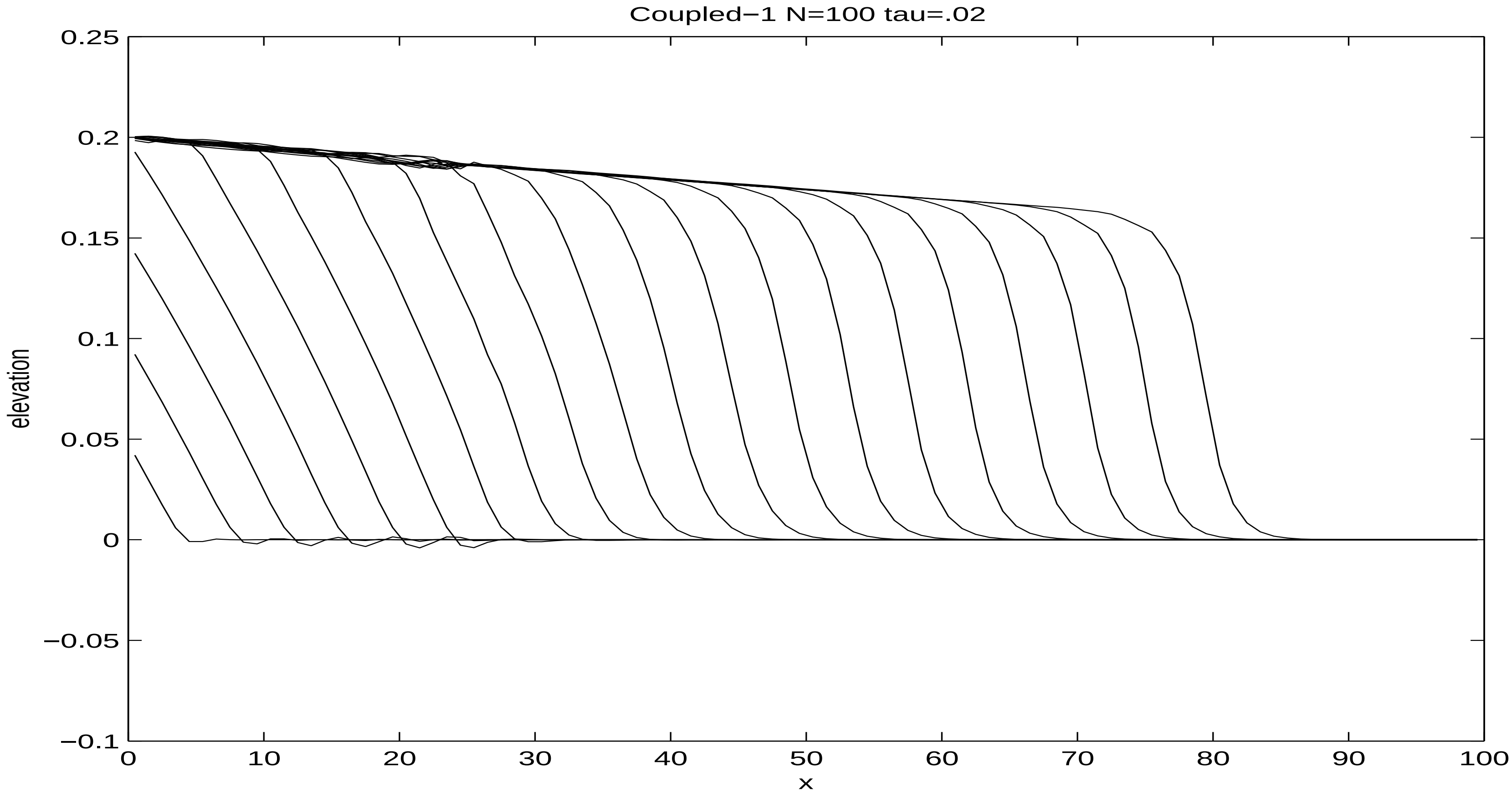

Coupled DG/CG, $\Omega_{CG} = (0, 25)$, $\Omega_{DG} = (25, 100)$.

Cost of simulation

- DG with $p = 1$: $4N$ degrees of freedom
- CG with $p = 1$: $2N$
- Coupled DG and CG: $2N/4 + 12N/4 = 7N/2$

Conclusions

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- Different methods can use different formulations which can complicate the coupling
- We have formulated and analyzed approaches for transport applications
- Preliminary numerical results are encouraging