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1 Introducion

In a previous paper, Ref.[1], results were derived for the growth rates due to intrabeam scattering for a beam with a bi-gaussian particle distrbution. The results found were complicated and required the evaluation of 3 dimensional integrals over the relative momemtum of the particles. To make the results useful, one needs a program that evaluates numerically the 3 dimensional integrals and other related quantities. With this goal in mind, a program was written, called BIG (for BI-Gaussian). This program can be used in the following two ways

1. As a stand alone program to find growth rates due to intrabeam scattering for a beam with a bi-gaussian distribution. The BIG program gives growth rates for the average emittances, $< \epsilon_x >, < \epsilon_y >, < \epsilon_s >$ and also of $< p_i p_j >$, $i = 1, 3, j = 1, 3$ where $p_i \quad i = 1, 3$ are the components of the particle momentum. This note defines what is meant by a bi-gaussian distribution and what information is required by BIG to compute the growth rates. The BIG program also computes growh rates for a simple gaussian distribution.

2. As a subroutine called by a tracking program like SIMCOOL which tracks a group of particles starting with a gaussian distributiion. In tracking the particles through the elements of the lattice, the particles receive kicks which are determined by the growth rates computed by BIG to represent the effect of IBS. In order to use the BIG program, one has to fit the current particle distribution in the tracking program with a bi- gaussian distribution.

2 The Gaussian Distribution

We will first consider the case of a gaussian particle distribution. Let $Nf(x,p)$ give the number of particles in d^3xd^3p, where N is the number of particles in a bunch.x, p represent the coordinates x, y, s and $p_x, p_y.p_s$

For a gaussian distribution, $f(x,p)$ ls given by

$$\begin{aligned} f(x,p) &= \frac{1}{\Gamma} exp[-S(x,p)] \\ \Gamma &= \int d^3xd^3p \; exp[-S(x,p)] \end{aligned} \tag{1}$$

One would like that $f(x,p)$ should show no beam growth if the coulomb interaction between the particles is switched off. This is achieved when $S(x,p)$ is a function of the 3 emittances $\epsilon_x, \epsilon_y, \epsilon_s$ as the emittances are constants of the motion. For an element of the lattice which has the orbit parameters $\beta_x, \alpha_x, \beta_y, \alpha_y, D, D'$, then $S(x,p)$ is given by

$$\begin{aligned} S &= S_x + S_y + S_s \\ \\ S_x &= \frac{1}{\bar{\epsilon_x}} \epsilon_x(x_\beta, x'_\beta) \\ x_\beta &= x - D(p-p_0)/p_0 \\ x'_\beta &= x' - D'(p-p_0)/p_0 \quad x' = p_x/p_0 \\ \epsilon_x(x_\beta, x'_\beta) &= [x_\beta^2 + (\beta_x x'_\beta + \alpha_x x_\beta)^2]/\beta_x \\ \\ S_y &= \frac{1}{\bar{\epsilon_y}} \epsilon_y(y, y') \quad y' = p_y/p_0 \\ \\ S_s &= \frac{1}{\bar{\epsilon_s}} \epsilon_s(s - s_c, (p-p_0)/p_0) \\ \epsilon_s(s - s_c, (p-p_0)/p_0)/\bar{\epsilon_s} &= \frac{(s-s_c)^2}{2\sigma_s^2} + \frac{((p-p_0)/p_0)^2}{2\sigma_p^2} \\ \epsilon_s(s - s_c, (p-p_0)/p_0) &= [(s-s_c)^2 + (\beta_s((p-p_0)/p_0))^2]/\beta_s \\ \beta_s &= \sigma_s/\sigma_p \\ \bar{\epsilon}_s &= 2\sigma_s\sigma_p \end{aligned} \tag{2}$$

Note that β_s is determined by the RF system used.

The above gaussian distribution contains 3 parameters, $\bar{\epsilon_x}, \bar{\epsilon_y}, \bar{\epsilon_s}$. One can show that for this distrbution, the 3 parameters $\bar{\epsilon_x}, \bar{\epsilon_y}, \bar{\epsilon_s}$ are related to the average values of $\epsilon_x, \epsilon_y, \epsilon_s$ by

$$\bar{\epsilon_i} \;=\; <\epsilon_i> \quad i = x, y, s \tag{3}$$

In order to use the BIG program to find the growth rates of the average emittances, $<\epsilon_i>$, $i = x, y, s$, one has to communicate to the BIG program the 3 parameters that specify the gaussian distribution, $\bar{\epsilon_x}, \bar{\epsilon_y}, \bar{\epsilon_s}$, and N where N is the number of particles in a bunch, and the orbit parameters of each element, $\beta_x, \alpha_x, \beta_y, \alpha_y, D, D'$.

Another interesting result for the gaussian distribution is

$$Nf(0,0) \;=\; \frac{N}{\pi^3 \bar{\epsilon_s} \bar{\epsilon_x} \bar{\epsilon_y} p_0^3} \tag{4}$$

which shows that $f(x,p)$ at the center of the distribution gives the particle density in 6 dimensional phase space.

3 Fitting the tracking distribution with a gaussiann distribution

A tracking program like SIMCOOL starts with a group of particles having a gaussian distributiion. The initial $\bar{\epsilon_x}, \bar{\epsilon_y}, \bar{\epsilon_s}$ are known and one can use an IBS program like BIG to compute the growth rates for the average values of $\epsilon_x, \epsilon_y, \epsilon_s$ for a particular element of the lattice. In tracking the particles through this element, the particles receive kicks which are determined by the growth rates computed by BIG to represent the effect of IBS. After passing through one element, the particle tracking distribution has changed because of the kicks. In order to track through the next element of the lattice, one might want to compute a new set of growth rates for the new tracking distribution. To do this , one has to fit the new tracking distribution with a gaussian. A simple way to do this to compute the average values of $\epsilon_x, \epsilon_y, \epsilon_s$ using the tracking distribution, and find new parameters of the gaussian using

$$\bar{\epsilon_i} \;=\; <\epsilon_i> \quad i = x, y, s \tag{5}$$

For an actual tracking program, one would not compute growth rates and kicks after each element.One would wait until the particle distribution has changed appreciably.

4 The Bi-Gauassian Distribution

The bi-gaussian distribution will be assumed to have the form given by the following.

$Nf(x,p)$ gives the number of particles in d^3xd^3p, where N is the number of particles in a bunch. For a bi-gaussian distribution, $f(x,p)$ ls given by

$$\begin{aligned} f(x,p) &= \frac{N_a}{N}\frac{1}{\Gamma_a}exp[-S_a(x,p)] + \frac{N_b}{N}\frac{1}{\Gamma_b}exp[-S_b(x,p)] \\ N_a + N_b &= N \\ \Gamma_a &= \pi^3\bar{\epsilon_{sa}}\bar{\epsilon_{xa}}\bar{\epsilon_{ya}}p_0^3 \\ \Gamma_b &= \pi^3\bar{\epsilon_{sb}}\bar{\epsilon_{xb}}\bar{\epsilon_{yb}}p_0^3 \end{aligned} \tag{6}$$

In the first gaussian,to find S_a in the expressions for S, given above for the gaussian distribution, replace $\bar{\epsilon_x}, \bar{\epsilon_y}, \bar{\epsilon_s}$ by $\bar{\epsilon_{xa}}, \bar{\epsilon_{ya}}, \bar{\epsilon_{sa}}$. In the second gaussian, in the expressions for S, replace $\bar{\epsilon_x}, \bar{\epsilon_y}, \bar{\epsilon_s}$ by $\bar{\epsilon_{xb}}, \bar{\epsilon_{yb}}, \bar{\epsilon_{sb}}$. This bi-gaussian has 8 parameters instead of the three parameters of a gaussian. One could equally well say the bi-gaussian has 7 parameters by using the simple relationship,$N_a + N_b = N$.

In order to use the BIG program to find the growth rates of the average emittances, $<\epsilon_i>$, $i = x, y, s$, one has to communicate to the BIG program the 8 parameters that specify the gaussian distribution, ϵ_{ia}, ϵ_{ib}, $i = x, y, s$, and N_a,N_b and the orbit parameters of each element, $\beta_x, \alpha_x, \beta_y, \alpha_y, D, D'$.

The following results can be proven for the bi-gaussian distribution

$$\begin{aligned} <\epsilon_i> &= \frac{N_a}{N}\bar{\epsilon_{ia}} + \frac{N_b}{N}\bar{\epsilon_{ib}} \quad i = x, y, s \\ <\epsilon_i^2> &= 2\frac{N_a}{N}\bar{\epsilon_{ia}}^2 + 2\frac{N_b}{N}\bar{\epsilon_{ib}}^2 \quad i = x, y, s \\ Nf(0,0) &= \frac{N_a}{\pi^3\bar{\epsilon_{sa}}\bar{\epsilon_{xa}}\bar{\epsilon_{ya}}p_0^3} + \frac{N_b}{\pi^3\bar{\epsilon_{sb}}\bar{\epsilon_{xb}}\bar{\epsilon_{yb}}p_0^3} \end{aligned} \tag{7}$$

These results are usefull for fitting the tracking distribution with a bi-gaussian.

5 Fitting the tracking distribution with a bi-gaussiann distribution

A tracking program like SIMCOOL starts with a group of particles having a gaussian distributiion. The initial ϵ_{ia}, ϵ_{ib}, $i = x, y, s$, and N_a,N_b are known and one can use BIG to compute the growth rates for the average values of $\epsilon_x, \epsilon_y, \epsilon_s$ for a particular element of the lattice. In tracking the particles through this element, the particles receive kicks which are determined by the growth rates computed by BIG to represent the effect of IBS. After passing through one element, the particle tracking distribution has changed because of the kicks. In order to track through the next element of the lattice, one might want to compute a new set of growth rates for the new tracking distribution. To do this , one has to fit the new tracking distribution with a bi-gaussian. A simple way to do this is to compute the average values of $\epsilon_x, \epsilon_y, \epsilon_s$ and the average values of $\epsilon_x^2, \epsilon_y^2, \epsilon_s^2$ using the tracking distribution, and find new parameters for the bi-gaussian using the 8 equations

$$\begin{aligned} < \epsilon_i > &= \frac{N_a}{N}\bar{\epsilon_{ia}} + \frac{N_b}{N}\bar{\epsilon_{ib}} \quad i = x, y, s \\ < \epsilon_i^2 > &= 2\frac{N_a}{N}\bar{\epsilon_{ia}}^2 + 2\frac{N_b}{N}\bar{\epsilon_{ib}}^2 \quad i = x, y, s \\ N f(0,0) &= \frac{N_a}{\pi^3 \bar{\epsilon_{sa}}\bar{\epsilon_{xa}}\bar{\epsilon_{ya}} p_0^3} + \frac{N_b}{\pi^3 \bar{\epsilon_{sb}}\bar{\epsilon_{xb}}\bar{\epsilon_{yb}} p_0^3} \\ N_a + N_b &= N \end{aligned} \tag{8}$$

6 Integration interval

One other parameter that one has to communicate to BIG is the integration interval. The IBS growth rates are given by the theory [1] as a 3 dimensional integral over Δ, which is the relative momentum of any two particles in a bunch. To numerically evaluate this 3 dimensional integral, one first does a rotation in $\Delta_x, \Delta_y, \Delta_s$ space to eliminate the $\Delta_x \Delta_s$ term in the exponent. One now makes a transformation to a new set of variables u_x, u_y, u_s to make the integration limits finite. By properly scaling the transformations one can make the integration limits equal for all 3 variables, u_x, u_y, u_s, and the integration interval can be chosen as equal for all 3 variables. I believe the results are relatively insensitive to the size of the interval , and the interval size can probably be left unchanged at the value set internally in the program.

References

1. G. Parzen BNL report C-A/AP No.169 (2004)