Chapter 4, the first on cooling, of lectures to be given at Acc School in Vanderbilt in Jan 99, by R.B.Palmer.

4 Capture & Phase Rotation

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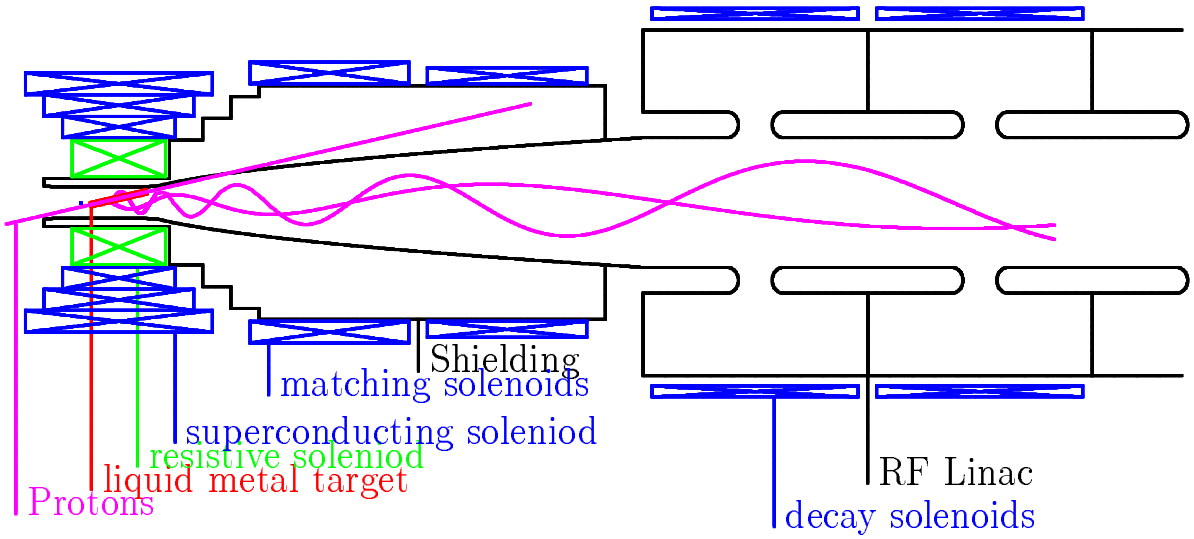
4 Capture & Phase Rotation



4.1 Li Lens Capture

Ref

Palmer, Gallardo, Fernow, Torun, Neuffer; BNL 61581; CAP 121-Muon-95c; Proc Workshop at Sausalido, CA Nov 17-19, 1994

$$\frac{B_{surface} = \mu_o I}{2\pi r}$$

$$B = B_{surface} \frac{r}{R}$$

$$\frac{1}{\rho} = \frac{d^2 r}{dz^2} = \frac{e B}{p_z} = \frac{B c}{(pc/e)}$$

note that (pc/e) is the momentum in eV's.

So the focusing strength per unit length $k' = 1/(\rho r)$ is

$$k' = \frac{B_{surface}}{R} \frac{c}{(pc/e)} \tag{1}$$

Motion, starting on the axis, is given by:

$$r = a \sin(\sqrt{k'}\, z) \tag{2}$$

$$\theta_{max} = R\sqrt{k'} = \sqrt{B_{surface}\, R} \frac{c}{(pc/e)} \tag{3}$$

e.g.

$B_{surface} = 10$ T

$p = 200$ MeV/c

$R = 10$ cm

$$I = 5\ 10^6 \quad (Amps)$$

$$\theta_{max} \; = \; 1.5$$

which is good, but such current and size is far larger than anything ever built. The focusing is only good at one momentum and from one point on the finite length of target. The use of multiple such lenses has been studies (cite BNL 61581). Interactions in the material also have to be considered. It was found that such a system was only competitive with a solenoid at high pion momenta ($p_\pi \; \geq \; 3$ GeV). Since the peak of pion production is much lower, ≈ 100 MeV/c, solenoids are much better.

problems

For a lithium rod with uniform current density, radius R, and length L, with a point source on axis at the front end:

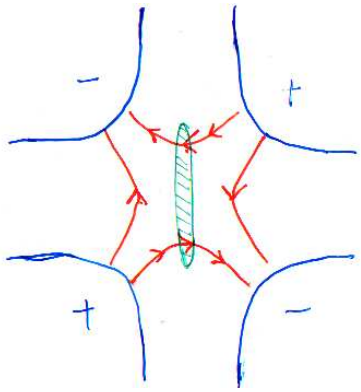
1. Calculate the radial surface pressure as function of $(R,\ B_{surface})$
2. If $p = 300$ MeV/c, $\theta = 1$, $B_{surface} = 10T$ then calculate the minimum R and L to focus to a parallel beam.
3. If, further, $dn/dz = 1/(39.7cm)$, what is the fractional pion loss?

4.2 Quadrupole Capture

Ref

Palmer, Gallardo, Fernow, Torun, Neuffer; BNL 61581; CAP 121-Muon-95c; Proc Workshop at Sausalido, CA Nov 17-19, 1994

Quadrupoles do not have material to absorb the pions, and pole tip fields of 10 T (as assumed above for the Li lenses, are probably just possible, but the focusing is in one plane and there is defocusing in the other.



$$B_y \propto x \quad \Delta x' = -k\,x \qquad \text{focusing}$$
$$B_x \propto -y \quad \Delta y' = +k\,y \qquad \text{defocusing}$$

***

Alternating quads must be used, and the required maximum aperture, for a given acceptance and peak field, is greater than for a Li lens by a factor of approximately 8 times (see Section 11). No matter what the maximum aperture, the maximum possible acceptance, in both directions, is only about 30^o. There is no such limit for lithium lenses, or solenoids.

4.3 Solenoid Capture

In the transverse plane:

$$r \;=\; \frac{p_\perp}{e\ B_z} \;=\; \frac{(p_\perp\ c/e)}{c\ B}$$

Noting that $(p_\perp\ c/e)$ is the momentum in units of electron Volts.

For particles generated in a thin target on the axis, inside a solenoid of inside radius R, the maximum transverse momenta captured will be:

$$(p_\perp(max)\ c/e) \;=\; \frac{c\ B_z\ R}{2} \qquad (4)$$

e.g. For a 20 T solenoid of 8 cm radius, $p_\perp(max) = 240 MeV/c$. These are the dimensions of an existing resistive solenoid at FSU.

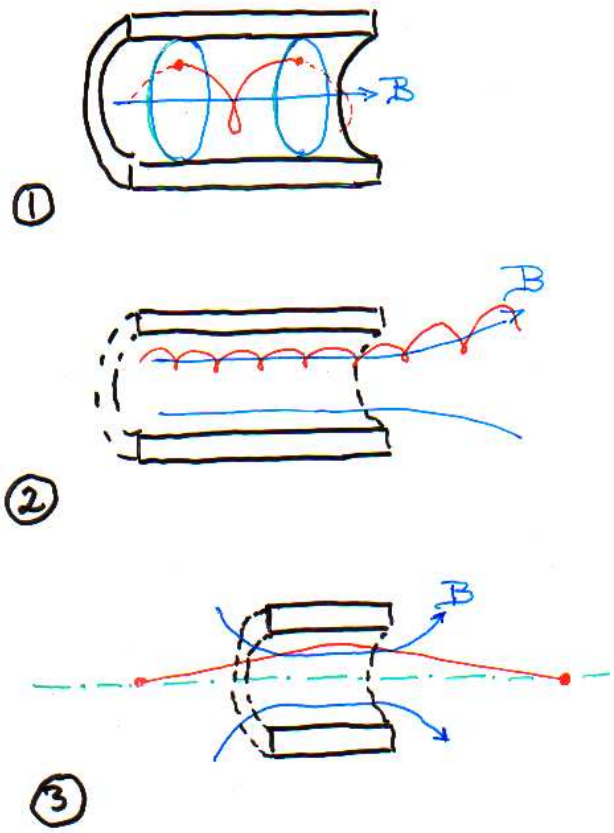
We must now match the captured beam into a lower field decay channel. To understand how to do this, and other uses of solenoids we digress on a tutorial on Solenoid focusing Theory.

4.4 Solenoid Focus Theory

4.4.1 Introduction

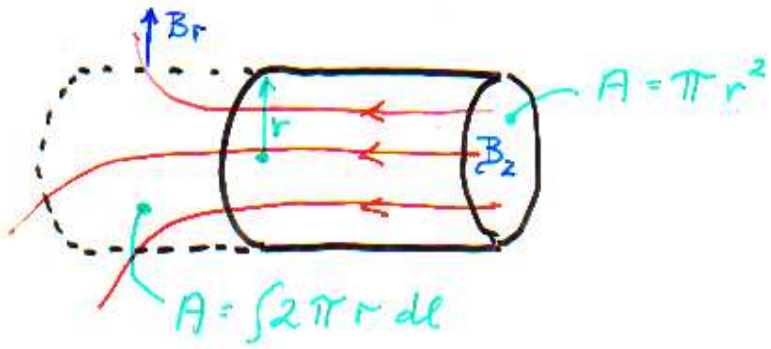
We are familiar with 3 kinds of solenoid focussing:

1. focusing in one or more lamor rotation in a uniform long solenoid (as in an image intensifier).
2. The trapping of articles along field lines.
3. focusing from poin to point by a thin solenoid (as in a TV tube).



We mostly understand # 1, but not # 2 and # 3.

4.4.2 Radial Fields

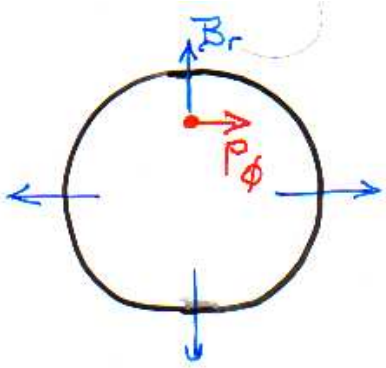


assuming $B_z \approx$ independent of r; i.e $dB/dz \ll B/r$:

$$2\pi\ r\ B_r\ \approx\ \frac{dB_z}{dz}\ \pi\ r^2$$

$$B_r\ =\ \frac{r}{2}\ \frac{dB_z}{dz} \tag{5}$$

4.4.3 Induced Angular Momentum



$$dp_\phi\ =\ B_r\ e\ v_z\ dt\ =\ B_r\ e\ dz\ \approx\ \frac{1}{2}\ r\ \frac{dB_z}{dz}\ e\ dz$$

$$dp_\phi\ \approx\ \frac{r\ e}{2}\ dB_z$$

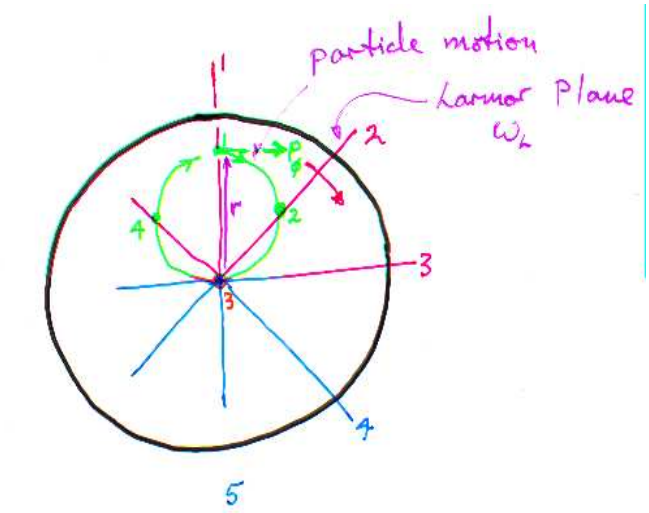
If B_z initial $=$ 0 then:

$$\Delta p_\phi \;=\; \frac{e}{2}\, r\; B_z \tag{6}$$

4.4.4 Canonical Angular Momentum

$$P_{canonical} \;=\; p_\phi \; c\; r \;=\; p_\phi \;-\; \frac{e}{2}\; B_z\; r^2 \tag{7}$$

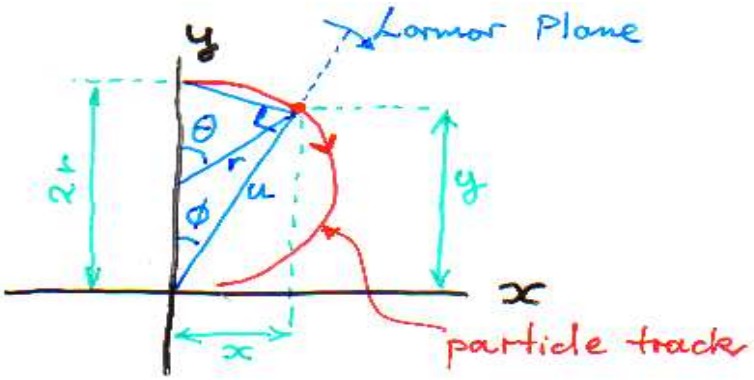
4.4.5 Larmor Rotation



$$\frac{p_\phi}{r} = \frac{p_s}{r}\frac{v_\phi}{v_s}\,\omega_L = \frac{p_s}{v_s}\,\omega_L$$
$$= \gamma\, m\, \omega_L = \frac{e}{2}\,\Delta B_z$$

$$\omega_L = \frac{e}{2m\gamma}\,B_z \tag{8}$$

4.4.6 Motion in Lamor Plane



$$x = r \sin(\theta)$$

$$y = r + r \cos(\theta)$$

$$u = 2r \cos(\phi)$$

$$\phi = \frac{\theta}{2}$$

$$\ddot{u} = \frac{d^2u}{dt^2} = -2r \cos(\phi) \left(\frac{d\phi}{dt}\right)^2$$

$$\frac{d^2u}{dt^2} = -u\,\omega_L^2 \tag{9}$$

$$= -u \left(\frac{e^2\,B_z^2}{4\,p_z^2}\right)$$

$$\frac{d^2u}{dz^2} = -u \left(\frac{e^2\,B_z^2}{4\,p_z^2}\right) \tag{10}$$

i.e. Motion in the rotating Larmor plane, and thus the radial motion, is that of a particle focused towards the

axis with a force proportional to the coordinate u , i.e. the radius r. This is a linear focusing force.

Note

- Focus $\propto$ B_z^2. Either direction of B_z focuses.
- Focus $\propto$ e^2. Either sign is focused.
- Focus $\propto$ $1/p_z^2$. c.f. axial lenses: Focus $\propto$ $1/p_z$

4.4.7 Matched β_o

If B_z is a constant:

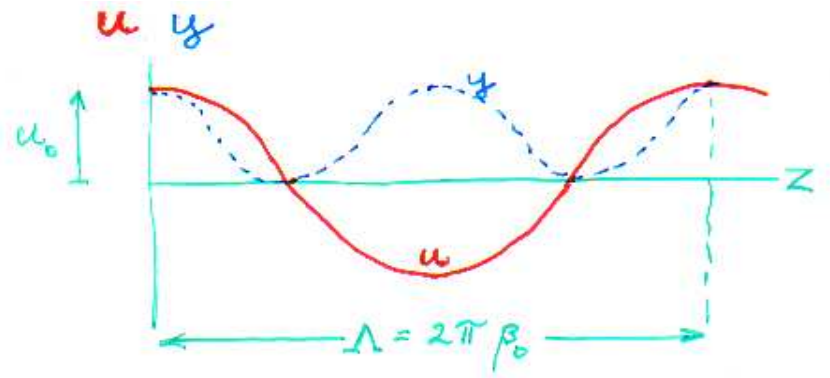
$$u = u_o \cos\left(\frac{z}{\beta_o}\right)$$

$$\frac{du}{dz} = -\frac{u_o}{\beta_o} \sin\left(\frac{z}{\beta_o}\right)$$

$$\beta_o = \frac{2\ p_z}{e\ B_z} \tag{11}$$

$$\Lambda = 2\pi\ \beta_o$$

i.e. motion is sinusoidal in the planewith a wavelength Λ. Note that this wavelength is DOUBLE that of the Larmor helix itself.



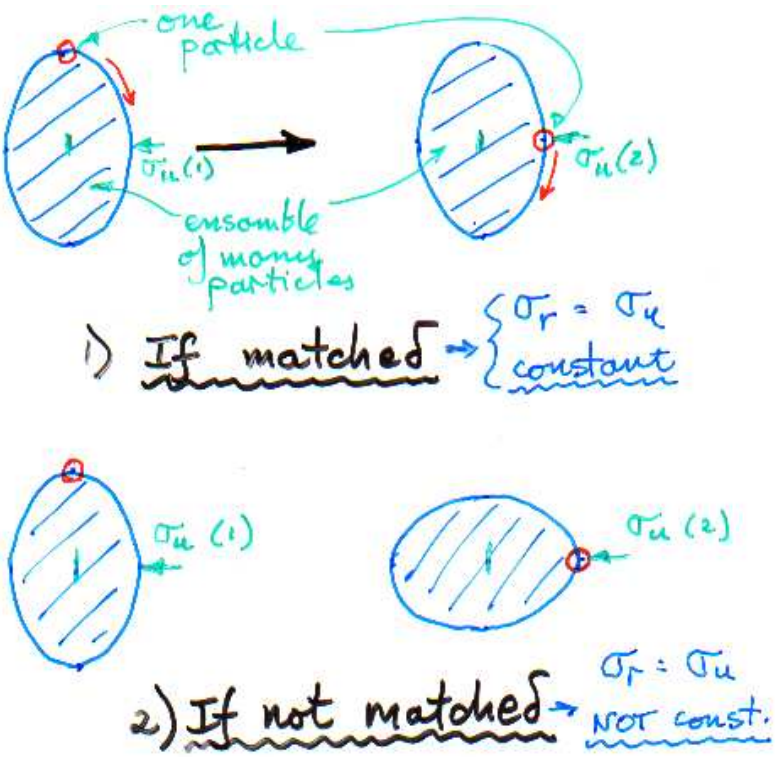
Remembering the definition of the Courant Snyder β

$$\beta = \frac{u}{u'}$$

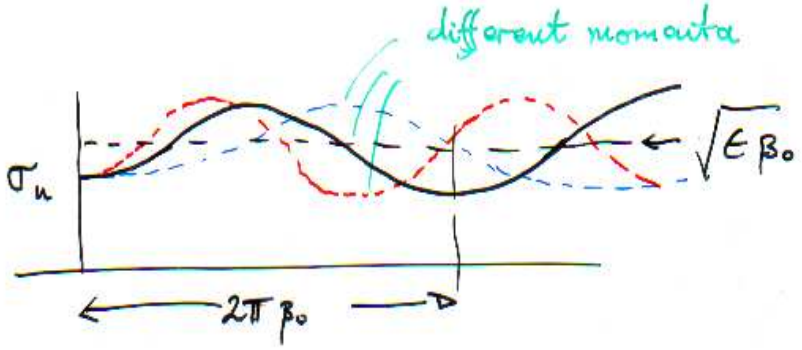
If the initial $\beta = \beta_o$ then the particles move around the ellipse without it changing shape. i.e. σ_r and β remains constant.

But if the initial $\beta \neq \beta_o$ then the ellipse itself rotates, and the σ_r and β wobbles about its equilibrium value of β_o.

$$\sigma_u = \sqrt{\epsilon\,\beta_o} + (\sigma_{u\,o} - \sqrt{\epsilon\,\beta_o})\,\cos\left(\frac{z}{\beta_o}\right)$$



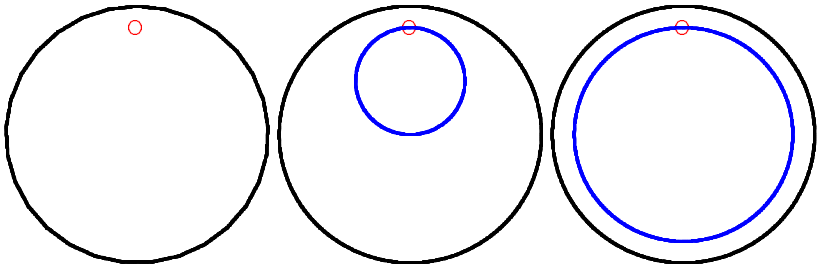
Different momenta have different β_o's so after a distance the wobbles get mixed which gives rise to an effectove increase in emittance.



4.4.8 Examples with $p_{\phi\ c} \neq 0$

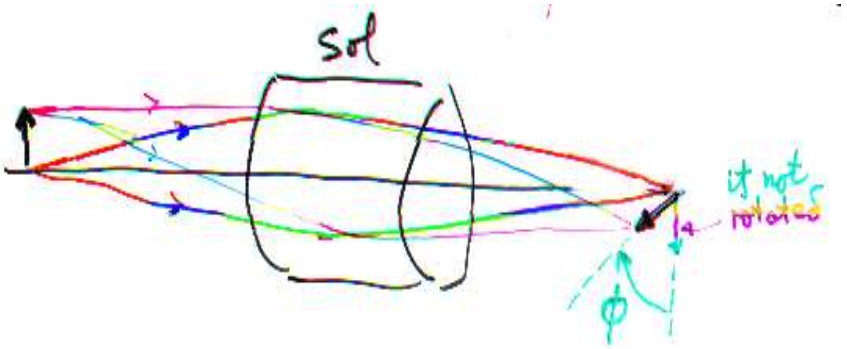
Motion in a rotating u and a perpendicular v planes each follow the trajectories of particles under a simple focus force towards the axis. If the motions in u and v are in phase then this is motion in an intermediate plane, but if they are out of phase, the motion is in an ellipse in the $u - v$ plane.

Examples shown in the $x - y$ plane:



$p_\phi = 0$	$p_\phi = \frac{eBr}{2}$	$p_\phi = 2\ \frac{eBr}{2}$
$p_{\phi\ c} = -\frac{eBr}{2}$	$p_{\phi\ c} = 0$	$p_{\phi\ c} = \frac{eBr}{2}$
a)	**b)**	**c)**

4.4.9 Image Rotation



$$\Phi = \frac{e}{2\ p_z} \int B_z\ dz \tag{12}$$

4.4.10 Aberrations

- B_z is not independent of r

  Stronger focus at large r : spherical aberation

- $\dot{u} = \frac{du}{dz} \neq \ll 1$

  Stronger focus at large θ : spherical aberation

- $p_z \neq$ constant

$$p_z^2 = p^2 - p_\perp^2$$

$p_\perp$ increases with B_z; p_z falls and can fall to zero; the particle is then reflected.

4.5 Adiabatic Matching

Refs

R. Chehab, J. Math. Phys. 5 (1978) 9.

The match between a target capture Solenoid and a decay channel solenoid can be made, with negligible loss, by gently tapering the magnetic field. The condition of suffciently "gentle" is that the relative change in the focusing β, i.e. the change in $d\beta/\beta$, is small in a distance equal to the current β:

$$\frac{d\beta}{\beta} \ll \frac{dz}{\beta}$$

or

$$\frac{d\beta}{dz} = \epsilon \ll 1$$

Since $\beta \propto 1/B_{solenoid}$:

$$\frac{d(1/B)}{dz} = \epsilon \ll 1$$

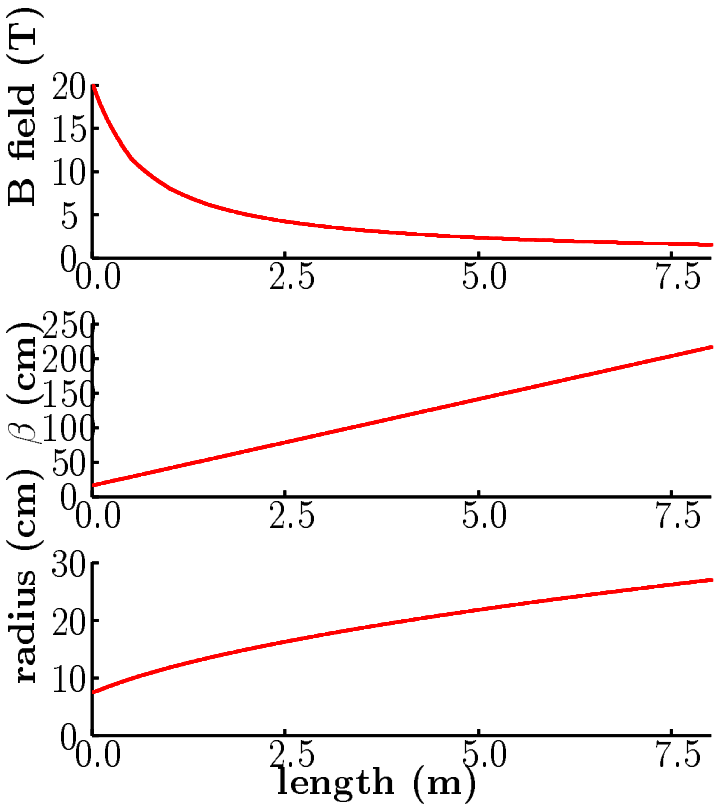
which gives:

$$B(z) = \frac{B_o}{1 + k\ z} \tag{13}$$

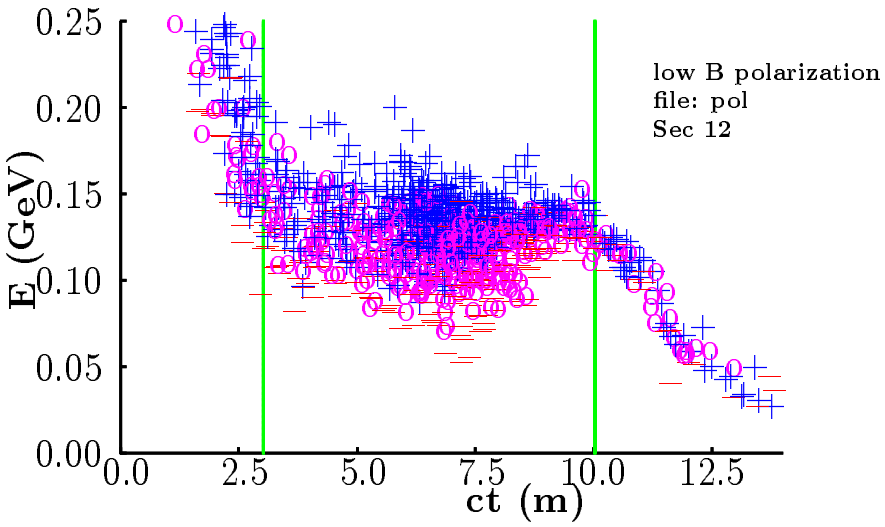
where

$$k = \epsilon \frac{B_o\ c}{2\ (pc/e)} \tag{14}$$

Note that the B drops initially very fast, corresponding to the short β's at the high initial filed, but falls much slower at the lower later fields where the β's are long. For a taper from 20T to 1.25T at momenta less than 1 GeV and $\epsilon = .5$, the taper length should be approximately 6 m.

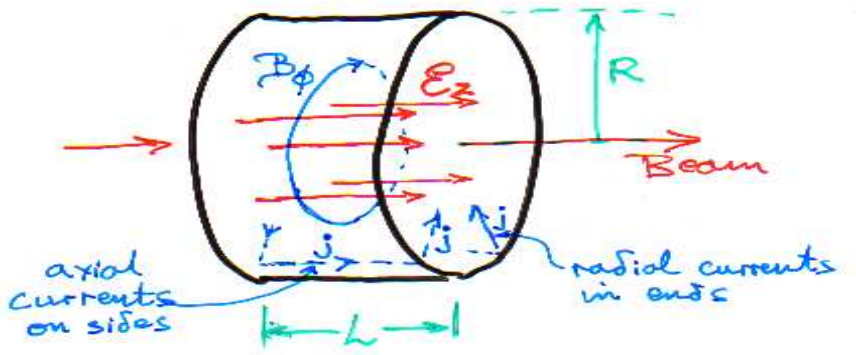

B field (T)
20
15
10
5
0
0.0
2.5
5.0
7.5
β (cm)
250
200
150
100
50
0
0.0
2.5
5.0
7.5
radius (cm)
30
20
10
0
0.0
2.5
5.0
7.5
length (m)

4.6 Phase Rotation



4.7 Introduction to RF

4.7.1 Fields in Pill Box Cavity



For a pillbox shaped cavity with an accelerating field along the axis:

$$\mathcal{E}_z \;=\; \mathcal{E}_o\; J_o(kr)\; e^{i\omega t} \tag{15}$$

$$B_\phi \;=\; -\frac{i\;\mathcal{E}_o}{c}\; J_1(kr)\; e^{i\omega t} \tag{16}$$

Where $k = 2\pi/\lambda$. Or, since $c^2 = 1/(\epsilon_o \mu_o)$:

$$B_\phi \;=\; -i\;\sqrt{\epsilon_o \mu_o}\;\mathcal{E}_o\; J_1(kr)\; e^{i\omega t}$$

At the outer wall we require $\mathcal{E}_z = 0$, and $J_o(2.4048) \approx 0$, so

$$R \;\approx\; 2.4048\;\frac{\lambda}{2\;\pi}$$

$$R \;\approx\; .3827\;\lambda \tag{17}$$

4.7.2 Stored Energy

The stored energy is

$$U = \frac{1}{\mu_o} \int_o^R \frac{B_\phi^2}{2}\, dV$$

$$U = L\, \frac{\epsilon_o\, \mathcal{E}_o^2}{2} \int_o^R J_1(kr)\; 2\pi\; r\; dr$$

and since

$$\int_o^R J_1^2(kr) r\; dr = \frac{R^2}{2}\; J_1^2(kR)$$

$$U = L\, \frac{\epsilon_o\, \mathcal{E}_o^2}{2}\, \pi\; R^2\; J_1^2(kR) \tag{18}$$

$$U \approx 0.55\, 10^{-12}\, \mathcal{E}_o^2\, \mathrm{L}\, \lambda^2$$

Defining Φ as the rf phase advance as a particle of velocity βc passes across a cavity of length L.

$$L = \beta_v\, \lambda\, \frac{\Phi}{2\pi} \tag{19}$$

For example:
$\lambda = .373$ m (805 MHz)
$\beta_{vel} = 0.87$
$\Phi = \pi/2$
$\mathcal{E} = 36$ MeV/m
Then

$$L = .107 \quad \mathrm{m}$$
$$U = 10.4 \quad \mathrm{J}$$

4.7.3 Transit Time Factor

The axial electric filed in the cavity is independent of z, but the field seen by a particle moving at a finite velocity $\beta_v\ c$ will vary:

$$\mathcal{E} \ = \ \mathcal{E}_{\rm o}\ \cos(\omega\ {\rm t}) \ = \ \mathcal{E}_{\rm o}\ \cos(\phi)$$

where

$$\phi \ = \ \omega\ t \ = \ \frac{\omega\ (z \ - \ z_{center})}{\beta_v\ c}$$

The energy gain is

$$\Delta E \ = \ \int_{-L/2}^{+L/2} \mathcal{E}_{\rm z}\ {\rm dz}$$

$$= \ \frac{\lambda \beta_v}{2\pi} \int_{-\Phi/2}^{+\Phi/2} \mathcal{E}_{\rm z}\ {\rm d}\phi$$

$$= \ \frac{\mathcal{E}_{\rm o}\ {\rm L}}{\Phi} \int_{-\Phi/2}^{+\Phi/2} \cos(\phi)\ d\phi$$

$$\Delta E \ = \ \mathcal{E}_{\rm o}\ {\rm L}\ \frac{\sin(\Phi/2)}{(\Phi/2)}$$

Defining a Transit Time Factor T_r

$$T_r \ = \ \frac{\Delta E}{\Delta E_{ideal}} \ = \ \frac{\Delta E}{\mathcal{E}\ {\rm L}} \tag{20}$$

$$T_r \ = \ \frac{\sin(\Phi/2)}{(\Phi/2)} \tag{21}$$

e.g. for

$\Phi = \pi$ as in most standing wave systems $T_r \approx 0.637$

$\Phi = 2\pi/3$ as in most traveling wave systems $T_r \approx 0.827$

$\Phi = \pi/2$ as in an interleived s.w. foil system $T_r \approx 0.90$

4.7.4 Energy Loss

The currents flowing in the cavity walls are:

$$j \;=\; H_\phi \;=\; \frac{B_\phi}{\mu_o} \;=\; -i\,\frac{\mathcal{E}_{\rm o}}{{\rm c}\;\mu_{\rm o}}\;J_1(kr)\;e^{i\omega t}$$

$$j \;=\; -i\,\sqrt{\frac{\epsilon_o}{\mu_o}}\;\mathcal{E}_{\rm o}\;J_1(kr)\;e^{i\omega t}$$

And the average heating from these currents:

$$\frac{dP{\rm ohmic}}{dA} \;=\; \frac{j^2}{2}\;\frac{\rho}{\delta}$$

$$\frac{dP{\rm ohmic}}{dA} \;=\; \frac{\epsilon_o}{\mu_o}\;\mathcal{E}_{\rm o}^2\;J_1^2(kr)\;\frac{\rho}{2\delta} \qquad (22)$$

where the $1/2$ comes from the time average, ρ is the resistivity, and the skin depth δ is given by:

$$\delta \;=\; \sqrt{\frac{2\;\rho}{\omega\;\mu_o}} \qquad (23)$$

For a cavity of length L and radius R, the total heating Pohmic will have two terms: from the cylindrical outer wall, and from the two ends.

$$P{\rm ohmic} \;=\; \frac{\rho}{2\delta}\,\frac{\epsilon_o}{\mu_o}\,\mathcal{E}_{\rm o}^2\,\left(2\pi{\rm R}\;{\rm L}\;{\rm J}_1^2({\rm kR}) \;+\; 2\;\int_0^{\rm R}{\rm J}_1^2({\rm kr})\;2\pi{\rm r}\;{\rm dr}\right)$$

and since

$$\int_o^R J_1^2(kr)r\;dr = \frac{R^2}{2}\;J_1^2(kR)$$

$$P{\rm ohmic} \;=\; \frac{\rho}{\delta}\,\frac{\epsilon_o}{\mu_o}\,\mathcal{E}_{\rm o}^2\,\pi{\rm R}^2{\rm J}_1^2({\rm kR})\;\left(1\;+\;\frac{\rm L}{\rm R}\right) \qquad (24)$$

4.7.5 Quality Factor Q

The Q of the cavity is defined a

$$Q = \frac{\omega U}{P\text{ohmic}}$$

$$Q = \omega \frac{\frac{1}{2} L \epsilon_o \mathcal{E}_o^2 \pi \mathrm{R}^2 \mathrm{J}_1^2(\mathrm{kR})}{\frac{\rho}{\delta \mu_o} \epsilon_o \mathcal{E}_o^2 \pi \mathrm{R}^2 \mathrm{J}_1^2(\mathrm{kR}) \left(1 + \frac{\mathrm{L}}{\mathrm{R}}\right)}$$

$$Q = \frac{\omega \mu_o}{2\rho} \delta L \frac{1}{\left(1 + \frac{L}{R}\right)}$$

$$Q = \frac{L}{\delta} \frac{1}{\left(1 + \frac{L}{R}\right)} \propto \sqrt{\lambda} \qquad (25)$$

For example:
$\lambda = .373$ m (805 MHz)
$\beta_{vel} = 0.87$
$\delta = 2.33\ \mu$m (Cu at oom temp)
$\Phi = \pi/2$
Then

$$L = .107 \quad \mathrm{m}$$

$$R = .143 \quad \mathrm{m}$$

$$Q = 27,000$$

4.7.6 Shunt Impedance R_s

The Shunt Impedance R_s can be defined as

$$R_s \;=\; \frac{V^2}{2\ P_{\rm ohmic}} \qquad (\Omega'{\rm s})$$

or more commonly:

$$\frac{R_s}{L} \;=\; \frac{V^2}{2\ L\ P_{\rm ohmic}} \qquad (\Omega'{\rm s}\ /{\rm m}) \tag{26}$$

$$\frac{R_s}{L} \;=\; \frac{\mathcal{E}^2\ {\rm L}}{2\ {\rm P_{ohmic}}} \;=\; \frac{\mathcal{E}^2\ {\rm L\ Q}}{2\ \omega\ {\rm U}}$$

$$\frac{R_s}{L} \;=\; \frac{\delta}{\rho}\,\frac{\mu_o}{\epsilon_o}\,\frac{1}{\pi\ R^2\ J_1(kR)\ (1/L\ +1/R)} \;\propto\; \frac{1}{\sqrt{\lambda}}$$

Or, substituting $Q \;=\; \omega\ U/P$:

$$\frac{R_s}{L} \;=\; \frac{\mathcal{E}^2\ {\rm L\ Q}}{2\ \omega\ {\rm U}} \tag{27}$$

For example: $Q \;=\; 27,000$; $L \;=\; .107$; $U \;=\; 10.4$; $\mathcal{E} \;=\; 36\ {\rm MV/m}; {\rm f} \;=\; 805\ {\rm MHz}$

$$\frac{R_s}{L} \;=\; 35.6 \quad {\rm M}\Omega/m$$

4.7.7 Geometric Factor R/Q

$$\frac{R_s}{Q} = \frac{V^2}{2\ \omega\ U} \tag{28}$$

$$\frac{R_s}{Q} = \frac{\mathcal{E}^2\ \mathrm{L}^2}{2\ \omega\ \mathrm{U}}$$

$$= \left(\frac{Z_o}{2\ \pi^2\ J_1^2(kR)}\ \frac{\lambda}{R}\right)\frac{L}{R} \tag{29}$$

which is independent of wavelength or resistivity. It is a geometric factor, dependent only on the shape of the cavity.

In our example:

$$\frac{R_s}{Q} \approx 138 \quad (\Omega)$$

$$R_s \approx 3.72 \quad M\Omega$$

$$P_{\mathrm{ohmic}} \approx 2 \quad MW$$

A related parameter, used at SLAC, which includes the transit time factor, is the elastance s. It has the dimensions of 1/capacitance.

$$s = \frac{\mathcal{E}_{\mathrm{a}}^2}{\frac{\mathrm{dU}}{\mathrm{dz}}} = \frac{2\ \omega}{L}\left(\frac{R}{Q}\right)\ T_r^2 \tag{30}$$

4.7.8 Irises

For relativistic particles acceleration is independent of r within the iris. So

$$\mathcal{E}_{\mathrm{a}} \approx \mathcal{E}\ \mathrm{J_o(k\ r_{iris})}$$

At lower energies, the loss is greater.

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4.7.9 Noses

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4.7.10 Filling Standing Wave Cavities

The filling of a standing wave cavity balances three terms:

- The rf power from the driver;
- the ohmic losses in the cavity; and
- the rf power reflected when there is a lack of matching between the rf driver and the cavity.

$$P_{\text{ohmic}} \;=\; \left(\frac{\omega}{Q}\right)\; U \;=\; \frac{U}{\tau}$$

where τ the cavity time constant:

$$\tau \;=\; \frac{U}{P_{\text{ohmic}}} \;=\; \frac{Q}{\omega} \tag{31}$$

The equilibrium stored energy, assuming perfect matching of the rf power P_{rf} into the cavity is:

$$U_o \;=\; \tau\; P_{rf}$$

so

$$P_{\text{ohmic}} \;=\; \frac{U}{U_o}\; P_{rf}$$

The reflected power arises from the difference of amplitudes between the reflected input, and the leaking stored fields. The magnitude of the leak is controlled by the coupling, and can be chosen to exactly cancel the reflected fields at any one value of the stored energy, defined here as U_{match}. The reflected power, being the square of the reflected minus leaking fields, is:

$$P_{\text{reflection}} \;=\; P_{rf}\; \left(1 \;-\; \sqrt{\frac{U}{U_{\text{match}}}}\right)^2$$

The rate of change of the stored energy is:

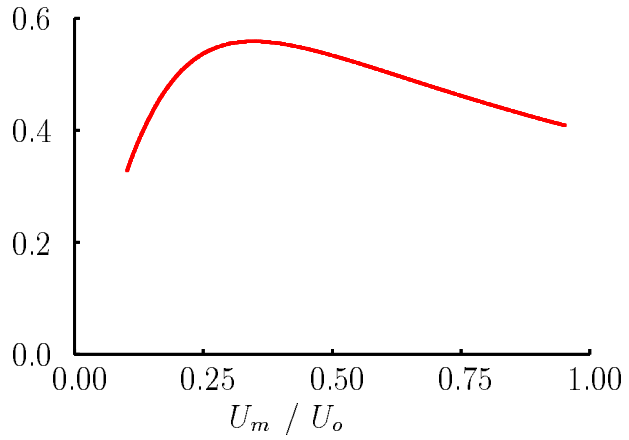
$$\frac{dU}{dt} \;=\; P_{rf} \;-\; P_{\text{reflection}} \;-\; P_{\text{ohmic}}$$

$$\frac{dU}{dt}(\text{rf on}) \;=\; P_{rf}\left(1 \;-\; \left(1 \;-\; \sqrt{\frac{U}{U_{\text{match}}}}\right)^2 \;-\; \frac{U}{U_o}\right) \tag{32}$$

When the rf is off, the field will fall as:

$$\frac{dU}{dt}(\text{rf off}) \;=\; P_{rf}\left(\;-\;\frac{U}{U_{\text{match}}} \;-\; \frac{U}{U_o}\right) \tag{33}$$

The following plot shows the achieved $U(t)$ after a filling time $t = 1\ \tau$, as a function of U_m/U_o, and shows $\approx$ 38% more stored energy with a value far less than that matched at the equilibrium field.



The following table shows the values of U_{match}/U_o that maximize the stored energy after time intervals varying from 0.5 τ to 3 τ.

t/τ	U_m/U_o	P_{rf}/P_{ideal}	U_{rf}/U_{cavity}	$U_{\text{ohmic}}/U_{\text{cavity}}$
0.5	0.19	3.02	1.51	0.39
1	0.35	1.79	1.79	0.75
1.5	0.49	1.41	2.11	1.11
2	0.61	1.24	2.47	1.48
2.5	0.70	1.14	2.85	1.87
3	0.78	1.09	3.26	2.27

For example:

$t = 3\tau$

$U_o = 10.4$ J

$f = 15$ Hz

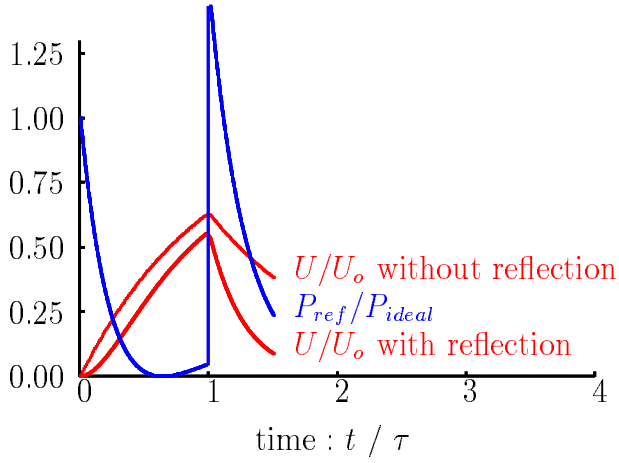
$P_{\text{ave ohmic}} \approx 354$ W (117 W, at $t = 1\ \tau$)

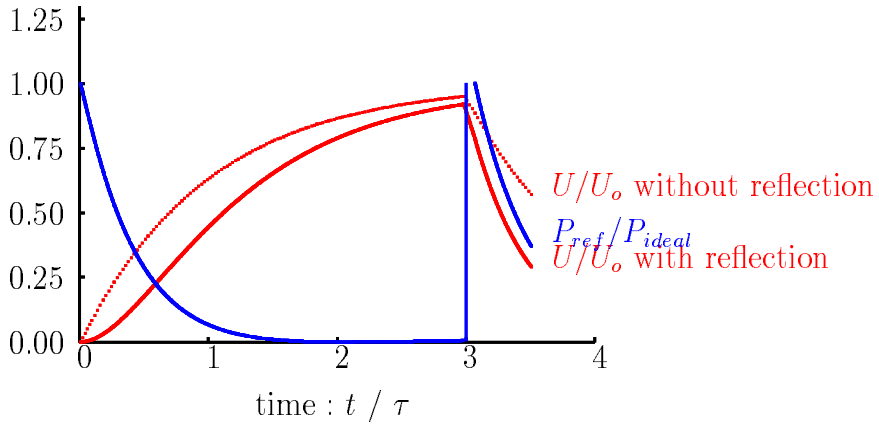
$P_{\text{ave rf}} \approx 508$ W (278 W, at $t = 1\ \tau$)

$P_{\text{peak rf}} \approx 2.2$ MW (3.58 MW, at $t = 1\ \tau$)

Problem: In a standing wave cavity filled with rf power P_{rf}, for a time t/τ, and then let to decay; derive analytic equations for P_{rf}/P_{ideal}, U_{rf}/U_{cavity} , and $U_{\text{ohmic}}/U_{\text{cavity}}$ in the imaginary case in which there is no reflection of the input rf power.

The following two plots show the stored energy as a function of time for two rf fill times (1 τ and 3 τ)). The line gives the values with reflection, the dotted line gives the values that would be obtained if the coupling could be varied so as to eliminate the reflection (although we do not know how to do this).



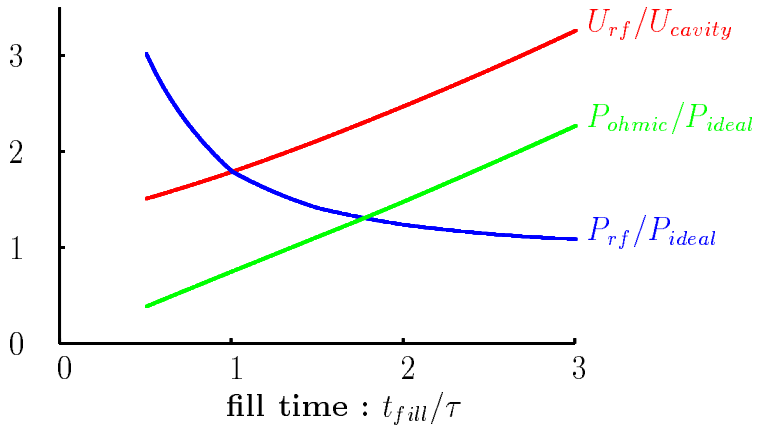


We define $U(t)$ as the stored energy after a fill time t, $P_{\rm ideal}$ as the matched equilibrium rf power for that stored energy,

$$P_{ideal} \;=\; \frac{U(t)}{\tau}$$

and $U_{rf} \;=\; (t\; P_{rf})$ as the total rf energy expended during the fill:

The above table and plot, below, show the peak rf power P_{rf} compared with the ideal equilibrium power P_{ideal} to achieve that field; and the total rf energy U_{rf} compared with cavity energy $U_{\rm cavity} \;=\; U(t)$.



It is seen that peak power requirement approaches the ideal value at large fill times (e.g. 1.09× ideal, at 3τ), but rises if the fill time is less (e.g. 1.79× ideal, at 1 τ: 1.64 ×). On the other hand, the required rf energy, and thus average rf power, go the other way: lower rf energy for short fill times (e.g. 1.79× ideal, at 1 τ), and larger, for long fill times (e.g. 3.26× ideal, at 3 τ: 1.82 ×). The improvement in ohmic loss in the cavity is even more dramatic: being only 0.75× the ideal input rf power, at 1 τ, but rising to 2.27× that ideal, at 3 τ: a difference of a factor of 3.

4.7.11 Be Foil Heating

Refs

Derun, Corlett, Turner ; EPAC 98

Holtkamp ; Mu Collider Note 13

The power dissipated in an end wall, or foil across an iris, up to a radius r

$$P(r) \;=\; \frac{\rho}{2\delta_{foil}}\;\frac{\epsilon_o}{\mu_o}\;\mathcal{E}_{\rm o}^2\;\left(\int_0^{\rm r} {\rm J}_1^2({\rm kr})\;2\pi {\rm r}\;{\rm dr}\right)$$

and, assuming only a small fraction of the power is disipated in the foil, as a fraction of the total power disipated in the cavity:

$$\frac{P(r)}{P} \;\approx\; \sqrt{\frac{\rho_{foil}}{\rho_{cavity}}}\;\frac{\int_0^r J_1^2(kr)\;r\;dr}{R\;J_1^2(kR)\;(L\;+\;R)}$$

At small kr: $J_o(kr) \;\approx\; 1$ and $J_1(kr) \;\approx\; kr/2$, so

$$\mathcal{E}_{\rm z} \;=\; \mathcal{E}_{\rm o}\;{\rm e}^{{\rm i}\omega {\rm t}}$$

$$B_\phi \;=\; -\frac{i\;\mathcal{E}_{\rm o}}{\rm c}\;\frac{kr}{2}\;e^{i\omega t}$$

So for small r/R:

$$\frac{P(r)}{P} \;=\; \sqrt{\frac{\rho_{foil}}{\rho_{cavity}}}\;\frac{\int_0^r \left(\frac{kr}{2}\right)^2\;r\;dr}{R\;J_1^2(kR)\;(L\;+\;R)}$$

$$\frac{P(r)}{P} \;=\; \sqrt{\frac{\rho_{foil}}{\rho_{cavity}}}\;\frac{\frac{k^2\;r^4}{16}}{R\;J_1^2(kR)\;(L\;+\;R)}$$

The temperature rise in a foil, of thermal conductivity κ, of thickness t, heated from both sides, cooled from the edge, in a radial interval dr is then

$$dT \;=\; -2\;P(r)\frac{dr}{2\pi\;r\;\kappa\;t}$$

and the central temperature difference from the outside is

$$\Delta T \;=\; 2\;P\;\sqrt{\frac{\rho_{foil}}{\rho_{cavity}}}\;\frac{k^2}{16\;R\;J_1^2(kR)\;(L\;+\;R)}\;\int_o^r \frac{r^4}{2\pi\;r\;\kappa\;t}dr$$

$$\Delta T \;=\; 2\;P\;\sqrt{\frac{\rho_{foil}}{\rho_{cavity}}}\;\frac{k^2}{64\;R\;J_1^2(kR)\;(L\;+\;R)}\;\;\frac{r^4}{2\pi\;\kappa\;t}$$

$$dT \;=\; P\;\sqrt{\frac{\rho_{foil}}{\rho_{cavity}}}\;\frac{\pi}{16\;\lambda^2\;R\;J_1^2(kR)\;(L\;+\;R)}\;\;\frac{r^4}{\kappa\;t} \qquad (34)$$

For example:
$\lambda \;=\; .373$ m (805 MHz)
$\beta_{vel} \;=\; 0.87$
$\rho_{cavity} \;=\; 1.54\;10^{-8}$ (Ωm) (Cu) ($0.2\;10^{-8}$ at 77^o)
$\rho_{foil} \;=\; 3.0\;10^{-8}$ (Ωm) ($\approx\; 0.07\;10^{-8}$ at 77^o)
$\Phi \;=\; \pi/2$
$\kappa \;=\; 200$ W/m/K
$t \;=\; 127\;\mu$m
$P_{\rm ohmic}(3\tau) \;=\; 354$ W (117 W, at $t = 1\;\tau$)

$$\delta_{cavity} \;=\; 2.33\;(\mu\;m) \qquad \text{(Cu at room temp)}$$

$$\Delta T \;=\; 106^o$$

To avoid the foil bulging, it must be initially tensioned greater than:

$$S \;=\; \alpha\;T\;Y$$

where α is the coefficient of expansion and Y is the youngs modulus. If this is not to break, this better be less than the tensile strength $\hat{S}$.

For Example
$\alpha \;=\; 11.3\;10{-}6$ 1/K ($1.3\;10^{-6}$ at 77^o)
$Y \;=\; 2.9\;10^{12}$ N/m^2
$\hat{S} \;=\; 1.2 - 1.5\;10^{9}$ N/m^2 (probably higher at 77^o)

$$S \;\approx\; 3.5\;10^{9}\quad N/m^2 \qquad (1.1\;10^{9}\quad N/m^2, att = 1\;\tau)$$

Which, for $t = 3\tau$ is much higher than the breaking stress.

The situation can be improved by:

- reducing the filling time ($S \propto \tau$). The use of $\tau = 1 \times \tau_f$ instead of $3 \times \tau_f$ would reduce S to 1.1 10^9.
- reducing the radius ($S \propto r^4$). If in addition to the aboce, r could be reduced to 7cm (from 8 cm) then S is 0.65 10^9 which is now below the breaking stress.
- by thickening the foil at its edges: It has been shown citeDerun,Corlett,Turner that doubling the thickness at the edge, tapered from $r = 4cm$, reduces the temperature by 35%, to 0.4 10^9, which is about 1/3 of the breaking strength.

So, if all three improvements are taken, a possible room temperature solution may exist.

But a far larger gain is obtained by lowering the temperature to 77^o: The Cu conductivity drops out because the fill time rises as $\sqrt{1/\rho}$ cancelling the lower heating, but since the Be conductivity falls faster than tat of the Cu, we gain a factor

$$S \propto \sqrt{\frac{\rho_{Be}\ (at77^o)}{\rho_{Cu}(at77^o)}} = \sqrt{\frac{0.07}{0.2}} \approx 0.59$$

In addition there is a gain from the better thermal conductivity ($S \propto \kappa^{-1}$) of $\approx$ 200/1500 $\approx$ .13

And finally, a factor from the lower coefficient of expansion ($S \propto \alpha$) of 1.3/11.3 $\approx$ .11.

Altogether, with no changes in filling time, radius and foil thickness, the reduction in S is a factor of approximately 100, lowering the stress to a very acceptable value.

4.7.12 Summary of foils

I like foils because

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1. I gain a factor 3 in peak gradient. I tend now to believe that we can make 100-200 MW tetrodes (more on that later). If that is so, then the ultimate gradient will be limited by break-down, not peak power.

2. I gain a factor 3-4 in peak power requirement, which I estimate:

   - from the worse transit factor for pi, vs pi/2 phase advance, $(.63\ /.9)^2 = .5$
   - from loss of acc due to iris rad r : $[J1(2\ pi\ r\ /lambda)=.62]^2 = .38$ (this assumes beta=1 so that acc=acc at iris edge. at less beta, the acc on axis is less than at the iris edge, giving a bigger loss of acc)
   - but corrected for loss in fewer walls $=2(L+R)/(2L+R) =2(.11+.15)/(.22+.15)=1.4$

   P(foil)/P(iris) = .5 .38 1.4 = .26 =1/3.7

3. The gain in average power is a little less because the Q is higher by the factor 1.4 giving longer filling times, but it should still be about a factor of 3.

So why make it cold?

1. Because the foils overheat and bow. I reject the idea of allowing it to bow. I think the tolerances will be hard to control. So enough presstress is required to stop the bow; and it is probably too much. Ok, you may be able to get by with a) short filling time, b) tapered foil thickness, c) smaller iris, but it is pushing things. At 77 deg, a prestess of 1

2. By further lowering the peak power by a factor 2-3, one is more inclined to trade all, or some, of this for reduced fill time and thus reduced average power. In fact you would do this in order to lower the refrigerator power. So, ok, this is a weak argument for low temp.

The only strong argument is the foil one, and this does NOT require 77 deg. 150 deg would probably be ok.

4.7.13 Problems

1. For a lithium rod with uniform current density, radius R, and length L, with a point source on axis at the front end:
   - a) Calculate the radial surface pressure as function of (R, $B_{surface}$)
   - b) If $p = 300$ MeV/c, $\theta = 1$, $B_{surface} = 10T$ then calculate the minimum R and L to focus to a parallel beam.
   - c) If, further, $dn/dz = 1/(39.7cm)$, what is the fractional pion loss?
2. What is $p_{\perp}(max)$ for a 20 T coild of 8 cm radius. These are the dimensions of an existing resistive solenoid at FSU.
3. In a standing wave cavity filled with rf power P_{rf}, for a time t/τ, and then let to decay; derive analytic equations for P_{rf}/P_{ideal}, U_{rf}/U_{cavity} , and $U_{\text{ohmic}}/U_{\text{cavity}}$ in the imaginary case in which there is no reflection of the input rf power.