Quantum Error Correction by Coding

Isaac L. Chuang[1] *and* R. Laflamme[2]

[1]*ERATO Quantum Fluctuation Project*
Edward L. Ginzton Laboratory, Stanford University, Stanford, CA 94305

[2]*Theoretical Astrophysics, T-6, MS B288*
Los Alamos National Laboratory, Los Alamos, NM 87545, USA

(October 13, 1995)

Recent progress in quantum cryptography and quantum computers has given hope to their imminent practical realization. An essential element at the heart of the application of these quantum systems is a quantum error correction scheme. We propose a new technique based on the use of coding in order to detect and correct errors due to imperfect transmission lines in quantum cryptography or memories in quantum computers. We give a particular example of how to detect a decohered qubit in order to transmit or preserve with high fidelity the original qubit.

Classical information theory tells us that messages may be communicated with high fidelity and at a finite rate, even along noisy channels, by using appropriate coding techniques [1,2]. Does this apply for transmitting *quantum* messages along *quantum* channels? Consider the following scenario: Alice wishes to send some qubit (quantum two-state system) $c_0\,|0\rangle + c_1\,|1\rangle$ to Bob, but their communication link is noisy – interaction with the line during transmission causes the information stored in the superposition state of the qubit to be lost to the environment (Figure 1). For example, Alice may prepare an electronic state of a single ion in some superposition state, then Federal Express it to Bob; unfortunately, field fluctuations along the way destroy part of the information stored in the superposition of the $|0\rangle$ and $|1\rangle$. Knowing the specific form of the microscopic interaction Hamiltonian, can Alice encode a qubit in some way such that Bob can detect and correct transmission errors?

The answer to this *quantum noisy coding problem* is presently unknown [3,4]. However, we have gained some insight by studying the special case of decoherence [5,6]: phase damping between the $|0\rangle$ and $|1\rangle$ states which leaves the relative probabilities intact. We have discovered a coding technique for partially detecting and correcting errors due to decoherence. Our idea is based on using a representation for a quantum bit which is sensitive to the quantum jumps induced by decoherence. Furthermore, the representation is constructed in such a manner that if no jump occurs, the state is left intact with high probability. The two cases are discriminated by using a projective measurement, which selects the original qubit with high probability, as the amount of redundancy in the code is increased. The key to why this works lies in an understanding of the decoherence process. The coding will also detect error due to amplitude damping, the process occurring when the excited state $|1\rangle$ decays to the ground state with a given probability.

High-fidelity transmission through a noisy channel is based on knowledge about the noise structure. For example, in the case of transmission of classical information where the $|1\rangle$ state can decay into $|0\rangle$, it is possible to make a code where every bit of the original message is mapped into a new message using an even number of $|1\rangle$ states. If we end up with an odd number of $|1\rangle$ states, we know that we have lost information during transmission [2].

The same principle is also true for a quantum channel. Unwanted environmental interactions along a channel cause the message $|\psi\rangle$ to *decohere* into some mixed state $\rho_{noisy} = \$(|\psi\rangle\langle\psi|)$, where \$ is a superscattering operator which describes the noise process [7,8]. It is not easy to find a way to make a code to correct for this interaction using the density matrix in the $(|0\rangle, |1\rangle)$ basis. However, useful hints can be obtained utilizing an equivalent *single wavefunction* picture of the noise process that is similar to the methods of quantum trajectories [9,10]. We will denote a phase-damped quantum trajectory "wavefunction" by the subscript *pd*. These mixed states will be expressed in the basis $|\phi_n\rangle$ in which all initial states remain diagonal during the noise process, such that $\rho_{noisy} = \sum_n |\phi_n\rangle\langle\phi_n|$. Such an identity always exists, for some ϕ_n, since a mixed state density matrix mathematically describes a statistical mixture of pure states.

The main gist of our coding technique is to represent a single qubit $|\psi_0\rangle = c_0|0\rangle + c_1|1\rangle$ as some state $|\psi\rangle = c_0|0_L\rangle + c_1|1_L\rangle$ using N qubits. Decoherence causes us to get a mixture of $|\phi_n\rangle$ states. However, our qubit representation is designed such that decoherence acts symmetrically upon the whole state, such that with probability p_0 we have $|\phi_0\rangle = \sqrt{p_0}|\psi\rangle$, which corresponds to having the qubit remain intact through the interaction. For $n \geq 1$, the states $|\phi_n\rangle$ describe cases when the qubit symmetry is disrupted by the noise process. These are undesirable final states which have to be rejected. Thus, if $|\phi_0\rangle$ were orthogonal to all the others, we could detect errors perfectly by distinguishing the two manifolds. Although we have not found such a perfect scheme, we can come close, as we show below.

Let us begin by describing the decoherence process. An assembly of N qubits, represented by the state

$$|\psi\rangle = \sum_{b\in\mathcal{B}} c_b\,|b\rangle \qquad (1)$$

decoheres due to interaction with the environment (with operators a_k^i) through the Hamiltonian

$$H_I = \lambda' \sum_{i=1}^{N}\sum_k \sigma_z^i {a_k^i}^\dagger a_k^i \qquad (2)$$

into the mixed state described by the density matrix

$$\rho_{pd} = \$_\Lambda(|\psi\rangle\langle\psi|) = \sum_{a\in\mathcal{B}}\sum_{b\in\mathcal{B}} c_a c_b^* e^{-\lambda h(a,b)}\,|a\rangle\langle b|\,, \qquad (3)$$

where $\$_\Lambda$ is the superscattering operator, λ parameterizes the amount of damping, σ_z^i rotates the i^{th} qubit about the computational basis, a and b are binary strings of N bits, and $\mathcal{B} = \{0,1\}^N$ is the set of 2^N bit-strings which span the Hilbert space. $h(a,b) = a$ xor b gives the Hamming distance [11] between a and b; it appears because the rate at which the off-diagonal element $\langle a|\rho_{pd}|b\rangle$ decays is proportional to the number of bits different between a and b. This assumes that each qubit is decohered by an independent bath, which is physically very reasonable.

The mixed state described by the density matrix ρ_{pd} can be decomposed into the explicit statistical mixture of pure states

$$\rho_{pd} = \sum_{n=0}^{2^N-1} |\phi_n\rangle\langle\phi_n| \qquad (4)$$

where

$$|\phi_n\rangle = \begin{cases} \displaystyle\sum_{b\in\mathcal{B}} c_b e^{-\lambda h(b)}\,|b\rangle & \text{for } n=0 \\ \displaystyle\sum_{b\in\mathcal{B}} \theta(n\wedge b)(1-\theta(n\wedge \overline{b})) & \\ \qquad\times\, c_b\, e^{-\lambda h(n,b)} \left[1-e^{-2\lambda}\right]^{\frac{h(n\wedge b)}{2}}\,|b\rangle & \text{for } n\geq 1 \end{cases} \qquad (5)$$

$\wedge$ denotes the bitwise binary AND function, $\theta(x) = 1$ for $x > 0$ and zero otherwise is the usual step function, $h(x) = h(x,0)$ is the Hamming weight of x, and $\overline{b}$ denotes the bitwise complement of b. The function of $\theta(n\wedge b)(1-\theta(n\wedge\overline{b}))$ is to select those values of n which have one's in its bit-string only where b does. Proof of the equivalence of Eqs. (4-5) to Eq. (3) is straightforward, and follows from showing that $\langle a|\rho_{pd}|b\rangle = c_a c_b^* e^{-\lambda h(a,b)}$.

In the quantum trajectory picture, the effect of phase damping on the state $|\psi$ can thus be described as

$$|\psi_{pd}\rangle = \$_\Lambda|\psi\rangle = \bigoplus_{n=0}^{2^N-1} |\phi_n\rangle\,. \qquad (6)$$

The $\oplus$ denotes a direct sum of the vector spaces (in contrast to a tensor product), such that $|\langle\alpha|\psi_{pd}\rangle|^2 = \sum_n |\langle\alpha|\phi_n\rangle|^2$ for an arbitrary pure state $\langle\alpha|$. For $p_n = |\langle\phi_n|\phi_n\rangle|^2$, we have that $\sum_n p_n = 1$, so we may understand p_n as being the weight of $|\phi_n\rangle$ in the mixture, and $|\phi_n\rangle/p_n$ as the possible final pure states after the decoherence. Physically, one may think of the decoherence process as occurring because of phase randomization due to interaction with a bath coordinate. In this picture, the $n=0$ state results when the interaction leaves the coordinate unchanged; otherwise, one of the $n>1$ states results. Keeping with the quantum trajectories idea, in the former case it can be said that the wavefunction is rotated by a non-unitary transform, and in the latter case, a quantum jump occurs.

The single wavefunction approach helps us in devising a code because we can think of the different results of the transmission as vectors in the Hilbert space, instead of having to use the space of density matrices. Geometrically, our coding technique works by first extending the Hilbert space by including ancilla qubits. A single qubit is then coded with the help of the ancilla in such a way that when the system has decohered we can decode and project the state so that the final qubit is as near to the original as possible.

The standard representation of a qubit encodes the logical zero and one states as $|0_L\rangle = |0\rangle$ and $|1_L\rangle = |1\rangle$, such that an arbitrary qubit is given by the state

$$|\psi\rangle = c_0\,|0\rangle + c_1\,|1\rangle\,. \qquad (7)$$

Using Eq. (5), we find that phase damping turns this pure state into the mixture

$$|\psi_{pd}\rangle = c_0\,|0\rangle + c_1 e^{-\lambda}\,|1\rangle \;\oplus\; c_1\left[1-e^{-2\lambda}\right]^{1/2}|1\rangle\,. \qquad (8)$$

Using the nomenclature of [10], decoherence either leads the original state to be rotated non-unitarily, or to a quantum jump into $|\phi_1\rangle$ (i.e. into the $|1\rangle$ state). The latter results with probability $|c_1|^2(1-e^{-2\lambda})$. Unfortunately, (1) the $|1\rangle$ state is in the space spanned by $|\psi\rangle$ and thus there is no way of detecting a jump, and (2) even if no jump occurs, the damping has deformed the original state and so $|\psi\rangle$ cannot be recovered intact.

Consider instead a single qubit which is represented by a sequence of N qubits, with the help of $N-1$ ancilla. Specifically, let $\mathcal{C} = \{0\cdots 001, 0\cdots 010, \ldots, 10\cdots 0\}$ be the set of all length $N/2$ bit-strings containing only one 1, such that we have the representation

$$|0_L\rangle = \sqrt{\frac{2}{N}}\sum_{b\in\mathcal{C}} |0\cdots 0b\rangle \qquad (9)$$

$$|1_L\rangle = \sqrt{\frac{2}{N}}\sum_{b\in\mathcal{C}} |b0\cdots 0\rangle\,, \qquad (10)$$

where the label of each ket contains N digits. We shall call the manifold defined by these two unit vectors the *representation manifold*, and say that as long as a state lives within this plane, it satisfies the *representation invariance* condition. The effect of phase damping on an arbitrary qubit superposition, $|\psi\rangle = c_0\,|0_L\rangle + c_1\,|1_L\rangle$ is found to be

$$|\psi_{pd}\rangle = e^{-\lambda}\,|\psi\rangle \;\oplus\; \left[\bigoplus_{b\in\mathcal{C}} c_0\sqrt{\frac{2}{N}}\left[1-e^{-2\lambda}\right]^{1/2}\,|0\cdots 0b\rangle\right] \oplus \left[\bigoplus_{b\in\mathcal{C}} c_1\sqrt{\frac{2}{N}}\left[1-e^{-2\lambda}\right]^{1/2}\,|b0\cdots 0\rangle\right]. \quad (11)$$

This time, when a jump occurs, it results in some state $|0\cdots 0b\rangle$ or $|b0\cdots 0\rangle$ for some $b\in\mathcal{B}$, and furthermore, this result contains a component orthogonal to the representation manifold. It thus violates the representation invariance condition, meaning that we can detect, with probability $1-|\langle 0_L|\psi_{pd}\rangle|^2 - |\langle 1_L|\psi_{pd}\rangle|^2$ when any jump has occured, using a projective measurement (which leaves the qubit intact). Furthermore, when a jump does not occur, because of the symmetry of the effect of decoherence on states in the representation manifold, the original qubit is left intact! This result is the basis for an error correction scheme against decoherence.

Note that in our model, we do not assume that the ancilla are "error-free." Rather, the ancilla qubits decohere along with the original qubit; this is important because that is the case for realistic systems.

The qubit code given in Eqs. (9-10) can be used to increase the probability of successful transmission of a qubit through an imperfect communication link (Figure 2). Alice prepares her single qubit $|\psi_0\rangle = c_0\,|0\rangle + c_1\,|1\rangle$, and introduces $N-1$ ancilla qubits (prepared in the ground state $|0\cdots 0\rangle$) to get $|\psi_1\rangle = c_0\,|e_0\rangle + c_1\,|e_1\rangle$, where $|e_0\rangle = |0\cdots 00\rangle$ and $|e_1\rangle = |0\cdots 01\rangle$. This is fed into the unitary coding transform U to give $|\psi_2\rangle = U|\psi_1\rangle = c_0\,|0_L\rangle + c_1\,|1_L\rangle$. For example, we may have

$$|\psi_2\rangle = \frac{c_0}{\sqrt{3}}\left[|000001\rangle + |000010\rangle + |000100\rangle\right] + \frac{c_1}{\sqrt{3}}\left[|001000\rangle + |010000\rangle + |100000\rangle\right]. \quad (12)$$

The result is transmitted to Bob, who decodes his received mixed state $|\psi_3^{pd}\rangle = \$_\Lambda|\psi_2\rangle$ to get $|\psi_4^{pd}\rangle = U^\dagger\,|\psi_3^{pd}\rangle$. Note that $|\psi_3^{pd}\rangle$ is given by Eq.(11). Bob will be interested in two probabilities. He will reject the entire transmission if any ancilla qubit is measured to be nonzero; otherwise, he will accept it. This happens with probability

$$P_{\text{accept}} = |\langle e_0|\psi_4^{pd}\rangle|^2 + |\langle e_1|\psi_4^{pd}\rangle|^2 \quad (13)$$

$$= \frac{2}{N} + \left(1-\frac{2}{N}\right)e^{-2\lambda}. \quad (14)$$

As $\lambda\to\infty$, $P_{\text{accept}}\to 2/N$ because even when the state becomes completely decohered, there is some probability of not detecting the error. Note that for small λ, the rejection rate $(1-P_{\text{accept}})^{-1}$ is essentially independent of N. When all the $N-1$ ancilla bits are found to be zero, then the qubit Bob receives is a "good" qubit, which is described by the density matrix

$$\rho_5 = \frac{1}{P_{\text{accept}}}\begin{bmatrix} |\langle 0_L|\psi_3\rangle|^2 & \langle 0_L|\psi_3\rangle\langle\psi_3|1_L\rangle \\ \langle 1_L|\psi_3\rangle\langle\psi_3|0_L\rangle & |\langle 1_L|\psi_3\rangle|^2 \end{bmatrix} \quad (15)$$

$$= \begin{bmatrix} |c_0|^2 & Jc_0c_1^* \\ Jc_0^*c_1 & |c_1|^2 \end{bmatrix}, \quad (16)$$

where J describes the decoherence which occurs despite the error correction scheme, and is found to be

$$J = \frac{N}{2e^{2\lambda}-2+N} \approx 1-\frac{4\lambda}{N}. \quad (17)$$

For small λ, the amount of decoherence suffered decreases inversely as the number of qubits N used in the code. In comparison, if the usual qubit representation Eq. (7) is used, the amount of decoherence suffered is $J_0 = e^{-\lambda} \approx 1-\lambda$, and thus the advantage of our scheme is that it causes the off-diagonal terms to decay less quickly, as long as $N > 2(1+e^\lambda)$ (recall that e^λ is the decoherence suffered by each qubit). Finally, Bob extracts the correct result with probability

$$P_{\text{correct}} = |\langle\psi_0|\rho_5|\psi_0\rangle|^2 = 1-2|c_0|^2|c_1|^2(1-J) \geq \frac{1+J}{2}, \quad (18)$$

so that $P_{\text{correct}} \geq 1-2\lambda/N$ for small λ. This probability is known as the transmission fidelity $\mathcal{F}$ [3].

One interesting question to ask is: in analogy to the watchdog effect [12], can the error detection probability and transmission fidelity be improved by periodic correction? The answer depends on the form of the errors suffered as a function of time. Instead of suffering decoherence λ, we may perform k corrections each with decoherence λ/k. In this case, we find that

$$P_{\text{accept}} = \left[\frac{2}{N} + \left(1-\frac{2}{N}\right)e^{-\lambda}\right]^k \quad\text{and}\quad J_k = \left[\frac{N}{2e^\lambda-2+N}\right]^k, \quad (19)$$

which, unfortunately, is worse than the result of Eqs.(14) and (17) for any $N>2$. Of course, this happens because we have assumed that phase damping occurs exponentially with time, in which case it is known that the watchdog effect is ineffective. Instead, of $e^{-\lambda t}$, if we have an error rate which is quadratic in time, $1-\epsilon t^2$, then we must compare

$$J = \frac{N}{2(1-k^2\epsilon)-2+N} \quad\text{and}\quad J_k = \left[\frac{N}{2(1-\epsilon)-2+N}\right]^k. \quad (20)$$

For small error per step ϵ, $J \approx 1 + 2k^2\epsilon/N$, and $J_k \approx 1 + 2k\epsilon/N$, so we find that periodic correction *is* effective. Our scheme works hand-in-hand with the principle of watchdog stabilization.

By increasing the size of the Hilbert space and using a coding which distributes unwanted transmission errors symmetrically, we have demonstrated how a single qubit can be coded to guarantee as high transmission fidelity as desired using additional ancilla bits. Another interesting characteristic about the scheme presented here is that it can also be used to perfectly detect errors due to amplitude decoherence. The coding in Eqs. (9-10) is a generalization of the dual-rail bit of [13,14] and their scheme can be adapted straightforwardly. Our result may be contrasted with that of [15]. Shor has a scheme that is able to reconstruct the initial state exactly assuming that only one of nine bits decohered. Our scheme is independent of the decoherence strength. It might be possible to adapt Shor's scheme to ours in order to get perfect fidelity. Bennett et. al. have devised a method to rejuvenate EPR pairs that have lost their purity [16]. Their method uses EPR bits to accomplish this process, while ours needs only ancilla qubits in their ground states and will be advantageous when EPR pairs are expensive.

We believe that other (better) coding schemes against decoherence, developed along the lines we have presented, may exist. In our search, we found an intriguing one using only one ancilla where

$$|0_L\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \tag{21}$$

$$|1_L\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}}\,, \tag{22}$$

for which the probability of acceptance and the fidelity are

$$P_{\rm accept} = \frac{1+e^{-2\lambda}}{2} \qquad \text{and} \qquad \mathcal{F} = 1 - 2\frac{|c_0|^2|c_1|^2}{\cosh\lambda}\,. \tag{23}$$

The interesting point here is that for small λ, $\mathcal{F}$ is quadratic in λ. However we were unable to generalize this to a scheme which would go as λ^2/N for a N-bit code. This is in contrast with the code in Eqs. (9-10).

In conclusion, our error correction technique uses a N-qubit representation of a single qubit to increase the transmission fidelity through a noisy quantum channel from $1-\lambda/2$ (for no error correction) to at least $1-2\lambda/N$ for the accepted qubit, using the code in Eqs. (9-10). This result provides an example of how coding can be used to construct representations which are robust against phase decoherence. The same general technique may be applied to construct error correcting codes for other sources of decoherence.

[1] C. E. Shannon, Bell System Technical Journal **27**, 379 (1948).

[2] T. Cover and J. Thomas, *Elements of Information Theory* (Wiley, New York, 1991).

[3] B. Schumacher, Phys. Rev. A **51**, 2738 (1995).

[4] P. Hausladen, B. Schumacher, M. Westmoreland, and W. Wootters, Preprint (1995).

[5] W. H. Zurek, Physics Today (1991).

[6] W. H. Zurek, Progress of Theoretical Physics **89**, 281 (1993).

[7] S. W. Hawking, Comm. Math. Phys. **87**, 406 (1982).

[8] C. W. Gardiner, *Quantum Noise* (Springer-Verlag, New York, 1991).

[9] H. J. Carmichael, Phys. Rev. Lett. **70**, 2273 (93).

[10] K. Molmer, Y. Castin, and J. Dalibard, J. Opt. Soc. Am. B **10**, 524 (1993).

[11] E. A. Lee and D. G. Messerschmitt, *Digital Communication* (Kluwer Academic Publishers, ADDRESS, 1988).

[12] W. H. Zurek, Phys. Rev. Lett. **53**, 391 (1984).

[13] I. L. Chuang and Y. Yamamoto, Phys. Rev. A (1995).

[14] I. L. Chuang and Y. Yamamoto, Submitted to Phys. Rev. Lett. (1995).

[15] P. Shor, Preprint (1995).

[16] C. Bennett *et al.*, Phys. Rev. Lett. to appear (1995).

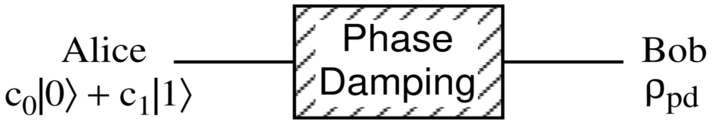

FIG. 1. Transmission of a single qubit down a noisy communication link.

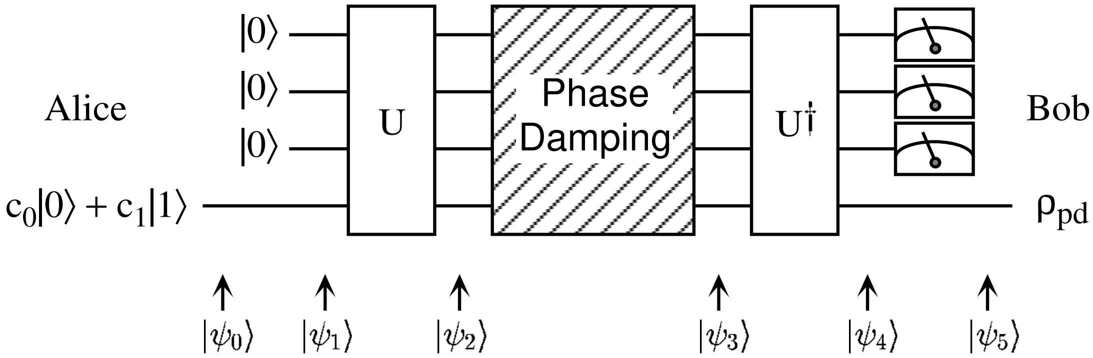

FIG. 2. Single qubit transmission using a code which is robust against phase decoherence. The $N = 4$ case is pictured.