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Charge conjugation invariance of the Spectator Equations

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Abstract

In response to recent criticism, we show how to define the spectator equations for negative energies so that charge conjugation invariance is preserved. The result, which emerges naturally from the application of spectator principles to systems of particles with negative energies, is to replace all factors of the external energies W_i by $\sqrt{W_i^2}$, insuring that the amplitudes are independent of the sign of the energies W_i .

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I. INTRODUCTION

In a recent set of papers [1,2], Pascalutsa and Tjon have criticized the spectator formalism by claiming that it violates charge conjugation invariance, C . When applied to the self energy of a Dirac particle, $\Sigma(p_0, \mathbf{p})$, this requirement is

$$C\Sigma(p_0, \mathbf{p})C^{-1} = \Sigma^T(-p_0, -\mathbf{p}), \quad (1)$$

where C is the Dirac charge conjugation matrix and the superscript T refers to the transpose in the Dirac space. The spectator equations have been previously applied only to the positive energy subspace, and the transformation (1) is the only one of all the transformations in the full Lorentz group that connects states of positive and negative energy. Before it can be tested, the definition of the spectator equations must be extended to negative energy. The claim of Pascalutsa and Tjon that the spectator formalism violates charge conjugation invariance (and hence Lorentz covariance) follows from their consideration of how the spectator equations should be extended to negative energy.

In this short paper we confirm that the extension of the spectator formalism to negative energies proposed by Pascalutsa and Tjon does indeed violate C invariance, but that a more natural extension does not. Since the equations have never been applied to negative energies before the work of Refs. [1,2], our discussion is, strictly speaking, a proposal for how the equations should be extended to negative energies in such a way as to preserve C invariance. We will show that this extension is a natural and a faithful application of the basic principles guiding the construction of the spectator theory.

In the next section we review the basic principles underlying the spectator theory [3], and apply these principles to the study of systems with negative energy. This leads naturally to the principle that all external energies, referred to collectively as W_i , should be interpreted as $|W_i|$ (or $\sqrt{W_i^2}$), insuring that the C invariance condition (1) is trivially satisfied. In Sec. III we present some examples in 1+1 dimension, where numerical results can be easily obtained without the use of form factors. We summarize our conclusions in a final section.

II. PRINCIPLES OF THE SPECTATOR THEORY AND ITS EXTENSION TO NEGATIVE ENERGIES

The principles of the spectator theory are illustrated by a simple $\lambda \phi^2 \Psi^\dagger \Psi$ theory, where Ψ is a “heavy” fermion field of mass m (referred to as a “nucleon”) and ϕ is a “light” self-conjugate boson with mass μ (referred to as a “meson”).

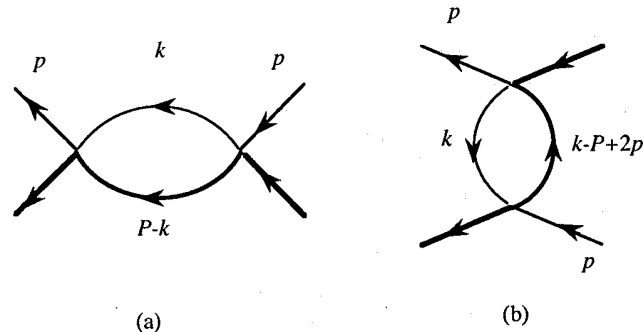


FIG. 1. The bubble (a) and “crossed bubble” (b) diagrams.

The one-loop diagrams in the theory are the second order bubble diagrams shown in Fig. 1. Here $P = (W, \mathbf{0})$ is the total four-momentum of the pair, and p is the external four-momentum of the heavy Dirac particle. While we assume a scalar interaction for simplicity, all of our results are independent of the Dirac structure of the interaction.

The Feynman integral for the bubble diagram (a) is

$$\Sigma_a(P) = i\lambda^2 \int \frac{d^4k}{(2\pi)^4} \frac{(m + \not{P} - \not{k})}{(\mu^2 - k^2)(m^2 - (P - k)^2)} F(k^2, k \cdot P, P^2), \quad (2)$$

where F is a function that depends on the form factors or regularization prescription used in the calculation, and must depend on the arguments $k^2, k \cdot P$, and P^2 . Changing $P \rightarrow -P$ and $k \rightarrow -k$ shows immediately that

$$\Sigma_a(-P) = C\Sigma_a^T(P)C^{-1}, \quad (3)$$

and the diagram is invariant under charge conjugation. A similar argument works for the crossed bubble (b). Hence the four-dimensional calculation of these diagrams is C invariant.

Now look at the spectator calculation of these two diagrams. The philosophy underlying the spectator approach, as *commonly stated*, is to approximate the diagram (a) by picking up the leading positive energy heavy particle pole, and to lump all other contributions from these diagrams with the higher order terms included if the calculation were to be carried out to third (or higher) order. Of course, if we only need the result to second order (for example, when calculating high energy scattering for a weak coupling when perturbation theory gives a reliable result) it is simple enough to obtain the exact answer in this case. But

in the more general case (for example, when the coupling is strong or an infinite sum of diagrams is needed at low energy or near bound state poles – even when the coupling is small) then we will need a systematic approach which sums all ladder and crossed ladder diagrams efficiently. For this simple theory, the bubble diagram (a) plays the role of a fourth order ladder diagram [where

$$\lambda = -\frac{g^2}{M^2 - q^2} \rightarrow -\frac{g^2}{M^2} \quad (4)$$

is the effective coupling from a very heavy meson exchange of very short range] and the crossed bubble plays the role of the fourth order crossed ladder. [This can be easily demonstrated by writing down these diagrams and letting the heavy meson mass, $M \rightarrow \infty$].

Understanding of the mathematical and physical motivation behind the spectator theory comes from a study of the singularities of the two bubble diagrams in the complex k_0 plane. In the next section we will give a numerical demonstration of the following discussion; here we focus on a qualitative understanding. The two bubbles each have four poles in the complex k_0 plane. In the rest frame of the two particles, the four poles for diagram 1(a) are at

$$\begin{aligned} k_0 &= \pm(\omega(k) - i\epsilon) \\ &= W - E(k) + i\epsilon \\ &= W + E(k) - i\epsilon, \end{aligned} \quad (5)$$

where $\omega(k) = \sqrt{\mu^2 + \mathbf{k}^2}$ and $E(k) = \sqrt{m^2 + \mathbf{k}^2}$. For diagram 1(b), the poles are at

$$\begin{aligned} k_0 &= \pm(\omega(k) - i\epsilon) \\ &= W - 2E(p) - E(k) + i\epsilon \\ &= W - 2E(p) + E(k) - i\epsilon. \end{aligned} \quad (6)$$

The location of these poles is shown in Fig. 2 for the case when $\mathbf{k} \simeq 0$, $\mathbf{p} \simeq 0$, and $|W| \simeq m + \mu$. When $W > 0$ [panels (a) and (b)] the contour shown in the figures is closed in the upper half plane, and encloses the negative energy meson pole and [in panel (a)] the positive energy nucleon pole or [in panel (b)] the negative energy nucleon pole. In panels (c) and (d) the opposite is true; the contour is closed in the lower half plane and encloses the positive energy meson pole and [in panel (c)] the negative energy nucleon pole or [in panel (d)] the positive energy nucleon pole.

First suppose that the external energy is positive, and $W \simeq m + \mu$, so that the location of the singularities is as given in panels (a) and (b). Closing the contour

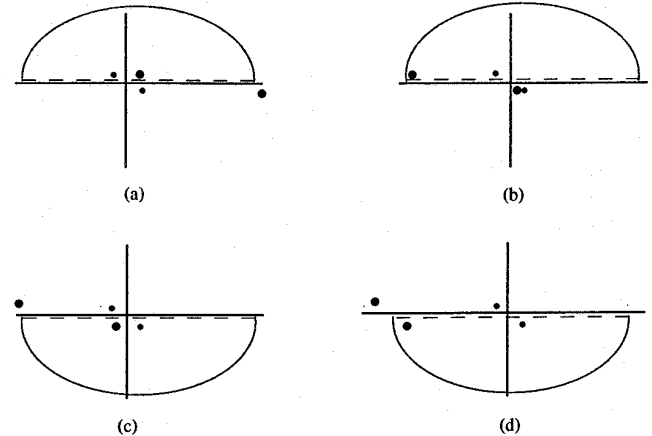


FIG. 2. The poles in the complex k_0 plane for the two diagrams shown in Fig. 1. Panel (a) shows the poles for diagram 1(a) and panel (b) the poles for diagram 1(b) when $W = |W| > 0$. Panels (c) and (d) show the corresponding locations of the poles when $W = -|W| < 0$.

for diagram (a) in the upper half plane gives two contributions: the positive energy nucleon pole and the negative energy meson pole. At first it looks like the negative energy meson pole will introduce a large correction (because it is so close to the nucleon pole), but it turns out that *the contribution from the negative energy meson pole is almost exactly cancelled by a similar contribution from the crossed bubble (b), and hence the nucleon pole alone gives a very accurate result.* This will be demonstrated numerically in the next section. Hence, for $W > 0$, the spectator result for *both* of the bubble diagrams in Fig. 1 is the nucleon contribution from (a)

$$\Sigma_S(W) \Big|_{W>0} = -\lambda^2 \int \frac{d^3k}{(2\pi)^3} \frac{(m + \mathcal{P} - \hat{k})}{2E(k)[\omega^2(k) - (W - E(k))^2]} F(\hat{k}^2, \hat{k} \cdot P, P^2), \quad (7)$$

where $\hat{k} = (W - E(k), -\mathbf{k})$. Since the internal energy of the nucleon is positive, the internal energy cannot be changed and the argument used to demonstrate C invariance for the four-dimensional calculation fails.

However, *the same physics which lead to the selection of the positive energy nucleon pole will yield a different result* if the bubbles are to be evaluated at a negative external energy $W \simeq -(m + \mu)$. Since negative external energies are unphysical, this case was not considered in the original formulation of the spectator theory. We look at it now.

For negative external energies $W \simeq -(m + \mu)$ the poles are as shown in panels (c) and (d). Now the role of the upper and lower half planes are changed, and the *negative energy* nucleon pole dominates diagram (a), with the *positive energy* meson pole giving the leading correction. Furthermore, as in the positive energy case, this leading correction (from the positive energy meson pole now) is cancelled by the contribution from the crossed bubble. Hence, for both diagrams *the same physical/mathematical argument* yields, for negative energy,

$$\Sigma_S(W)\Big|_{W<0} = -\lambda^2 \int \frac{d^3k}{(2\pi)^3} \frac{(m + \mathbb{P} - \tilde{k})}{2E(k)[\omega^2(k) - (W + E(k))^2]} F(\tilde{k}^2, \tilde{k} \cdot P, P^2), \quad (8)$$

where $\tilde{k} = (W + E(k), -\mathbf{k})$. Comparing Eqs. (7) and (8) [after changing $\mathbf{k} \rightarrow -\mathbf{k}$] shows that

$$\Sigma_S(W)\Big|_{W<0} = \Sigma_S(-|W|)\Big|_{W<0} = C \Sigma_S^T(W)\Big|_{W>0} C^{-1}. \quad (9)$$

This is the proof of C invariance we seek. Note that the natural extension of the spectator equations to negative energy has lead to a result which can be obtained from the positive energy result by using the transformation (9).

If we separate the Σ_S into scalar functions according to

$$\begin{aligned} \Sigma_S(W)\Big|_{W>0} &= \mathbb{P}A(W)\Big|_{W>0} + B(W)\Big|_{W>0} \\ \Sigma_S(W)\Big|_{W<0} &= \mathbb{P}A(W)\Big|_{W<0} + B(W)\Big|_{W<0}, \end{aligned} \quad (10)$$

then, from Eqs. (7) and (8)

$$\begin{aligned} B(W)\Big|_{W>0} &= -\lambda^2 \int \frac{d^3k}{(2\pi)^3} \frac{m F(\hat{k}^2, \hat{k} \cdot P, P^2)}{2E(k)[\omega^2(k) - (W - E(k))^2]} \\ B(W)\Big|_{W<0} &= -\lambda^2 \int \frac{d^3k}{(2\pi)^3} \frac{m F(\tilde{k}^2, \tilde{k} \cdot P, P^2)}{2E(k)[\omega^2(k) - (W + E(k))^2]} \\ A(W)\Big|_{W>0} &= -\lambda^2 \int \frac{d^3k}{(2\pi)^3} \frac{F(\hat{k}^2, \hat{k} \cdot P, P^2)}{2W[\omega^2(k) - (W - E(k))^2]} \\ A(W)\Big|_{W<0} &= +\lambda^2 \int \frac{d^3k}{(2\pi)^3} \frac{F(\tilde{k}^2, \tilde{k} \cdot P, P^2)}{2W[\omega^2(k) - (W + E(k))^2]}. \end{aligned} \quad (11)$$

Hence

$$\begin{aligned} B(W)\Big|_{W<0} &= B(W)\Big|_{W>0} \\ A(W)\Big|_{W<0} &= A(W)\Big|_{W>0}. \end{aligned} \quad (12)$$

These are precisely the properties of the scalar functions A and B required by C invariance. They are possible because A and B for $W < 0$ are *different algebraic functions* of W . The simple relationships (12) between the functions for $W < 0$ and $W > 0$ permits us to write them as a single function of $|W|$:

$$\begin{aligned} B(W)\Big|_{W<0} &= B(W)\Big|_{W>0} \equiv B(|W|) = B(\sqrt{W^2}) \\ A(W)\Big|_{W<0} &= A(W)\Big|_{W>0} \equiv A(|W|) = A(\sqrt{W^2}), \end{aligned} \quad (13)$$

as stated in the introduction. While the rule (13) was only derived in this section for a simple φ^4 -type theory, examination of the details of the derivation will convince one that it can be extended to the general case.

We turn now to short numerical study of these results.

III. NUMERICAL EXAMPLES IN 1+1 DIMENSION

The discussion in the last section showed that the natural extension of the spectator equations to negative energies preserves charge conjugation invariance. In this section show that

- failure to use the prescription $W \rightarrow |W|$ when applying the spectator theory to negative energies does indeed lead to very serious numerical errors, as pointed out in Refs. [1,2], and
- the spectator approximation to the *sum* of the bubble and the crossed bubble is a better approximation than the exact bubble diagram itself.

In order to keep the discussion simple and to the point, we limit these numerical examples to the B function in 1+1 dimensions, where the integrals converge without form factors [4]. Extension of the results to the A function, and to higher dimensions yields similar results, but is complicated by the need for form factors or cutoffs.

In 1+1 dimension, the B functions for diagrams 1(a) and (b) are

$$\begin{aligned}
B_a(P) &= i\lambda^2 \int \frac{d^2k}{(2\pi)^2} \frac{m}{(\mu^2 - k^2)(m^2 - (P - k)^2)} \\
B_b(P) &= i\lambda^2 \int \frac{d^2k}{(2\pi)^2} \frac{m}{(\mu^2 - k^2)(m^2 - (P - 2p - k)^2)},
\end{aligned} \tag{14}$$

where the form factor function has been set to unity. These integrals are easily evaluated. Numerical results for the case when $M = m/\mu = 7$, $\lambda^2/(2\pi\mu^2) = 3$, $p = (E(p), p)$, and $W = E(p) + \omega(p)$, corresponding to scattering in the forward direction, are shown in in Fig. 3. Note that the bubble (a) and crossed bubble (b) are comparable in size, and that their sum (the heavy dotted line) is almost identical to the positive energy (because $W > 0$!) nucleon pole contribution from diagram (a) alone. This latter is

$$\begin{aligned}
B_S(W) &= -\frac{m\lambda^2}{4\pi} \int \frac{dk}{E(k)[\mu^2 - m^2 - W^2 + 2E(k)W]} \\
&= -\frac{m\lambda^2}{2\pi\mu^2} \int_0^\infty \frac{d\kappa}{e(\kappa)[1 - M^2 - (W/\mu)^2 + 2e(\kappa)(W/\mu)]},
\end{aligned} \tag{15}$$

where $k = \mu\kappa$ and $e(\kappa) = \sqrt{M^2 + \kappa^2}$ is the dimensionless form of $E(k)$.

The Figure shows clearly that the spectator contribution gives a much better description of the sum of the direct and crossed bubbles than that given by the direct bubble alone. The reason is that the (large) contribution from the nearby negative energy meson pole is cancelled by a similar contribution from the crossed bubble diagram. Such a cancellation between ladder and crosses ladder diagrams is the foundation of the spectator theory [3], but it was initially assumed that this cancellation would only be important when light mesons were exchanged between the scattered particles. Since a four point interaction is equivalent to the exchange of an infinitely heavy meson [cf. Eq. (4) above], this discussion demonstrates that the same cancellation is also important *even if the effective mesons being exchanged are very heavy*. This has been recently noted by Pascalutsa and Tjon [2] and by the author [5].

What should we do if there is no exchange term (exchange bubble in our example)? This situation could arise in a φ^3 -type theory, for example the familiar theory $\Psi^\dagger \gamma_5 \Psi \phi$. In this case the lowest order self energy diagram would be a bubble of the type shown in Fig. 1(a), and there would be no contribution similar to diagram 1(b). Hence there is *no diagram to cancel the negative energy meson pole*, and the taking the positive energy nucleon pole will not give a good description of the full result. For this reason, Surya and I [6,7] decided to use spectator equations based on the positive energy *meson* pole, which would emerge by closing the contour in Fig. 2(a) in the *lower* half plane. The positive energy

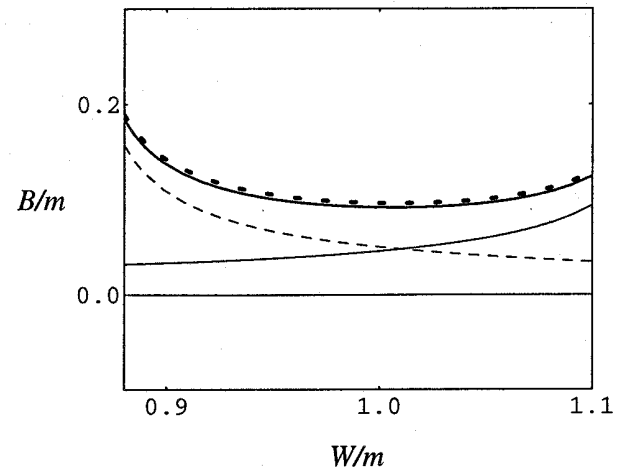


FIG. 3. The scalar functions B/m for the bubble and “crossed bubble” shown in Fig. 1. The light solid line is the exact bubble, B_a , the dashed line is the crossed bubble, B_b (for forward scattering), the heavy dotted line is the sum, and the heavy solid line is the positive energy nucleon pole contribution, B_S .

meson pole which is isolated in this way is very distant from the negative energy nucleon pole and gives a good approximation to the exact bubble. Of course one could just as well calculate the bubble exactly, but we intended to eventually embed the equations in the $NN\pi$ system, and wanted to preserve the spectator formalism in a three body system [8].

As shown in Fig. 4, the positive energy meson pole does an excellent job approximating the exact result for the direct bubble 1(a) [the solid line and the dotted line agree very well]. However, if $W < 0$, the positive energy meson pole gives a very different result. This *positive* energy meson pole contribution for negative W is *identical* to the *negative* energy meson pole contribution for *positive* W , and this is the result shown (the dashed line) in the figure. We see, in agreement with Refs. [1,2], that using the positive energy pole for *both* $W > 0$ and $W < 0$ violates C invariance very significantly. However, if the positive energy pole is used when $W > 0$ and the negative energy pole when $W < 0$, as suggested by the spectator philosophy, the results are identical and C invariance

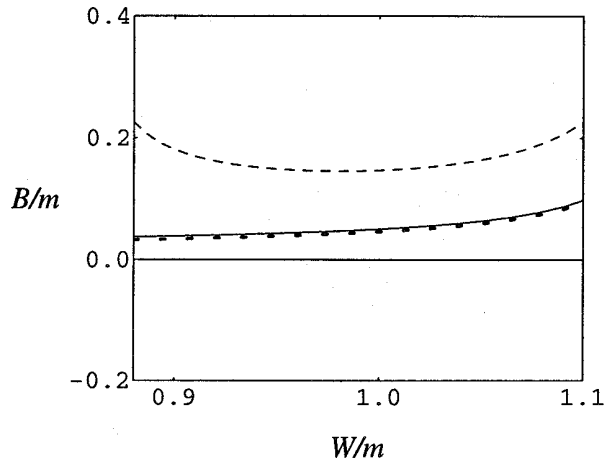


FIG. 4. The scalar function B/m as a function of the scaled energy W/m . The solid line is the positive energy meson pole contribution to $B_a(W)$; the dashed line is the *negative energy* meson pole contribution, and the dotted line is the exact result (identical to Fig. 3).

is satisfied. This brings us back to the discussion and derivation in the previous section.

IV. CONCLUSIONS

In this paper we have shown that

- the spectator equations satisfy charge conjugation invariance exactly provided the on-shell internal particle(s) are on their positive energy mass-shell when the external energies are positive, and on their negative energy mass-shell when the external energies are negative;
- this requirement is equivalent to extending the positive energy spectator theory to negative energies by replacing all external energies W_i by $|W_i| = \sqrt{W_i^2}$, and is consistent with the spectator philosophy;

- spectator equations with the the heavy particle on-shell should be used in all cases when there are exchange forces; and
- spectator equations with the light particle on-shell (or the Bethe-Salpeter equation) should be used if there are no exchange forces.

We close this discussion by emphasizing that the simplified *one channel* spectator theory described in this paper cannot be used when it is important to get an accurate description of the self energies or scattering amplitudes in a region where the external energy W is near zero. The simplified treatment described here has unphysical singularities at $W = 0$ [clearly evident in Eq. (8)], and unphysical cuts for $W^2 < 0$. In studies of the pion, where chiral symmetry requires an accurate description near $m_\pi = W \simeq 0$, and C invariance requires that the $q\bar{q}$ system be treated symmetrically, the *four channel* spectator equation originally introduced in Ref. [9] must be used.

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