Stochastic Filtering of Rain Profiles using Radar, Surface-Referenced Radar, or Combined Radar/Radiometer Measurements

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Abstract

This paper describes a computationally efficient nearly-optimal Bayesian algorithm to estimate rain (and drop-size-distribution) profiles, given a radar reflectivity profile at a single attenuating wavelength. In addition to estimating the averages of all the mutually ambiguous combinations of rain parameters that can produce the data observed, the approach also calculates the r.m.s. uncertainty in its estimates (this uncertainty thus quantifies the "amount of ambiguity" in the "solution"). The paper also describes a more general approach that can make estimates based on a radar reflectivity profile together with an approximate measurement of the path-integrated attenuation, or a radar reflectivity profile and a set of passive microwave brightness temperatures. This more general "combined" algorithm is currently being adapted for the Tropical Rainfall Measuring Mission.

1 Introduction

It is well-documented (Hitschfeld and Bordan 1954, Meneghini 1978, Haddad et al 1993) that there are significant ambiguities inherent in the determination of a particular vertical rain intensity profile from a given time profile of radar echo powers measured by a downward–looking (spaceborne or airborne) radar at a single attenuating frequency. Indeed, in addition to deriving the formulas that generate all *deterministic* mutually ambiguous rain rate profiles from a given profile of received radar reflectivities, we have derived (Haddad et al 1994) a quantitative measure to assess how likely each of these deterministic profiles is, what the appropriate "average" profile should be, and what the "variance" of these multiple solutions is. In order to do this, *stochastic* constraints allowing one to make sense of the words "average" and "variance" in a mathematically rigorous way had to be postulated. The quantitative approach which then estimated the rain would be particularly well-suited for such systems as the spaceborne Ku-band Precipitation Radar of the Tropical Rainfall Measuring Mission (TRMM – see Kawanishi et al, 1993), if it were more efficiently implementable in real-time. Indeed, this stochastic approach had so far relied on calculating the full density function for the rain variables conditioned on the radar observations in order to estimate the rain. The resulting algorithm was very cumbersome and computationally intensive. In this paper, a new extended-Kalman algorithm to calculate the first and second moments of the density function directly is described. The approach is then extend in order to account for other observations such as one additional rain-modified surface-reflectivity measurement, or multiple-microwave-frequency radiometer measurements.

2 Mathematical approach

For simplicity, it is assumed as usual that the observed reflectivity $Z_m(r)$, measured from range r by a downward-looking mono-static narrow-band radar such as the TRMM Precipitation Radar, is proportional to the reflectivity factor Z of the rain at range r, and to the accumulated attenuation from range 0 (the top of the cloud) to range r. Calling $k(r)$ (resp. $R(r)$) the attenuation coefficient (resp. rain rate) at range r, it is assumed for simplicity that $Z = aR^b$ and $k = \alpha R^\beta$ for some value of the parameters a, b, α and β, and that the measured reflectivity is therefore given by

$$Z_m(r) = aR(r)^b 10^{-0.1(2\int_0^r \alpha R(t)^\beta dt)} \tag{1}$$

Treating a, b, α and β as parameters, the solution to equation (1) can be written as

$$R(r) = \frac{Z_m(r)^{1/b}}{\left(a^{\beta/b} - \frac{0.2\log(10)\alpha\beta}{b}\int_0^r Z_m(t)^{\beta/b}dt\right)^{1/\beta}} \tag{2}$$

Equation (2) suggests that if the rain parameters are not known exactly, multiple solutions for R can exist. Indeed, Haddad et al (1993) describe just how mutually ambiguous these multiple solutions can get. In the same paper, it is also shown that using the surface return as a reference does not solve the ambiguity problem. Since one has to "live with" these ambiguities, it is very important

to know how likely each of the multiple solutions is: specifically, given some a-priori "statistical" constraints on the variables involved, one would like to find the "average" solution to (2).

Thus one is naturally led to a stochastic filtering approach. One would like to introduce a "measure" on the set of all ambiguous profiles giving rise to the same measured reflectivity profile, and try to find the "average" profile with respect to this measure on this set, along with an estimate of the mean difference between the members of this set of mutually ambiguous profiles. In (Haddad et al, 1994), we described an algorithm to compute the joint probability density function $\mathcal{P}$ for $\{R(r), a, b, \alpha, \beta\}$ given measurements of $Z_m(r)$. The "average" rain profile and the "mean deviation" with all the mutually ambiguous profiles can then be obtained from the moments of $\mathcal{P}$. Indeed, the results reported in (Haddad et al, 1994) have been very encouraging. In particular, in the case where a, b, α and β are assumed known, *this approach yields a stable inversion algorithm which does not require any surface reference information.* But calculating the full density function requires large amounts of computer memory and CPU time, too large to make the algorithm useful in anywhere near real-time. In order to reduce the amount of computer resources required, rather than calculating $\mathcal{P}$ itself, one can try to compute its mean and covariance directly. This amounts to deriving the extended Kalman filter appropriate to the problem at hand.

To do so, one needs to specify the a-priori constraints on the "state variables" $R(r)$, a, b, α, β and $c(r) = \int_0^r \alpha R(t)^\beta dt$. For simplicity, we shall assume that a, b, α and β are constant, that the only constraint on c is that it be the integral with respect to r of $\alpha R(r)^\beta$, and we express the requirement that R itself be positive and continuous by writing

$$R(r) = e^{x(r)+\lambda r} \tag{3}$$

thus, in effect, replacing R with a new state variable x and a (possibly 0) "drift" parameter λ. More specifically, we assume that the a-priori constraints on the evolution of $\log(R)$ with range r are those of standard Brownian motion, up to a possible drift term λr. These are the simplest constraints that allow R to be any continuous positive function of r. How the new parameter λ (along with the three implicitly introduced new parameters m_0 = the mean of $\log(R(0))$, σ_0^2 = the variance of $\log(R(0))$, and σ^2 = essentially the a-priori variance of $\log(R(r))$) should be chosen is discussed in Appendix B.

Now that the variables have been defined, and the a-priori constraints on their dynamics have been postulated, there remains to make explicit the function $h(r)$ modeling the dependence of the measurement from range r on the state variables. From equation (1), one can see that

$$h(r) = \log(a) + b(x(r) + \lambda r) - 0.2\log(10)c(r) + \text{Noise}. \tag{4}$$

Write σ_N for the r.m.s. noise level in the measurements, which, for simplicity, we shall attribute here to Rayleigh fading only (system noise can be taken into account, at the expense of making the exposition somewhat more cumbersome). Since the data consist of the averaged power of M independent pulses, the noise term in (4) would be the logarithm of the average of the squared-magnitudes of M independent standard complex Gaussian variables. Hence, as soon as $M > 4$, it is quite reasonable to assume that this noise term is itself approximately a 0-mean normal variable with variance $\sigma_N^2 \simeq 1/M$.

One can now apply the standard machinery of stochastic filtering to obtain the best estimate $\hat{R}(r)$ of the rain rate at range r given all the observations. Since the relation $dc/dr = \alpha R^{\beta}$ is non-linear, one cannot use a straightforward Kalman filter to solve the problem. We chose to use an extended Kalman filter approach, using a first-order Taylor series linearization to obtain both the downward estimate (starting from the top of the cloud $r = 0$) and the upward estimate (starting from the ocean surface). The theory and details behind the technique can be found for example in Øksendal, 1985, or Jazwinski, 1970. For completeness, the flow of the particular algorithm in the case at hand, when the parameters a, b, α and β are assumed known, is summarized in appendix A. The optimal estimate $\hat{R}_{radar}(r)$ of the rain rate itself is then given by the mean

$$\hat{R}_{radar}(r) = e^{\hat{x}(r)+0.5v(r)+\lambda r} \tag{5}$$

where $v(r)$ is the variance of $\hat{x}(r)$, and the r.m.s. uncertainty $\sigma_{radar}(r)$ in the rain rate estimate is given by

$$\sigma_{radar}(r) = \hat{R}_{radar}(r) \cdot \sqrt{e^{v(r)} - 1} \tag{6}$$

As expected, the resulting algorithm turns out to be orders of magnitude more efficient than the full density function approach described in (Haddad et al, 1994). For completeness, there remains to describe how one specifies values for the parameters m_0, σ_0, σ and λ. Although, as we shall see, in practice, the exact values do not affect the estimation algorithm significantly, one should certainly try to give them at least physically reasonable values. This issue is discussed in appendix B.

3 Applications – radar only

To test the approach, radar reflectivity profiles were synthesized, starting with each of three assumed rain rate profiles, and using fading noise corresponding to the average of 50 independent pulses. The three input rain rate profiles were

Profile A (linear-then-constant): the rain increases linearly from 1 mm/hr at range 0 up to 5 mm/hr at range $r = 2$ km, then remains constant at 5 mm/hr until the surface range $r_s = 5$ km (see figure 1).

Profile B (sawtooth-linear): the rain increases linearly from 1 mm/hr at range 0 up to 25 mm/hr at range $r = 2$ km, then decreases linearly down to 12.5 mm/hr at range $r = 3$ km, then back up to 25 mm/hr at $r = 4$ km, and finally back down to 12.5 mm/hr at $r_s = 5$ km (see figure 2a).

Profile C (sawtooth-linear): the rain increases linearly from 1 mm/hr at range 0 up to 75 mm/hr at range $r = 2$ km, then decreases linearly down to 37.5 mm/hr at range $r = 3$ km, then back up to 75 mm/hr at $r = 4$ km, and finally back down to 37.5 mm/hr at $r_s = 5$ km (see figures 3a and 3b).

The synthesized reflectivity profiles were then supplied as input to the extended-Kalman algorithm in order to estimate the associated rain rates. The values $a = 255$, $b = 1.31$, and the typical Ku-band values $\alpha = 0.0285$ and $\beta = 1.1$ were assumed given. To determine the parameters σ_0 and m_0, we use equations (37) and (38), except that rather than specifying R_{min} and its variance directly, we specify the minimum threshold-onset reflectivity Z_{min} and its variance. Practically, we set $10\log_{10}(Z_{min}) = 26$ dB, so that nominally $R_{min} = (Z_{min}/a)^{1/b} \simeq 1.4$ mm/hr in our case, then we used this value in (38), with the following values in equation (37): $\sigma_0^2 = \log(10^{(0.1)\cdot(2.3)})$. Finally, for σ, we used the value given by equation (41) with $\mathcal{E}\{R(r+1)/R(r)\} = 1.5$, r in km.

Figure 1 shows the estimated rain rate profile and the associated r.m.s. uncertainty when the input was profile A. The estimates are manifestly quite accurate. We tried changing the value of Z_{min} over the interval $20 \leq Z_{min} \leq 32$ dB, with results that were less than 1% off the ones in figure 1. Similarly, changing the value of σ by allowing $\mathcal{E}\{R(r+1)/R(r)\}-1$ to vary between 10 % and 200 % did not significantly alter the estimates in this case. Figure 2a shows the estimates of our algorithm when the input profile was profile B. Figure 2b shows the estimates when $\mathcal{E}\{R(r+1)/R(r)\}$ is increased up to 3 ($= 1 + 200\%$), and figure 2c shows the estimates when $\mathcal{E}\{R(r+1)/R(r)\}$ is given the value 1.1 – the effects are smaller than the r.m.s. uncertainty in the estimates. Figures 3a and 3b show what can happen at larger rain rates, specifically when the input is profile C. The estimates in both cases are quite close to the actual values up to a range of 3 km. Beyond that, the algorithm begins to over- or under-estimate noticeably. The reason for the discrepancy is that the intrinsic ambiguities in the rain-rate–reflectivity-profile relation can be quite pronounced, especially at higher rain rates. In fact, in general, by changing the rain parameters a, b, α or β by small amounts, one can succeed in constructing substantially different rain profiles that produce identical reflectivity profiles. As was shown in (Haddad et al, 1993), at high rain rates, the exponential contribution of these ambiguitites can grow very quickly. Figure 3a shows our algorithm's estimates with the original parameter values as above, and figure 3b shows the estimates when b is changed from 1.31 to 1.313, i.e. by a mere 0.2% ! The ambiguity is responsible for the increasing r.m.s. uncertainty in the estimates: the algorithm recognizes that its estimates are less and less precise as the potential for ambiguity increases with range. One conclusion that can be drawn from these observations is that it would be obviously unwise in practice to fix values for the rain parameters a, b, α and β, especially in cases where one expects sustained high rain rates. Instead, one should modify the model to allow these parameters to be themselves stochastic, as was done in our previous full-density-function approach (Haddad et al, 1994). How this can be done while still keeping the algorithm efficient is described in the next three sections.

We end our radar-only test cases with measured data. Figure 4a shows the estimated rain rate obtained using our extended-Kalman algorithm with the original parameter values, when the input was one of the radar reflectivity profiles measured by JPL's ARMAR radar (Durden et al, 1992) over the Western Pacific Ocean during the TOGA-COARE experiment on February 4, 1993, at 15:35 local time. Details of the participation of ARMAR in COARE can be found in (Li et al, 1993). Figure 4b shows our algorithm's estimates when the rain parameter values were changed to $a = 300$, $b = 1.4$, $\alpha = 0.026$, $\beta = 1.08$. The relative difference in the estimates increases, especially over the last two kilometers where it goes from 100 % up to close to 300 % . For comparison, figure 4c shows the estimates obtained using our full-density-function code (from Haddad et al, 1994),

where we had again assumed that $a = 300$, $b = 1.4$, $\alpha = 0.026$, $\beta = 1.08$. The difference between the two models in the stochastic constraints on the rain rate explain the difference in the r.m.s. uncertainty values between the two graphs. Otherwise, the estimates are remarkably close. Yet while the results of our old full-density-function algorithm took several hours of computer CPU time to produce, the estimates of the extended-Kalman algorithm took less than a second.

We shall return to the ARMAR data in the last section in order to illustrate how radiometer measurements can help determine the correct parameter values to use, and reduce the uncertainty in the estimates.

4 Coupled DSD-based Z–R and k–R relations

Rather than use the independent power-law Z–R and k–R relations above, and allow the parameters a, b, α and β to vary independently, we will use more realistic *coupled* relations, based on the drop size distribution (DSD). Specifically, following Ulbrich, 1983, we shall assume

- that the DSD at range r is given by $N(D;r) = N_0(r) D^{\mu} e^{-\Lambda(r) D}$, for all $D > 0$, with $-1 < \mu$,
- that N_0 is related to the rain rate R by $N_0(r) \simeq 140 R(r) \Lambda(r)^{4.67+\mu} / \Gamma(4.67 + \mu)$, when D is expressed in mm, R in mm/hr, and $N(D)dD$ in number of drops per m^3 – this follows directly from the assumption that all drops are falling at their power-law terminal velocity (see Haddad et al, 1993, for details),
- that the mass-weighted mean drop diameter, $(\mu + 4)/\Lambda(r)$, is related to the rain-rate by a power-law $(\mu + 4)/\Lambda(r) = R(r)^{\delta}$, where the parameter δ can vary over the interval $0.08 \leq \delta \leq 0.23$ (again, see Haddad et al, 1995, for the details),
- that μ, δ and R are a priori independent,
- and that μ and δ are (for now) constant over the rain column.

Thus we are describing the rain column by specifying the DSD at every range R, using the rain rate profile $R(r)$ and two additional parameters, μ and δ. As stated above, we shall start by assuming that these two additional parameters are constant over the rain column. As we shall see later, this simplifying assumption is not crucial to the derivation of the algorithm. Using this description of the rain, one can replace the power-law Z–R and k–R relations by any more realistic physical model such as Mie-scattering or a T-matrix calculation. For this paper, however, we shall stay with the simple power-law models. Under the Rayleigh hypothesis (for Z), and using a power law to relate the total scattering cross-section of a drop to its diameter (for k), the assumptions above produce the approximate relations

$$Z = 140 \frac{\Gamma(7.26 + \mu)}{\Gamma(4.67 + \mu) \cdot (4 + \mu)^{2.59}} R^{1+2.59\delta} \tag{7}$$

$$\text{and} \quad k = 0.026 \frac{\Gamma(5.33 + \mu)}{\Gamma(4.67 + \mu) \cdot (4 + \mu)^{0.86}} R^{1+0.86\delta} \tag{8}$$

Although still rather simplistic, these relations are preferable to the earlier power laws because they involve two parameters instead of four, both of which directly describe the drop size distribution itself. One can readily verify from our new Z–R and k–R equations that the values $(a, b, \alpha, \beta) = (255, 1.31, 0.0285, 1.1)$ used earlier correspond to $\mu \simeq 1.5$, $\delta \simeq 0.12$, values that are well within the range of both parameters.

5 Incorporating additional measurements

The previous sections described how to estimate the rain variables given the radar reflectivities only. While that is interesting in itself, one would like to be able to account for additional measurements, such as passive microwave brightness temperatures, to reduce the inherent ambiguity. The Bayesian approach which we have taken so far can guide us in performing this data fusion. Indeed, calling $\vec{T}$ the vector of multiple-frequency-radiometer-measured brightness temperatures, we need to compute the conditional density function $\mathcal{P}(\{R(r)\}_{0\leq r\leq r_s}, \mu, \delta \mid \vec{T}, Z_m(r))$ for the rain profile $\{R(r)\}_{0\leq r\leq r_s}$ and the rain variables μ and δ, given a reflectivity profile $Z_m(r)$ and a set of brightness temperatures $\vec{T}$. By Bayes's theorem, this probability is given by

$$\mathcal{P}(\{R(r)\}_{0\leq r\leq r_s}, \mu, \delta \mid \vec{T}, Z_m(r)) = \mathcal{P}(\vec{T} \mid \{R(r)\}_{0\leq r\leq r_s}, \mu, \delta, Z_m(r)) \cdot \mathcal{P}(\{R(r)\}_{0\leq r\leq r_s}, \mu, \delta \mid Z_m(r)) \cdot K_0 \tag{9}$$

where K_0 is the over-all normalization constant making the integral of the left-hand-side equal to 1. Let us now examine (9) carefully.

The first term in the right-hand-side of (9) is the probability of the brightness temperatures $\vec{T} = (T_1, ..., T_j, ...)$ given all the other data. This figure can be obtained using a "forward" model to compute the brightness temperature from the physical rain data. In fact, we shall assume that we have an efficient deterministic forward algorithm to compute the "mean" brightness temperatures $\hat{T}_j$ at the relevant frequencies that correspond to a given rain profile, along with an approximate formula giving the r.m.s. deviation $\sigma_{\hat{T}_j}$ to be expected about each of these average brightness temperatures. Given such a forward model, we can replace the first probability in the right-hand-side of (9) by the product of Gaussians $\mathcal{G}_{\sigma_{\hat{T}_j}}(T_j - \hat{T}_j)$ with mean $\hat{T}_j$ and variance $\sigma^2_{\hat{T}_j}$. Thus,

$$\mathcal{P}(\vec{T} \mid \{R(r)\}_{0\leq r\leq r_s}, \mu, \delta, Z_m(r)) \simeq \prod_j \mathcal{G}_{\sigma_{\hat{T}_j}(\mu,\delta,Z_m(r))}\left(T_j - \hat{T}_j(\mu, \delta, Z_m(r))\right) \tag{10}$$

where we have made explicit the dependence of $\hat{T}_j$ and $\sigma_{\hat{T}_j}$ on the rain data $(\mu, \delta, Z_m(r))$.

The second term in the right-hand-side of (9) can be further split into the product of two simpler terms, thanks again to Bayes's theorem:

$$\mathcal{P}(\{R(r)\}_{0\leq r\leq r_s}, \mu, \delta \mid Z_m(r)) = \mathcal{P}(\{R(r)\}_{0\leq r\leq r_s} \mid \mu, \delta, Z_m(r)) \cdot \mathcal{P}(\mu, \delta | Z_m(r)) \tag{11}$$

In the right-hand-side of (11), the first term is exactly what our extended Kalman filter described in the earlier sections computes: the probability function for the rain rate profile, assuming μ and

δ and the radar reflectivities are known. More precisely, our extended-Kalman-filtering algorithm computes the first two moments of this density function. The second term in the right-hand-side is the density function of μ and δ given the radar reflectivities. We shall approximate it with the a priori density function $\mathcal{P}(\mu,\delta)$ for μ and δ, namely a product of two independent uniform distributions.

Putting (9), (10) and (11) together, one thus finds that

$$\mathcal{P}(\{R(r)\}_{0\le r\le r_s},\mu,\delta|\vec{T},Z_m(r)) \simeq \mathcal{P}(\{R(r)\}_{0\le r\le r_s}|\mu,\delta,Z_m(r))\cdot\mathcal{P}(\mu,\delta)\cdot K_0\cdot\prod_j \mathcal{G}_{\sigma_{\hat{T}_j}}\left(T_j-\hat{T}_j(\mu,\delta,Z_m(r))\right) \tag{12}$$

where the first moments of the first term on the right are computed by the extended-Kalman-filter described in the previous sections. We are finally ready to write down explicit expressions for the optimal estimates of μ, δ and $R(r)$ given $Z_m(r)$ *and* $\vec{T}$. Indeed, all we need to do is calculate the various means of (12). Thus, the optimal estimate $\hat{R}_{radar+passive}(r_0)$ for $R(r_0)$ at any range r_0 is obtained by taking the mean of $R(r_0)$:

$$\begin{aligned}\hat{R}_{radar+passive}(r_0) &= \int R(r_0)\cdot\mathcal{P}\left(\{R(r)\}_{0\le r\le r_s},\mu,\delta\mid\vec{T},Z_m(r)\right)\,d\{R(r)\}d\mu d\delta \qquad (13)\\ &\simeq K_0\cdot\int\int\hat{R}_{radar}(r_0;\mu,\delta)\cdot\mathcal{P}(\mu,\delta)\cdot\left(\prod_j\mathcal{G}_{\sigma_{\hat{T}_j}(\mu,\delta,Z_m(r))}\left(T_j-\hat{T}_j(\mu,\delta,Z_m(r))\right)\right)\,d\mu d\delta\end{aligned}$$

where $\hat{R}_{radar}(r_0;\mu,\delta)$ is the radar-only estimate obtained as in formula (5). Thus, the optimal estimation of the rain rate proceeds in two steps: first, we compute the radar-only estimates using an efficient extended-Kalman-filtering approach, for each possible pair (μ,δ); next, again for each (μ,δ), we calculate the corresponding brightness temperatures. Finally, at each range r_0, we perform the double integration specified in (13) to estimate the rain rate itself at that range. Similarly, the estimates of the DSD parameters at range r_0 are given by

$$\begin{aligned}\hat{\Lambda}_{radar+passive}(r_0) &= \int(\mu+4)R(r_0)^{-\delta}\cdot\mathcal{P}\left(\{R(r)\}_{0\le r\le r_s},\mu,\delta\mid\vec{T},Z_m(r)\right)\,d\{R(r)\}d\mu d\delta \qquad (14)\\ &\simeq K_0\cdot\int\int(\mu+4)\hat{R}_{radar}(r_0;\mu,\delta)^{-\delta}\cdot\mathcal{P}(\mu,\delta)\cdot\left(\prod_j\mathcal{G}_{\sigma_{\hat{T}_j}(\mu,\delta,Z_m(r))}\left(T_j-\hat{T}_j(\mu,\delta,Z_m(r))\right)\right)\,d\mu d\delta\ ,\end{aligned}$$

$$\begin{aligned}\hat{\mu}_{radar+passive} &= \int\mu\cdot\mathcal{P}\left(\{R(r)\}_{0\le r\le r_s},\mu,\delta\mid\vec{T},Z_m(r)\right)\,d\{R(r)\}d\mu d\delta \qquad (15)\\ &\simeq K_0\cdot\int\int\mu\cdot\mathcal{P}(\mu,\delta)\cdot\left(\prod_j\mathcal{G}_{\sigma_{\hat{T}_j}(\mu,\delta,Z_m(r))}\left(T_j-\hat{T}_j(\mu,\delta,Z_m(r))\right)\right)\,d\mu d\delta\ ,\end{aligned}$$

and similarly for $N_0(r_0)$.

If, instead of passive microwave brightness temperatures, our additional measurement consisted of the path-integrated attenuation (PIA), we would use a similar algorithm to improve our radar-only estimates and condition them further on this additional measurement. Indeed, in this case, the optimal estimate would be

$$\hat{R}_{radar+PIA}(r_0)\simeq K'\cdot\int\int\hat{R}_{radar}(r_0;\mu,\delta)\cdot\mathcal{P}(\mu,\delta)\cdot\mathcal{G}_{\sigma_{\hat{c}}}\left(C-\hat{c}(r_s;\mu,\delta)\right)\,d\mu d\delta \tag{16}$$

where C is the measured PIA, $\hat{c}(r_s;\mu,\delta)$ the path-integrated attenuation from the top of the rain column to the surface range r_s as estimated by the extended Kalman algorithm, for each (μ,δ), and $\sigma_{\hat{c}}$ the associated standard deviation.

The estimates (13) and (16) can be compared with the radar-only estimate

$$\hat{R}_{radar}(r_0) \simeq K'' \cdot \int\int \hat{R}_{radar}(r_0;\mu,\delta) \cdot \mathcal{P}(\mu,\delta)\, d\mu d\delta. \tag{17}$$

In addition, in each of these three cases (radar-only, radar+passive, radar+PIA), one can compare the variance of $\hat{R}$, i.e. the error bar that the algorithm itself imposes on its own estimate, in order to decide how the ambiguity is being reduced, if at all. Similarly, one can compare the estimates (14) and (15) of μ and Λ with the radar-only estimates given by the corresponding equations similar to (17), and compare the error bars obtained using radar-only and radar+passive data.

6 Applications

Figure 5 shows the graph of the "forward" function we used to compute the average 13.8 GHz brightness temperature $\hat{T}_j(\mu,\delta,Z_m(r))$, namely

$$\hat{T}_j(\mu,\delta,Z_m(r)) = T + \epsilon(t-T)10^{-0.1\hat{c}(r_s)} - T(1-\epsilon)10^{-0.2\hat{c}(r_s)} \tag{18}$$

where ϵ = the surface emissivity $\simeq 0.38$, T = the atmosphere (physical) temperature $\simeq 280K$, t = the surface (physical) temperature $\simeq 300K$, and where $\hat{c}(r_s)$ is the estimated integrated attenuation at the surface $r = r_s$, which is computed by our extended-Kalman algorithm as a function of μ, δ and the input profile $Z_m(r)$. For every value of c, earlier calculations (Durden et al, 1994) have quantified the expected spread of the associated brightness temperatures: the minimum and maximum values are shown by the dotted curves in figure 5. We added this inherent variance in the forward formula to the variance due to the uncertainty $\sigma_{\hat{c}}$ in our estimate of $\hat{c}(r_s)$ to obtain an approximation to the m.s. deviation $\sigma_{\hat{T}_j}$ of the brightness temperature predicted by (18).

Starting with independent uniform distributions for μ over the interval $[-0.5,+3.5]$ and for δ over $[0.08,0.23]$, we then used formulas (17), (16) and (13) to obtain radar-only, radar+PIA and radar+radiometer estimates for each of our sample profiles considered in section 3. Figure 6a shows the result of the radar-only estimation (17) in the case of profile B. Figure 6b shows the result of the radar+PIA estimation (16) assuming that the path-integrated attenuation is the one corresponding to profile B, namely 3.1 dB, up to an r.m.s. uncertainty of 0.1 dB. Figure 6c shows the radar+PIA estimates when the r.m.s. uncertainty in the PIA is assumed to be 0.5 dB. The uncertainty in the resulting rain-rate estimates decreases (figure 6b) then increases (figure 6c) accordingly. Figure 6d shows the radar+radiometer estimates using (13), where the passive microwave brightness temperatures are handled as described above. In this case, figure 5 leads one to expect that the uncertainty in the relation between the brightness temperature $T_b \simeq 242.5K$ and the radar attenuation is relatively small. Indeed, the accuracy of the radar+radiometer estimates is quite good, and their uncertainty quite small. Figures 7abcd show what happens at higher rain

rates. Figure 7a shows the radar-only estimates in the case of profile C. The r.m.s. uncertainty in the estimates exceeds 50% rapidly at ranges above 1.5 km, as could have been expected from the ambiguity calculations in (Haddad et al, 1993). Figure 7b shows the improved result when one uses the radar+PIA estimates assuming that the path-integrated attenuation is the one corresponding to profile C, namely 10.3 dB, up to an r.m.s. uncertainty of 0.1 dB again. Figure 7c shows the estimates when the PIA uncertainty is assumed to be 0.5 dB. The uncertainty in the resulting rain-rate estimates increases slightly, but remains far smaller than that of the radar-only estimates. In fact, even at the surface range where it is largest, the r.m.s. uncertainty amounts to a reasonable 30%. Figure 7d shows the radar+radiometer estimates. In this case, figure 5 shows that the uncertainty in the relation between the brightness temperature $T_b \simeq 279K$ and the radar attenuation is quite large, leading one to expect little additional information from the low-frequency passive measurement. Indeed, the accuracy of the radar+radiometer estimates at ranges beyond about 3 km is quite unsatisfactory, and their r.m.s. uncertainty is correspondingly large.

Finally, we return to the case where our radar reflectivity profile is the ARMAR TOGA-COARE profile of figures 4abc. Figure 8a shows our radar-only algorithm's estimates, when μ and δ are independently uniformly distributed over $-0.5 \leq \mu \leq +3.5$ and $0.08 \leq \delta \leq 0.23$. The uncertainty in these estimates is unacceptably high. Figure 8b shows the radar+PIA estimates, assuming that the path-integrated attenuation is the one estimated by comparing the surface return with the "clear-air surface return" as explained in (Durden et al, 1994), namely 3.9 dB, up to a postulated r.m.s. uncertainty of 0.5 dB. It is most interesting to compare this estimated profile to the one in figure 8c, obtained using the radar+radiometer algorithm with the measured brightness temperature $T_b = 254.3K$. Indeed, the estimated profiles in both cases are remarkably close. Since the calibration of ARMAR's passive microwave measurements is not quite complete, we have computed the estimated profiles corresponding to $T_b = 250K$ (figure 8c), and $T_b = 260K$ (figure 8d). One can clearly see that, as the measured microwave temperature decreases or increases, the estimated profile follows suit appropriately.

Future work will concentrate on

- making the forward passive microwave calculation more realistic,
- incorporating additional passive microwave channels,
- modeling the full covariance between μ, δ and R in order to reduce the ambiguities in the radar-only model.

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Appendix A - Simple case of the extended-Kalman algorithm

The extended-Kalman estimation algorithm in the case where the parameters a, b, α and β are assumed known can be described relatively simply. First, one must obtain "downward" estimates $\hat{x}_d(r)$ and $\hat{c}_d(r)$ of the state variables x and c at all ranges r based on all earlier measurements obtained for $r' < r$, along with their covariances $p_{xx}(r)$, $p_{cx}(r)$ and $p_{cc}(r)$. To do this, one must start with

$$\hat{x}_d(0) = m_0 \tag{19}$$
$$\hat{c}_d(0) = 0 \tag{20}$$
$$p_{cc}(0) = 0 \tag{21}$$
$$p_{cx}(0) = 0 \tag{22}$$
$$p_{xx}(0) = 0 \tag{23}$$

Then, given our estimates at range r, the estimates at range $r + \Delta$ can be obtained in two steps, by first accounting for the changes in the dynamics using the formulas

$$\tilde{x}(r+\Delta) = \hat{x}_d(r) \tag{24}$$
$$\tilde{c}(r+\Delta) = \hat{c}_d(r) + \alpha \int_r^{r+\Delta} e^{\beta(\bar{x}(t)+\lambda t)} dt \tag{25}$$
$$\tilde{p}_{xx}(r+\Delta) = p_{xx}(r) + \sigma^2 \Delta$$
$$\tilde{p}_{cx}(r+\Delta) = p_{cx}(r) + \alpha\beta \int_r^{r+\Delta} e^{\beta(\bar{x}(t)+\lambda t)} \tilde{p}_{xx}(t) dt \tag{26}$$
$$\tilde{p}_{cc}(r+\Delta) = p_{cc}(r) + 2\alpha\beta \int_r^{r+\Delta} e^{\beta(\bar{x}(t)+\lambda t)} \tilde{p}_{cx}(t) dt, \tag{27}$$

then by accounting for the measurement $z(r + \Delta)$ obtained from range $r + \Delta$ using the formulas

$$\hat{x}_d(r+\Delta) = \tilde{x}(r+\Delta) + \frac{b\tilde{p}_{xx}(r+\Delta) - 0.2\log(10)\tilde{p}_{cx}(r+\Delta)}{D}\Xi \tag{28}$$
$$\hat{c}_d(r+\Delta) = \tilde{c}(r+\Delta) + \frac{b\tilde{p}_{cx}(r+\Delta) - 0.2\log(10)\tilde{p}_{cc}(r+\Delta)}{D}\Xi \tag{29}$$

where $D = (0.2\log(10))^2\tilde{p}_{cc}(r+\Delta) - 2(0.2\log(10))b\tilde{p}_{cx}(r+\Delta) + b^2\tilde{p}_{xx}(r+\Delta) + \sigma_N^2$, and $\Xi = z(r+\Delta) - (\log(a) + b(\tilde{x}(r+\Delta) + \lambda r) - 0.2\log(10)\tilde{c}(r+\Delta))$ measures the amount by which the observation that would correspond to the preliminary estimates $\tilde{x}$ and $\tilde{c}$ differs from the actual measurement. Finally, the covariance matrix itself is updated using

$$p_{xx}(r+\Delta) = \frac{(0.2\log(10))^2(\tilde{p}_{cc}(r+\Delta)\tilde{p}_{xx}(r+\Delta) - \tilde{p}_{cx}(r+\Delta)^2) + \sigma_N^2\tilde{p}_{xx}(r+\Delta)}{D} \tag{30}$$
$$p_{cx}(r+\Delta) = \frac{0.2\log(10)b(\tilde{p}_{cc}(r+\Delta)\tilde{p}_{xx}(r+\Delta) - \tilde{p}_{cx}(r+\Delta)^2) + \sigma_N^2\tilde{p}_{cx}(r+\Delta)}{D} \tag{31}$$
$$p_{cc}(r+\Delta) = \frac{b^2(\tilde{p}_{cc}(r+\Delta)\tilde{p}_{xx}(r+\Delta) - \tilde{p}_{cx}(r+\Delta)^2) + \sigma_N^2\tilde{p}_{cc}(r+\Delta)}{D} \tag{32}$$

The upward portion of the algorithm, to obtain "upward" estimates $\hat{x}_u(r)$ and $\hat{c}_u(r)$ of the state variables x and c at all ranges r based on all "later" measurements obtained for $r' > r$, proceeds in a similar fashion, except for the fact that one initializes $\hat{x}_u(r_s)$, $\hat{c}_u(r_s)$ and $P_u(r_s)$ with the downward estimates $\hat{x}_d(r_s)$, $\hat{c}_d(r_s)$ and $P_d(r_s)$, and one makes the obvious sign changes that are then necessary. Finally, one must combine downward and upward estimates to obtain the optimal estimates $\hat{x}$ and $\hat{c}$ based on *all* the reflectivity data. This is done using the formula

$$\begin{pmatrix} \hat{c} \\ \hat{x} \end{pmatrix} = \left(P_d^{-1} + P_u^{-1}\right)^{-1} \left(P_d^{-1} \begin{pmatrix} \hat{c}_d \\ \hat{x}_d \end{pmatrix} + P_u^{-1} \begin{pmatrix} \hat{c}_u \\ \hat{x}_u \end{pmatrix}\right) \tag{33}$$

at every range r, with P denoting the appropriate covariance matrix (we have tacitly assumed that all variables are evaluated at the same range r). The optimal estimate $\hat{R}_{radar}(r)$ of the rain rate itself is then given by the mean $\mathcal{E}\{e^{\hat{x}(r)+\lambda r}\}$, which, since x itself is Brownian, becomes

$$\hat{R}_{radar}(r) = e^{\hat{x}(r)+0.5p_{xx}(r)+\lambda r} \tag{5}$$

and its r.m.s. uncertainty $\sigma_{radar}(r)$ by

$$\sigma_{radar}(r) = \hat{R}_{radar}(r) \cdot \sqrt{e^{p_{xx}(r)} - 1} \tag{34}$$

Appendix B - Choosing the algorithm's "free" parameters

Without further a-priori information about the variation of R with range r, it was assumed that $x(r) = \log(R(r)) - \lambda r$ can be expressed as $x(r) = x(0) + \sigma w(r)$, where $x(0)$ and $w(r)$ are independent, $x(0)$ itself is Gaussian with mean m_0 and variance σ_0^2, and the process $w(r)$ is the standard Wiener process, i.e. it has independent 0-mean Gaussian increments with variance equal to the extent in range of the increment interval. To determine physically reasonable values for the four parameters m_0, σ_0, σ and λ, one must re-examine the constraints already imposed and interpret them physically. It follows from (3) that the expected value of the rain-rate $R(0)$ at the top of the rain column is

$$\mathcal{E}\{R(0)\} = e^{m_0 + \frac{1}{2}\sigma_0^2} \tag{35}$$

and its relative variance is

$$\mathcal{E}\left\{\left(\frac{R(0)}{\mathcal{E}\{R(0)\}} - 1\right)^2\right\} = \frac{\mathcal{E}\{R(0)^2\}}{\mathcal{E}\{R(0)\}^2} - 1 = e^{\sigma_0^2} - 1 \tag{36}$$

In practice, we set a minimum "threshold onset" value R_{min} for the smallest significant rain rate we expect at the top of the rain column, along with some estimate for the associated mean relative uncertainty $\mathcal{E}\{R(0)^2\}/R_{min}^2$. Equation (36) then implies that we should choose

$$\sigma_0^2 = \log\left(\frac{\mathcal{E}\{R(0)^2\}}{R_{min}^2}\right) \tag{37}$$

and (35) in turn implies that we should then choose

$$m_0 = \log(R_{min}) - \frac{1}{2}\sigma_0^2 \tag{38}$$

The choice of λ is somewhat more problematic. We do know that, a priori, by definition, the rain rate should initially increase with range from the a-priori threshold-onset value R_{min}. This would imply a positive drift λ. To get a specific value for λ, we look at the terminal behavior of R. Writing $R'(r)$ for $R'(r) = R(r_s - r)$, where r_s is the range of the surface, and if we reverse the constraint (3) in time to apply it to $R'(r)$, one finds that the "a-priori" (with time reversed) expected value for R' will be given by

$$\mathcal{E}\{R'(r)\} = e^{\hat{x}_d(r_s)+\frac{1}{2}p_{xx}(r_s)+\lambda r_s} \cdot e^{(\frac{1}{2}\sigma^2-\lambda)r} \tag{39}$$

Since we have a priori no reason to expect the rain rate to increase or decrease as one moves up from the bottom of the rain column, it is natural to choose the value

$$\lambda = \frac{1}{2}\sigma^2 \tag{40}$$

Last, we must decide on a value for σ. In practice, this parameter controls the a priori variation $\mathcal{E}\{(x(r+\Delta) - x(r))^2\} = \sigma^2\Delta$. In fact, the average a priori rate-of-change of R is given by

$$\mathcal{E}\left\{\frac{R(r+\Delta)}{R(r)}\right\} = e^{\sigma^2\Delta} = \frac{\mathcal{E}\{R(r+\Delta)\}}{\mathcal{E}\{R(r)\}} \tag{41}$$

We therefore choose an a priori value for $\mathcal{E}\{R(r+\Delta)/R(r)\}$ and use (41) to deduce the corresponding value of σ.

Figure captions

Figure 1: Radar-only rain estimates for profile A, assuming $a = 255$, $b = 1.31$, $\alpha = 0.0285$, $\beta = 1.1$

Figure 2a: Radar-only rain estimates for profile B, assuming $a = 255$, $b = 1.31$, $\alpha = 0.0285$, $\beta = 1.1$, and $\mathcal{E}\{R(r+1)/R(r)\} = 1.5$

Figure 2b: Same as 2a, except $\mathcal{E}\{R(r+1)/R(r)\} = 3$

Figure 2c: Same as 2a, except $\mathcal{E}\{R(r+1)/R(r)\} = 1.1$

Figure 3a: Radar-only rain estimates for profile C, assuming $a = 255$, $b = 1.31$, $\alpha = 0.0285$, $\beta = 1.1$

Figure 3b: Same as 3a, except $b = 1.313$

Figure 4a: Radar-only rain estimates for an ARMAR profile measured during TOGA COARE, assuming $a = 255$, $b = 1.31$, $\alpha = 0.0285$, $\beta = 1.1$

Figure 4b: estimates of TOGA COARE profile assuming $a = 300$, $b = 1.4$, $\alpha = 0.026$, $\beta = 1.08$

Figure 4c: Full-density function estimates of the profile of figure 4ab assuming $a = 300$, $b = 1.4$, $\alpha = 0.026$, $\beta = 1.08$

Figure 5: Simplified PIA-T_b relation

Figure 6a: Radar-only rain estimates for profile B, assuming $-0.5 \leq \mu \leq 3.5$ and $0.08 \leq \delta \leq 0.23$

Figure 6b: Radar+PIA rain estimates for profile B, assuming PIA $= 3.13 \pm 0.1$ dB

Figure 6c: Radar+PIA rain estimates for profile B, assuming PIA $= 3.13 \pm 0.5$ dB

Figure 6d: Radar+radiometer rain estimates for profile B, assuming $T_b = 242.5K$

Figure 7a: Radar-only rain estimates for profile C, assuming $-0.5 \leq \mu \leq 3.5$ and $0.08 \leq \delta \leq 0.23$

Figure 7b: Radar+PIA rain estimates for profile C, assuming PIA $= 10.4 \pm 0.1$ dB

Figure 7c: Radar+PIA rain estimates for profile C, assuming PIA $= 10.4 \pm 0.5$ dB

Figure 7d: Radar+radiometer rain estimates for profile C, assuming $T_b = 279K$

Figure 8a: Radar-only rain estimates ARMAR / TOGA-COARE profile, assuming $-0.5 \leq \mu \leq 3.5$ and $0.08 \leq \delta \leq 0.23$

Figure 8b: Radar+PIA rain estimates for ARMAR / TOGA-COARE profile, assuming PIA $= 3.9 \pm 0.5$ dB

Figure 8c: Radar+radiometer rain estimates for ARMAR / TOGA-COARE profile, assuming $T_b = 254.3K$

Figure 8d: Radar+radiometer rain estimates for ARMAR / TOGA-COARE profile, if $T_b = 250K$

Figure 8e: Radar+radiometer rain estimates for ARMAR / TOGA-COARE profile, if $T_b = 260K$