Lattice QCD at High Temperature and the QGP

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Abstract. We review recent progress in studies of bulk thermodynamics of strongly interacting matter, present results on the QCD equation of state and discuss the status of studies of the phase diagram at non-vanishing quark chemical potential.

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INTRODUCTION

During recent years our knowledge of the thermodynamics of strongly interacting elementary particles greatly advanced. Lattice calculations now allow to study also the thermodynamics at non-zero quark chemical potential (μ_q). The different approaches developed for this purpose [1, 2, 3, 4, 5] are still limited to the regime of high temperature and small values of the chemical potential, $T \gtrsim 0.9T_c,\ \mu_q/T \lesssim 1$. They, however, allow already to analyze the density dependence of the equation of state in a regime relevant for a wide range of energies explored by heavy ion experiments and may even be sufficient to establish or rule out the existence of a second order phase transition point in the QCD phase diagram. The existence of such a critical point as endpoint of a line of first order phase transition that separates at low temperature the low and high density regions in the QCD phase diagram, is suggested by many phenomenological models. For small values of μ_q/T lattice calculations suggest that the transition from low to high temperature is not a phase transition; the transition during which bulk thermodynamic quantities, e.g. the energy density, change rapidly and the chiral condensate drops suddenly, is a continuous, rapid crossover. It thus has been speculated [6] that a 2^{nd} order phase transition point exists somewhere in the interior of the QCD phase diagram.

The generic form of the QCD phase diagram is shown in Fig. 1. Although this phase diagram is well motivated by model calculations little is known quantitatively about it from lattice calculations. All calculations performed so far for QCD with two light quarks with or without the inclusion of a heavier strange quark find a smooth but rapid crossover from the low to high temperature regime. However, these calculations also did not provide evidence for the expected universal critical behavior in the light quark mass limit, which for 2-flavor QCD should be that of a 3-dimensional $O(4)$ or $O(2)$ spin model. It generally is expected that calculations closer to the continuum limit are needed to unreveal these universal features of the QCD transition. In view of this missing evidence for $O(N)$ scaling, it has been argued recently that the transition in 2-flavor QCD could also be a weak first order transition [7], in which case the entire transition line in the μ-T phase diagram could be a line of first order transitions.

We will start our survey of lattice calculations on QCD thermodynamics in the next

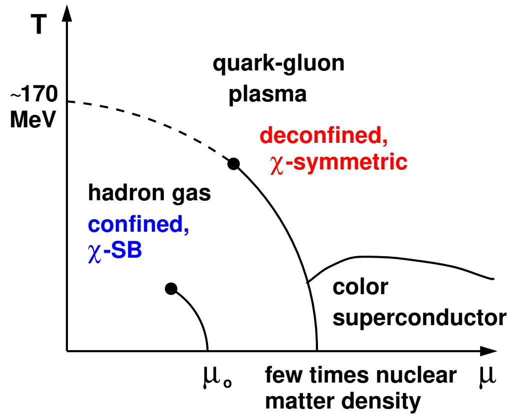

FIGURE 1. Sketch of the QCD phase diagram.

Section by discussing recent studies of the QCD equation of state at vanishing and non-vanishing chemical potential. Section 3 is devoted to a discussion of the status of studies of the 2^{nd} order phase transition point (*chiral critical point*) in the QCD phase diagram.

THE QCD EQUATION OF STATE

Vanishing baryon number density

Most information on the structure of the high temperature phase of QCD and the nature of the transition itself has been obtained through lattice calculations performed in the limit of vanishing baryon number density or vanishing quark chemical potential ($\mu_q = 0$). This limit is most relevant for our understanding of the evolution of the early universe and also is the regime which can be studied experimentally in heavy ion collisions at RHIC (BNL) and soon also at the LHC (CERN). The experimental accessibility of this regime of dense matter also drives the desire to go beyond a qualitative analysis of the QCD phase transition and to aim at a numerically accurate determination of basic parameters that characterize the thermodynamics of dense matter at high temperature.

The transition to the high temperature phase of QCD is related to the restoration of chiral symmetry as well as deconfinement; the vanishing of the chiral condensate at the transition temperature T_0 and the sudden liberation of quark and gluon degrees of freedom is clearly visible in Fig. 2. As can be seen the temperature and quark mass dependence of the light quark chiral condensate in QCD with two light and a heavier strange quark is similar to that of QCD with 3 degenerate (light) quarks. In the high temperature phase, on the other hand, bulk thermodynamic observables, e.g. the pressure shown in Fig. 2(left), clearly reflect the number of light degrees of freedom and also are sensitive to the heavier strange quark mass[1].

[1] In the calculation of the pressure of (2+1)-flavor QCD shown in Fig. 2(right) the strange quark mass has actually been taken to be proportional to the temperature, *i.e.* m_s/T =const. If one keeps instead m_s

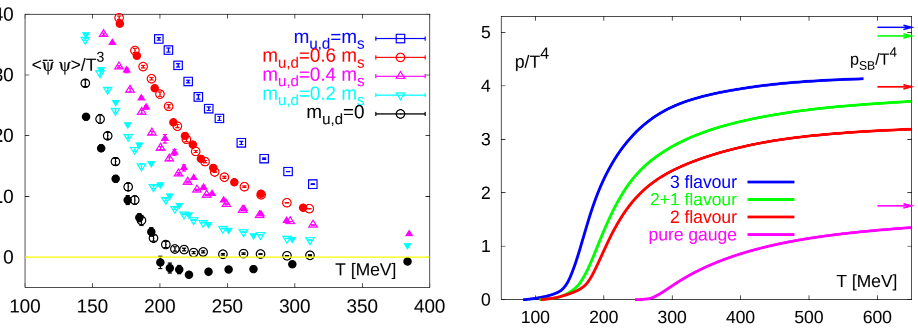

FIGURE 2. The light quark chiral condensate in QCD with 2 light up, down and a heavier strange quark mass (open symbols) and in 3-flavor QCD with degenerate quark masses (full symbols) [8]. The right hand part of the figure shows the pressure calculated in QCD with different number of flavors as well as in a pure gauge theory [9]. Note that (2+1)-flavor QCD here refers to QCD with two light quarks and a heavier (strange) quark with a mass proportional to the temperature, $m_s \sim T$.

Recent studies of the equation of state concentrated on calculations with an almost realistic quark mass spectrum performed along lines of constant physics, *i.e.* with light and strange quark masses fixed in units of hadron masses rather than the temperature [10, 11]. To reduce cut-off effects induced by the finite lattice spacing larger temporal lattices ($N_\tau = 6$) and so-called fat links have been used. At present these calculations still have been performed with rather small spatial volumes, $V^{1/3}T \simeq 2$. Close to T_0, where the correlation length becomes large, a more detailed analysis of the approach to the thermodynamic limit thus still has to be performed in the future.

Partly these calculations still have been performed with unimproved fermion actions [10] which makes it difficult to control cut-off effects and draw quantitative conclusions on the approach to the ideal gas limit at high temperature. Nonetheless, these calculations support earlier findings on the temperature dependence of the pressure and energy density in the transition region. In particular, they show that the presence of strange quarks has little influence on the thermodynamics in the vicinity of T_0. In Fig. 3 we compare the recent calculation of the pressure in (2+1)-flavor QCD [10] and earlier results for 2-flavor QCD [12, 13] which both have been performed with unimproved staggered fermions. The good agreement in the vicinity of T_0 suggests that the strange quark contribution to the pressure is small. Only for $T \gtrsim 1.5T_0$ differences show up; the positive strange quark contribution to the pressure in $(2+1)$-flavor QCD becomes sizeable[2].

The good agreement between recent calculations in (2+1)-flavor QCD, performed with smaller quark masses and smaller lattice spacings [10, 11], and earlier results [9, 12, 13] is quite reassuring. These calculations confirm that thermodynamics in the

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fixed to its physical value the strange quark effectively becomes light in units of the temperature and the pressure will gradually approach the high temperature limit of 3-flavor QCD.

[2] As the relative size of the strange quark contribution to thermodynamic quantities changes with temperature it is obvious that cut-off effects can not be taken into account by rescaling the pressure or energy density with a universal, temperature independent factor [10]. This underlines the need of improved actions for thermodynamic calculations.

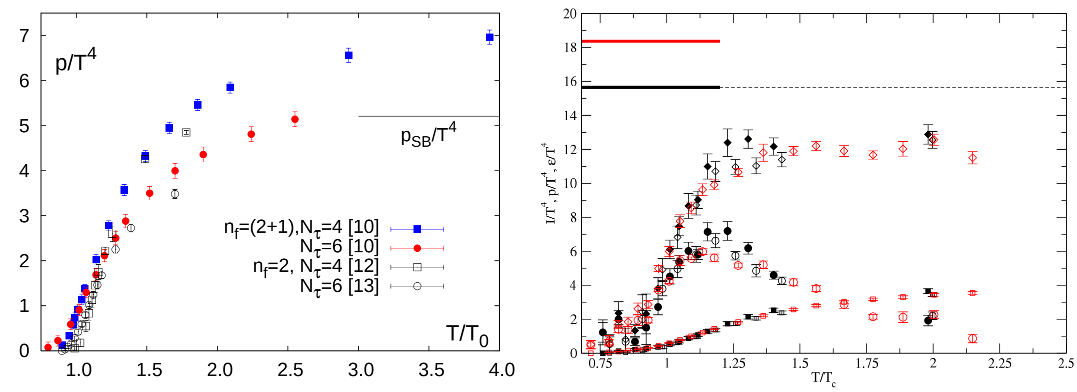

FIGURE 3. Cut-off dependence of the pressure calculated with the standard staggered fermion action on lattices with temporal extent $N_\tau = 4$ and 6 in 2-flavor QCD [12, 13] and (2+1)-flavor QCD [10] (left). The right hand figure shows energy density, pressure and $I = \varepsilon - 3p$ for (2+1)-flavor QCD calculated on lattices with temporal extent $N_\tau = 4$ and 6 with an $\mathcal{O}(a^2)$ improved staggered fermion action [11].

high temperature phase is rather insensitive to changes of the quark mass; a reduction of the light quark masses by almost an order of magnitude does not lead to drastic changes in the energy density at the transition point and in the high temperature phase. Moreover, they show that the transition itself is not strongly influenced by discretization errors, which in the staggered fermion formulation show up prominently in the distortion of the Goldstone modes; reducing m_q and thus the masses of light hadrons as well as reducing flavor symmetry breaking effects drastically [10] does not significantly change the energy density at the transition temperature. The estimate, $\varepsilon_c/T_0^4 = 6 \pm 2$ [9], is consistent with the recent calculations in (2+1)-flavor QCD performed with lighter quark masses. The weak dependence of transition parameters on the quark mass reflects the importance of numerous heavy resonances that are necessary to build up the particle and energy density needed for the transition to occur [14].

Also the recent studies of the equation of state performed at vanishing quark chemical potential suggest that for physical values of the quark masses the transition to the high temperature phase of QCD only is a rapid crossover rather than a phase transition which on finite lattices would be signaled by metastabilities in bulk thermodynamic observables or the chiral condensate. None of the calculations performed so far for QCD with two light quarks with or without the inclusion of a heavier strange quark gave direct evidence for a first order phase transition.

The missing guidance from any universal behavior in the vicinity of the transition also influences the determination of the transition temperature itself. As the transition temperature is determined at quark mass values which are usually larger than those realized in nature one has to extrapolate to the physical regime. Depending on the ansatz used for the quark mass dependence of T_c the extrapolations can differ by about 5%. A similar systematic uncertainty arises from the calculation of a zero temperature observable that is used to set the physical units for T_c. Thus also the recent determinations of the transition temperature [11, 15], which lead to values $T_c \simeq 170$ MeV still suffer from systematic errors of about 10%.

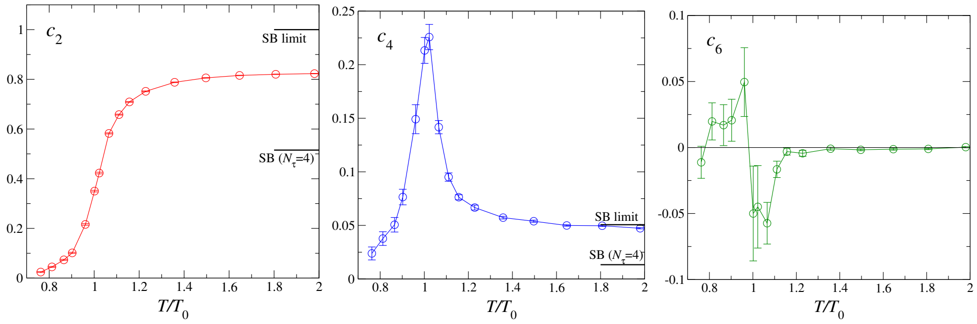

FIGURE 4. Temperature dependence of expansion coefficients for p/T^4 in 2-flavor QCD and for quark masses corresponding at T_c to a pseudo-scalar (pion) mass of about 770 MeV.

Non-zero baryon number density

Studies of the QCD equation of state have recently been extended to the case of non-zero quark chemical potential (μ_q). Calculations of bulk thermodynamic quantities for $\mu_q > 0$ based on the reweighting approach [16], using the Taylor expansion of the partition function [17, 18, 19], as well as analytic continuation of calculations performed with imaginary values of the chemical potential [20] show that the μ_q-dependent contributions to energy density and pressure are dominated by the leading order $(\mu_q/T)^2$ correction for parameters relevant for the description of dense matter formed at RHIC ($\mu_q/T \simeq 0.1$). Higher order contributions are still only a few percent at SPS energies ($\mu_q/T \lesssim 0.6$).

We will focus in the following on a discussion of Taylor expansions of the partition function of 2-flavor QCD around $\mu_q = 0$. At fixed temperature and small values of the chemical potential the pressure may be expanded in a Taylor series around $\mu_q = 0$,

$$\frac{p}{T^4} = \frac{1}{VT^3}\ln Z = \sum_{n=0}^{\infty} c_n(T,m_q)\left(\frac{\mu_q}{T}\right)^n \quad , \tag{1}$$

where the expansion coefficients are given in terms of derivatives of $\ln Z(T,\mu_q)$, *i.e.* $c_n(T,m_q) = \dfrac{1}{n!}\dfrac{\partial^n \ln Z}{\partial(\mu_q/T)^n}$. The series is even in (μ_q/T) which reflects the invariance of $Z(T,\mu_q)$ under exchange of particles and anti-particles. The Taylor series for the energy density can then be obtained using the relation, $(\varepsilon - 3p)/T^4 = T\mathrm{d}(p/T^4)/\mathrm{d}T$,

$$\frac{\varepsilon}{T^4} = \sum_{n=0}^{\infty}\left(3c_n(T,m_q) + c_n'(T,m_q)\right)\left(\frac{\mu_q}{T}\right)^n \tag{2}$$

with $c_n'(T,m_q) = T\mathrm{d}c_n(T,m_q)/\mathrm{d}T$. A similar relation holds for the entropy density [19]. The coefficients $c_n(T,m_q)$ calculated for a fixed value of the bare quark mass up to $n = 6$ are shown in Fig. 4.

Knowing the dependence of the energy density and the pressure on the quark chemical potential one can eliminate μ_q in favor of a variable that characterizing the thermodynamic boundary conditions for the system under consideration [19]. In the case of dense

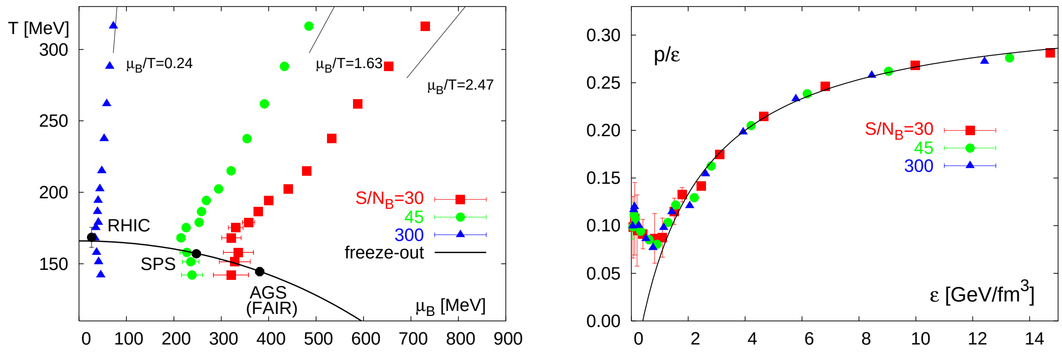

FIGURE 5. Equation of state of 2-flavor QCD on lines of constant entropy per baryon number. The left hand figure shows three lines of constant S/N_B in the QCD phase diagram relevant for the freeze-out parameters determined in various heavy ion experiments. The right hand figure shows the equation of state on these trajectories using $T_c = 175$ MeV to set the scale. The solid curve in the right hand figure is the parametrization of the high temperature part of the equation of state given in Eq. 3.

matter created in heavy ion collisions this is a combination of entropy and baryon number. Both quantities stay constant during the expansion of the system. In Fig. 5 we show the resulting isentropic equation of state as function of temperature as well as energy density obtained from a 6^{th} order Taylor expansion of pressure and energy density [19]. The three different entropy and baryon number ratios, $S/N_B = 30,\ 45$ and 100, correspond roughly to isentropic expansions of matter formed at the AGS, SPS and RHIC, respectively. It is quite remarkable that $p(\varepsilon)$ is to a good approximation independent of S/N_B; for temperatures $T > T_0$, or equivalently $\varepsilon \gtrsim 0.8$ GeV/fm^3, it is well described by

$$\frac{p}{\varepsilon} = \frac{1}{3}\left(1 - \frac{1.2}{1 + 0.5\ \varepsilon\ \mathrm{fm}^3/\mathrm{GeV}}\right) , \tag{3}$$

which for large ε agrees with a bag equation of state with $B^{1/4} \simeq 260$ MeV.

The insensitivity of the isentropic equation of state on S/N_B also implies that the velocity of sound, $v_S^2 = \mathrm{d}p/\mathrm{d}\varepsilon$ is similar along different isentropic expansion trajectories. In fact, the parametrization given in Eq. 3 suggests that the velocity of sound approaches rather rapidly the ideal gas value, $v_S^2 = 1/3$. In Fig. 6 we summarize results for v_s^2 obtained in lattice calculations for a pure $SU(3)$ gauge theory [21], for 2-flavor QCD with Wilson fermions [22] and $(2+1)$-flavor QCD with staggered fermions [10].

THE CHIRAL CRITICAL POINT

Various model calculations [6] suggest that a second order phase transition point (chiral critical point) exists in the QCD phase diagram which separates a region of first order phase transitions at high baryon number density and low temperatures from a cross-over region at low baryon number density and high temperature. Evidence for the existence of such a critical point may come from lattice calculations at non-zero quark chemical potential by either determining the location of Lee-Yang zeroes [1] or by determining

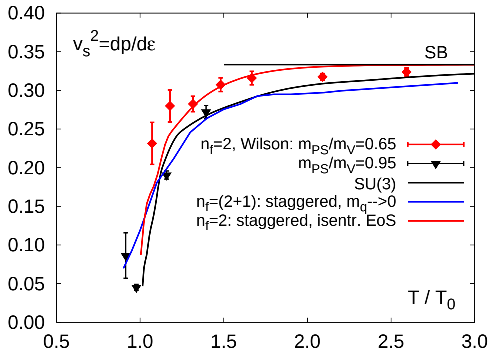

FIGURE 6. The velocity of sound in QCD vs. temperature expressed in units of the transition temperature T_0. Shown are results from calculations with Wilson [22] and staggered fermions [10] as well as for a pure SU(3) gauge theory [21]. Also shown is the resulting v_s^2 deduced from Eq. 3 [19].

the convergence radius of the Taylor series for the logarithm of the partition function which directly yields the pressure, $p/T = V^{-1} \ln Z$ [17].

If there exists a 2^{nd} order phase transition point in the QCD phase diagram, this could be determined from an analysis of the volume dependence of Lee-Yang zeroes of the QCD partition function. In any finite volume zeroes of $Z(V,T,\mu_q)$ only exist in the complex μ_q plane with $\mathrm{Im}\mu_q \neq 0$. Only for $V \to \infty$ some of these zeroes may converge to the real axis and will then give rise to singularities in thermodynamic quantities. The relation between phase transitions and zeroes of the partition function has been exploited using a reweighting technique to extend lattice calculations at $\mu_q = 0$ to $\mu_q > 0$ [1]. Recent results based on this approach [23] suggest that a critical point indeed exists and occurs at $\mu_B = 3\mu_q \simeq 360$ MeV. This estimate is about a factor two smaller than earlier estimates [1] which have been obtained on smaller lattices and with large quark masses. This suggests that a detailed analysis of the quark mass and volume dependence [24] still is needed to gain confidence in the analysis of Lee-Yang zeroes.

The radius of convergence of the Taylor series is controlled by a singular point in the complex μ_q plane closest to the origin. It is related to the location of the critical point only if this singularity lies on the real axis. A sufficient condition for this is that all expansion coefficients in the Taylor series are positive. For temperatures below the transition temperature at $\mu_q = 0$ this is indeed the case for all expansion coefficients calculated so far. The first coefficient, c_0, just gives the pressure at $\mu_q = 0$ shown in Fig. 2(right) and thus is positive for all temperatures. This also is the case for c_2, which is proportional to the quark number susceptibility at $\mu_q = 0$ [25],

$$\frac{\chi_q}{T^2} = \frac{\partial^2 p/T^4}{\partial(\mu_q/T)^2} = \sum_{n=0}^{\infty} d_n \left(\frac{\mu_q}{T}\right)^n \quad \text{with} \quad d_n = (n+2)(n+1)c_{n+2} \, . \tag{4}$$

As can be seen in Fig. 3 also the next-to-leading order coefficient, c_4, is strictly positive.

A new feature shows up in the expansion coefficients at $\mathcal{O}(\mu^6)$. The coefficient c_6 is positive only below T_0 and changes sign in its vicinity. If this pattern persists for higher order expansion coefficients one may conclude that the irregular signs of the expansion coefficients for $T > T_0$ suggest that the radius of convergence of the Taylor series is not related to critical behavior at these temperatures, whereas it determines a critical point for $T < T_0$ if also the higher order expansion coefficients stay positive.

Ratios of subsequent expansion coefficients provide an estimate for the radius of convergence of the Taylor expansion,

$$\rho(T) = \lim_{n\to\infty} \rho_n \equiv \lim_{n\to\infty} \sqrt{\left|\frac{c_n}{c_{n+2}}\right|} = \lim_{n\to\infty} \sqrt{\left|\frac{d_n}{d_{n+2}}\right|} \quad . \tag{5}$$

The expansion coefficients d_n for the quark number susceptibility have been analyzed recently for unimproved staggered fermions [26] up to $n = 6$. It has been shown that an accurate determination of these expansion coefficients requires large physical volumes. Based on a finite volume analysis the radius of convergence has been estimated from the Taylor series of the quark number susceptibility to be $\mu_B \simeq 180$ MeV [26]. As the expansion coefficients d_n are directly related to the expansion coefficients c_n of the pressure the radius of convergence coincides in the limit $n \to \infty$. For finite n, *i.e.* $n \simeq 6$, estimates based on the Taylor series for the pressure will be about 30% higher than those based on the Taylor series for susceptibilities.

While the quark number susceptibility is expected to diverge at the chiral critical point, the pressure, of course, will stay finite. Although χ_q rises rapidly with increasing μ_q/T, this is partly due to the rapid increase of the pressure itself. A quantity reflecting the relative magnitude of fluctuations is given by the ratio,

$$\frac{\Delta p}{T^2\chi_q} = \frac{1}{2}\left(\frac{\mu_q}{T}\right)^2 \frac{1+\frac{c_4}{c_2}\left(\frac{\mu_q}{T}\right)^2+\frac{c_6}{c_2}\left(\frac{\mu_q}{T}\right)^4+\mathcal{O}\left(\left(\frac{\mu_q}{T}\right)^6\right)}{1+6\frac{c_4}{c_2}\left(\frac{\mu_q}{T}\right)^2+15\frac{c_6}{c_2}\left(\frac{\mu_q}{T}\right)^4+\mathcal{O}\left(\left(\frac{\mu_q}{T}\right)^6\right)} \tag{6}$$

This quantity only depends on ratios of Taylor expansion coefficients, and should vanish at a 2^{nd} order phase transition point. It is shown in Fig. 7. As can be seen the ratio rises rapidly across T_0 and approaches the ideal gas value. It, however, does not show any sign of a drop in the vicinity of T_0 that could be taken as evidence for the existence of a second order transition. For this reason it has been argued in Ref. [18] that conclusions on the radius of convergence drawn from estimators ρ_n may be premature at least for the quark masses used in this calculations. As these masses were significantly larger than those used in Ref. [26] both results are not necessarily in contradiction. Like in the case of studies of Lee-Yang zeroes a more detailed analysis of the quark mass dependence also is needed in this case.

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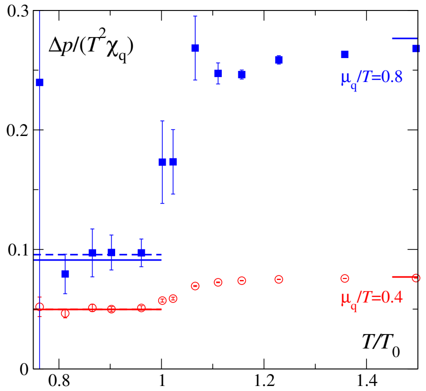

FIGURE 7. The dimensionless ratio $\Delta p/T^2\chi_q$ for two values of the chemical potential μ_q/T. Horizontal lines show the expected results for a hadronic resonance gas and an ideal quark-antiquark gas below and above T_0, respectively.

REFERENCES

1. Z. Fodor and S. Katz, *JHEP* **0203**, 014 (2002).
2. C.R. Allton, S. Ejiri, S.J. Hands, O. Kaczmarek, F. Karsch, E. Laermann, C. Schmidt and L. Scorzato, *Phys. Rev. D* **66**, 074507 (2002).
3. R.V. Gavai and S. Gupta, *Phys. Rev. D* **64**, 074506 (2001).
4. M.-P. Lombardo, *Phys. Rev. D* **67**, 014505 (2003).
5. P. de Forcrand and O. Philipsen, *Nucl. Phys. B* **642**, 290 (2002).
6. M.A. Stephanov, K. Rajagopal and E.V. Shuryak, *Phys. Rev. Lett.* **81**, 4816 (1998).
7. M. D'Elia, A. Di Giacomo and C. Pica, *Phys. Rev. D* **72**, 114510 (2005).
8. C. Bernard *et al.* [MILC Collaboration], *Phys. Rev. D* **71**, 034504 (2005).
9. F. Karsch, E. Laermann, and A. Peikert, *Phys. Lett. B* **478**, 447 (2000).
10. Y. Aoki, Z. Fodor, S. D. Katz and K. K. Szabo, hep-lat/0510084.
11. C. Bernard *et al.*, *PoS* **LAT2005**, 156 (2005).
12. T. Blum, L. Karkkainen, D. Toussaint and S. A. Gottlieb, *Phys. Rev. D* **51**, 5153 (1995).
13. C. W. Bernard *et al.*, *Phys. Rev. D* **55**, 6861 (1997).
14. F. Karsch, K. Redlich and A. Tawfik, Eur. Phys. J. C **29**, 549 (2003).
15. V. G. Bornyakov et al., *PoS* **LAT2005**, 157 (2005).
16. Z. Fodor, S. D. Katz and K. K. Szabo, *Phys. Lett. B* **568**, 73 (2003);
    F. Csikor, G. I. Egri, Z. Fodor, S. D. Katz, K. K. Szabo and A. I. Toth, *JHEP* **0405**, 046 (2004).
17. C. R. Allton, S. Ejiri, S. J. Hands, O. Kaczmarek, F. Karsch, E. Laermann and C. Schmidt, *Phys. Rev. D* **68**, 014507 (2003).
18. C. R. Allton, M. Doring, S. Ejiri, S.J. Hands, O. Kaczmarek, F. Karsch, E. Laermann, K. Redlich, *Phys. Rev D* **71**, 054508 (2005).
19. S. Ejiri, F. Karsch, E. Laermann and C. Schmidt, *The isentropic equation of state of 2-flavor QCD*, hep-lat/0512040.
20. M. D'Elia and M.-P. Lombardo, *Phys. Rev. D* **64**, 014505 (2003).
21. G. Boyd, J. Engels, F. Karsch, E. Laermann, C. Legeland, M. Lütgemeier and B. Petersson, *Nucl. Phys. B* **469**, 419 (1996).
22. A. AliKhan et al. (CP-PACS), *Phys. Rev. D* **64**, 074510 (2001).
23. Z. Fodor and S. Katz, *JHEP* **0404**, 050 (2004).
24. S. Ejiri, *Lee-Yang zero analysis for the study of QCD phase structure*, hep-lat/0506023.
25. S. Gottlieb, W. Liu, D. Toussaint, R. L. Renken and R. L. Sugar, *Phys. Rev. Lett.* **59**, 2247 (1987).
26. R.V. Gavai and S. Gupta, *Phys. Rev. D* **71**, 114014 (2005).