The Performance of Noncoherent Orthogonal *M*-FSK in the Presence of Timing and Frequency Errors[1]

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Abstract

Practical M-FSK systems experience a combination of time and frequency offsets (errors). This paper assesses the deleterious effect of these offsets, first individually and then combined, on the average bit error probability performance of the system. Exact expressions for these various error probability performances are derived and. evaluated numerically for system parameters of interest. Also presented is an upper (Chernoff-type) bound on average symbol error probability for the case of frequency error alone which is useful in assessing the relative performance of the system. Both continuous and discontinuous phase *M*-FSK cases are considered when timing error is present, the latter being much less robust to this type of offset.

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1.0 Introduction

Noncoherent orthogonal M-ary frequency-shift-keying (*M*-FSK) is a simple and robust form of digital communication when the transmission channel is such that fast reliable carrier recovery is difficult or impractical to achieve and the bandwidth requirements are not overly stringent. Most studies of this modulation/demodulation technique for the additive white Gaussian noise (AWGN) channel have focused on the error probability performance when the receiver is assumed to be perfectly time and frequency synchronized. That is, the receiver is assumed to have perfect knowledge of the instants of time at which the modulation can change state and also perfect knowledge of the received carrier frequency. In practical systems, such perfect knowledge is never available and thus the receiver must derive this information from the received signal imbedded in the AWGN. Since the estimates of the time epoch and received carrier frequency derived at the receiver are, in general, random variables (because of the presence of the AWGN), there will exist an error between these estimates and their true values. This lack of perfect time and frequency synchronization gives rise to a degradation in error probability performance relative to that corresponding to the ideal case where perfect knowledge of time and frequency is assumed known.

The purpose of this paper is to evaluate this performance degradation, first by treating the two sources of degradation separately, and then by considering their simultaneous effect. In particular, we shall present exact expressions for the symbol and bit error probability performances of noncoherent orthogonal *M*-FSK conditioned on the presence of time and frequency errors, These expressions involve integrals of Marcum-Q functions and, as such, their numerical evaluation is cumbersome. Thus, for the case of frequency only, we present an upper (Chernoff-type) bound on error probability performances that, because of its exponential behavior, is simpler to evaluate. Numerical results are obtained for cases of practical interest.

Before going into the details of the analysis, we wish to point out the existence of several papers that relate to the subject at hand [1-5]. The paper that, in principle, bears the closest resemblance to what we are trying to accomplish here is a paper by Nakamoto, Middlestead, and Wolfson [1]. The one primary difference is that the authors of [1] considered frequency-hopped M-FSK whereas here we are not allowing for any spread spectrum modulation. Despite this difference, however, many of the results in [1] could still be used if it were not for the following. In evaluating the bit error probability in terms of the expressions the authors of [1] derive for the symbol error probability, they assume that the signals remain orthogonal in the presence of timing and frequency error which facilitates the use of a well-known result [see [5], Eq. (5-54)] relating bit and symbol error probabilities of orthogonal M-FSK. Unfortunately, however, this assumption is not valid and hence the bit error probability results found in [1] are incorrect. In fact, to properly evaluate the bit error probability y in the presence of synchronization errors, one must specify an appropriate mapping, for example, a Gray code of the symbols to bits. In the perfectly synchronized case, the bit error probability performance is completely independent of the symbol-to-bit mapping since all errors are *equally likely* to occur. The significance of these statements will become apparent later on in the paper.

The organization of the paper is as follows, Section 2 exactly treats the effect of frequency error alone (perfect time synchronization is assumed) on orthogonal M-FSK noncoherent detection. Section 3 exactly treats the effect of timing error alone (perfect frequency synchronization is assumed) on the same detection scheme. Section 4 exactly treats the combined effect of timing and frequency errors. Finally, Section 5 presents an upper bound on the performance in the presence of frequency error,

2.0 Effect of Frequency Error on Orthogonal M-FSK Noncoherent Detection

Consider the transmission of orthogonal M-FSK over an AWGN channel where the signal set has a one-to-one correspondence with the set of M equiprobable messages $m_0, m_1, \ldots, m_{M-1}$. The optimum receiver (assuming perfect synchronization) is illustrated in Fig. 1. When the received frequency is not perfectly known, the observed signal, assuming that message m_i was sent, is given by

$$r(t) = \sqrt{2P}\cos\left(2\pi\left(f_c + f_i + \Delta f\right)t + \theta\right) + n(r),\ O \le t \le T \quad (1)$$

where P denotes the signal power in Watts, T denotes the symbol time in seconds, f_c is the carrier frequency in Hertz, $f_i = i/T$ is the transmitted frequency corresponding to message m_i, Δf is the error in the carrier frequency, and θ is the unknown carrier phase assumed to be uniformly distributed, Also, $n(t)$ denotes the AWGN with single-sided power spectral density N_0 Watts/Hertz, Alternatively, the received signal can be interpreted as a carrier at frequency f_c shifted by the appropriate signal frequency $f_i + \Delta f$, rather than f_i. This scenario is illustrated in Fig. 2 where the dashed lines denote start and end of integration times with the depicted frequencies. The inphase integrator output, $z_{c,k}$, matched to signal Sk(t) (corresponding to message m_k) becomes[1]

$$\begin{aligned} z_{c,k} &\triangleq \int_0^T r(t)\sqrt{2P}\cos\left(2\pi\left(f_c + f_k\right)t\right)dt \\ &= 2P\int_0^T \cos\left(2\pi\left(f_c + f_i + \Delta f\right)t\right)\cos\left(2\pi\left(f_c + f_k\right)t\right)dt + n_{c,k} \end{aligned} \quad (2)$$

where $n_{c,k}$ is a zero-mean Gaussian random variable with variance $\sigma^2 = N_0 / 2E_s$. Here, N_0 denotes the single-sided power spectral density of the AWGN in Watts/Hertz and $E_s \triangleq PT$ is the symbol energy in joules, Simplifying (2) reduces to

$$z_{c,k} = E_s \frac{\sin\left(2\pi\left(f_{i,k} + \Delta f\right)T\right)}{2\pi\left(f_{i,k} + \Delta f\right)T} + n_{c,k} \quad (3)$$

where $f_{i,k}$ denotes the difference between the frequencies representing messages m_i and m_k, that is,

$$f_{i,k} \triangleq f_i - f_k \quad (4)$$

Similarly, the quadrature integrator output, $z_{s,k}$, is given by

$$\begin{aligned} z_{s,k} &\triangleq \int_0^T r(t)\sqrt{2P}\sin\left(2\pi\left(f_c + f_k\right)t\right)dt \\ &= E_s \frac{\left[\cos\left(2\pi\left(f_{i,k} + \Delta f\right)T\right) - 1\right]}{2\pi\left(f_{i,k} + \Delta f\right)T} + n_{s,k} \end{aligned} \quad (5)$$

The envelope statistic $\xi_k = \sqrt{z_{c,k}^2 + z_{s,k}^2}$ will then be Rician distributed with parameter s_k^2 given by

$$\begin{aligned} s_k^2 &= \left(E_s\right)^2 \frac{\left\{\left[\sin\left(2\pi\left(f_{i,k} + \Delta f\right)T\right)\right]^2 + \left[\cos\left(2\pi\left(f_{i,k} + \Delta f\right)T\right) - 1\right]^2\right\}}{\left[2\pi\left(f_{i,k} + \Delta f\right)T\right]^2} \\ &= \left(E_s\right)^2 \frac{\sin^2\left(\pi\left(f_{i,k} + \Delta f\right)T\right)}{\left[\pi\left(f_{i,k} + \Delta f\right)T\right]^2} \end{aligned} \quad (6)$$

Normalizing $z_{c,k}$ and $z_{s,k}$ by $1 / \sigma = \sqrt{2 / N_0 E_s}$, the parameter s_k^2 is then normalized by

[1]Since we are dealing with noncoherent detection, we can, without any loss in generality, set $\theta = 0$.

$2/N_0E_s$ and since, as mentioned above, $f_i = i/T$ for orthogonal signals, then

$$s_k^2 = \left(\frac{2E_s}{N_0}\right)\frac{\sin^2(\pi(i-k+\rho))}{[\pi(i-k+\rho)]^2} \tag{7}$$

where $\rho \triangleq \Delta fT$ denotes the normalized frequency error, The probability y density function (pdf) of ξ_k is given by (note that $\sigma^2 = 1$ after the above normalization)

$$f_{\xi_k}(\xi) = \xi \exp\left\{-\left(\frac{\xi^2}{2} + \frac{E_s}{N_0} f_\rho^2(i,k)\right)\right\} I_0\left(\xi\sqrt{\frac{2E_s}{N_0} f_\rho^2(i,k)}\right), \quad 0 \le \xi \le \infty \tag{8}$$

where we let

$$f_\rho^2(i,k) = \frac{\sin^2(\pi(i-k+\rho))}{[\pi(i-k+\rho)]^2} \tag{9}$$

First, note that the detector matched to the incoming signal suffers from signa' attenuation equal to

$$f_\rho^2(i,i) = \frac{\sin^2 \pi\rho}{\pi^2\rho^2} \tag{10}$$

which, as expected, reduces to unity if $\rho = 0$. Simultaneously, loss of signal orthogonality occurs as a result of signal spill-over into the remaining M-1 detectors; hence, the nonzero means and the resulting Rician (as opposed to Rayleigh for $\rho = 0$) pdfs. Note that for zero frequency error ($\rho = 0$), then

$$f_{\rho=0}^2(i,k) = \frac{\sin^2(\pi(i-k))}{[\pi(i-k)]^2} = \begin{cases} 0, & i \neq k \\ 1, & i = k \end{cases} \tag{11}$$

and signal orthogonality is restored. Despite loss of orthogonality, the variables $\xi_0, \xi_1, \ldots, \xi_{M-1}$ remain independent since the Gaussian random variables resulting from the noise integration are still independent as the local signals remain orthogonal. In this case, the probability of correct symbol detection, assuming that message m_i is transmitted, is given by

$$\begin{aligned} P_s(C|m_i) &= \Pr\left\{\xi_i = \max_m \xi_m,\ m = 0,1,\ldots,M-1\right\} \\ &= \int_0^\infty f_{\xi_i}(x_i)\left(\prod_{\substack{m=0 \\ m\neq i}}^{M-1} \int_0^{x_i} f_{\xi_m}(x_m)\,dx_m\right) dx_i \end{aligned} \tag{12}$$

In terms of the Marcum Q-function [6] defined by

$$Q(a,b) \triangleq \int_b^{\infty} x \exp\left(-\frac{x^2+a^2}{2}\right) I_0(ax)dx \tag{13}$$

we have

$$\int_0^{x_i} f_{\xi_m}(x_m)dx_m = 1 - Q\left(\frac{2E_s}{N_0} f_\rho^2(i,m), x_i\right) \tag{14}$$

Hence, the conditional probability of symbol error, assuming that message m_i is transmitted, is given by

$$P_s(E|m_i) = 1 - \int_0^{\infty} x_i \exp\left\{-\left(\frac{x_i^2}{2} + \frac{E_s}{N_0}\left(\frac{\sin^2 \pi\rho}{\pi^2\rho^2}\right)\right)\right\} I_0\left(x_i\sqrt{\frac{2E_s}{N_0}\left(\frac{\sin^2 \pi\rho}{\pi^2\rho^2}\right)}\right)$$
$$\times \prod_{\substack{m=0 \\ m\neq i}}^{M-1}\left[1 - Q\left(\frac{2E_s}{N_0} f_\rho^2(i,m), x_i\right)\right] dx_i \tag{15}$$

and the unconditional probability of symbol error becomes

$$P_s(E) = \frac{1}{M}\sum_{i=0}^{M-1} P_s(E|m_i) \tag{16}$$

As previously mentioned, the average bit error probability cannot be obtained directly from the average symbol error probability as is customary in perfectly synchronized M-FSK systems, the reason being that, for a given transmitted message, the symbol errors are not equally likely. To compute the average bit error probability we must first compute the probability of a *particular* symbol error for a given transmitted message. Analogous to (12), the probability of choosing m_k when message m_i is transmitted is given by

$$P_{sk}(E|m_i) = \Pr\left\{\xi_k = \max_m \xi_m, \quad m = 0,1,\ldots,M-1\right\}$$
$$= \int_0^{\infty} f_{\xi_k}(x_k)\left(\prod_{\substack{m=0 \\ m\neq k}}^{M-1}\int_0^{x_k} f_{\xi_m}(x_m)dx_m\right)dx_k, \; k \neq i \tag{17}$$

If $w(k,i)$ denotes the Hamming weight of the difference between the code words (bit mappings) assigned to messages (symbols) m_i and m_k, that is, the number of bits in which the two differ, then the average bit error probability is

$$P_b(E) = \frac{1}{M\log_2 M}\sum_{i=0}^{M-1}\sum_{\substack{k=0\\k\neq i}}^{M-1} w(k,i)P_{sk}\left(E|m_i\right) \tag{18}$$

We now discuss the mapping from which the set of Hamming weights $w(k,i)$, $k,i = 0,1,\ldots,M-1$, $k \neq i$ is computed.

It is clear that if a symbol error occurs, it is more likely to occur in an adjacent frequency than in any other. Thus, a Gray code mapping is appropriate to this type of modulation. Figure 3 depicts the average bit error probability versus E_b/N_0 in dB with p as a parameter for binary, 4-ary and 8-ary FSK and a conventional Gray code assignment.

3.0 Effect of Timing Error on Orthogonal M-FSK Detection

When the receiver carrier frequency is precisely known but the symbol epoch is not, the receiver implements its integrate-and-dump (I&D) circuits using its own estimate of the symbol epoch which is offset from the true epoch by Δt sec. This phenomenon is depicted in Fig. 4 where the integration overlaps two successive symbol intervals. This lack of time synchronization results in signal attenuation in the detector matched to the incoming frequency and moreover, loss of orthogonality due to signal spillover into the remaining detectors, In the presence of timing error, the received signal can be modeled as

$$r(t) = \begin{cases}\sqrt{2P}\cos\left(2\pi\left(f_c + f_i\right)t + \theta_1\right) + n(t), & 0 \leq t \leq T\\ \sqrt{2P}\cos\left(2\pi\left(f_c + f_j\right)t + \theta_2\right) + n(t), & T \leq t \leq 2T\end{cases} \tag{19}$$

where we have assumed that signal $s_i(t)$ is transmitted folowed by signal s,(t) and have allowed for the possibility of a carrier phase discontinuity from symbol to symbol (so-called *discontinuous phase M-FSK modulation)*. Since, for noncoherent detection, the *absolute* carrier phase is inconsequential, we can, without loss in generality, set θ_1 = O and $\theta_2 = \theta$ for $i \neq j$ or $\theta_2 = 0$ for $i = j$. For so-called *continuous phase M-FSK* (CPFSK), we can, in addition, set θ = O. Since the local epoch estimate is not perfect, the receiver I&Ds operate in the interval $(\Delta t, \Delta t + T)$ to obtain at the kth detector

$$\begin{aligned} z_{c,k} &\triangleq \int_{\Delta t}^{\Delta t+T} r(t)\sqrt{2P}\cos\left(2\pi\left(f_c + f_k\right)t\right)dt \\ &= 2P\left\{\int_{\Delta t}^{T}\cos\left(2\pi\left(f_c + f_i\right)t\right)\cos\left(2\pi\left(f_c + f_k\right)t\right)dt\right. \\ &\quad \left. + \int_{T}^{\Delta t+T}\cos\left(2\pi\left(f_c + f_j\right)t + \theta\right)\cos\left(2\pi\left(f_c + f_k\right)t\right)dt\right\} + n_{c,k} \end{aligned} \tag{20}$$

and

$$
\begin{aligned}
z_{s,k} &\triangleq \int_{\Delta t}^{\Delta t+T} r(t)\sqrt{2P}\sin\left(2\pi\left(f_c+f_k\right)t\right)dt \\
&= 2P\left\{\int_{\Delta t}^{T}\cos\left(2\pi\left(f_c+f_i\right)t\right)\sin\left(2\pi\left(f_c+f_k\right)t\right)dt \right.\\
&\left. +\int_{T}^{\Delta t+T}\cos\left(2\pi\left(f_c+f_j\right)t+\theta\right)\sin\left(2\pi\left(f_c+f_k\right)t\right)dt\right\}+n_{s,k}
\end{aligned}
\tag{21}
$$

Normalizing $z_{c,k}$ and $z_{s,k}$ by $1/\sigma=\sqrt{2/N_0E_s}$, the pdf of ξ_k can be expressed as

$$
f_{\xi_k}(\xi)=\xi\exp\left\{-\left(\frac{\xi^2}{2}+\frac{E_s}{N_0}f_\lambda^2(i,j,k)\right)\right\}I_0\left(\xi\sqrt{\frac{2E_s}{N_0}f_\lambda^2(i,j,k)}\right),\quad 0\le\xi\le\infty \tag{22}
$$

where

$$
\begin{aligned}
f_\lambda^2(i,j,k) &= \frac{\sin^2\left(\pi(i-k)(1-\lambda)\right)}{\pi^2(i-k)^2}+\frac{\sin^2\left(\pi(j-k)\lambda\right)}{\pi^2(j-k)^2} \\
&\quad \frac{\sin^2\left(\pi\left[(i-j)-(j-k)\lambda\right]-\frac{\theta}{2}\right)}{\pi^2(i-k)(j-k)}-\frac{\sin^2\left(\pi\left[(i-k)\lambda-(j-k)\right]-\frac{\theta}{2}\right)}{\pi^2(i-k)(j-k)} \\
&\quad +\frac{\sin^2\left(\pi\left[(k-j)-(j-i)\lambda\right]-\frac{\theta}{2}\right)}{\pi^2(i-k)(j-k)}+\frac{\sin^2\left(\pi(i-j)-\frac{\theta}{2}\right)}{\pi^2(i-k)(j-k)},\quad i\ne k,\ j\ne k
\end{aligned}
\tag{23}
$$

with $\lambda\triangleq\Delta t/T$ denoting the normalized timing error. Some special cases of (23) are

$$
f_\lambda^2(i,i,k)=\frac{\sin^2\left(\pi(i-k)\right)}{\pi^2(i-k)^2}\begin{cases}1, & i=k\\ 0, & i\ne k\end{cases} \tag{24a}
$$

$$
\begin{aligned}
f_\lambda^2(i,j,i) &= \frac{1}{\pi^2(i-j)^2}\left\{\pi^2(i-j)^2(1-\lambda)^2+\sin^2\left(\pi(i-j)\lambda\right)\right. \\
&\quad +\pi(i-j)(1-\lambda)\sin\left(2\pi(i-j)(1+\lambda)-\theta\right) \\
&\quad \left.-\pi(i-j)(1-\lambda)\sin\left(2\pi(i-j)\right)-\theta\right\},\ i\ne j
\end{aligned}
\tag{24b}
$$

$$
\begin{aligned}
f_\lambda^2(i,k,k) &= \frac{1}{\pi^2(i-k)^2}\left\{\sin^2\left(\pi(i-k)(1-\lambda)\right)+\pi^2(i-k)^2\lambda^2\right. \\
&\quad -\pi\lambda(i-k)\cos\theta\sin\left(2\pi(i-k)\lambda\right) \\
&\quad \left.-\pi\lambda(i-k)\sin\theta\left[1-\cos\left(2\pi(i-k)\lambda\right)\right]\right\},\ i\ne k
\end{aligned}
\tag{24c}
$$

$$
f_\lambda^2(i,i,i)=1,\ \forall i \tag{24d}
$$

From (15), the conditional probability of symbol error, assuming that message m_i was sent followed by message m_j, is then given by

$$P_s\left(E \middle| m_i, m_j, \theta\right) = 1 - \int_0^{\infty} x_i \exp\left\{-\left(\frac{x_i^2}{2} + \frac{E_s}{N_0} f_\lambda^2(i,j,i)\right)\right\} I_0\left(x_i \sqrt{\frac{2E_s}{N_0} f_\lambda^2(i,j,i)}\right)$$
$$\times \prod_{\substack{m=0 \\ m \neq i}}^{M-1} \left[1 - Q\left(\frac{2E_s}{N_0} f_\lambda^2(i,j,m), x_i\right)\right] dx_i \tag{25}$$

The unconditional (with respect to the data) probability of symbol error is then

$$P_s(E|\theta) = \frac{1}{M} \sum_{i=0}^{M-1} \left[\frac{1}{M} \sum_{j=0}^{M-1} P_s\left(E \middle| m_i, m_j, \theta\right)\right] \tag{26}$$

As was the case for frequency error in Section 2.0, the presence of timing error produces a lack of orthogonality which results in the symbols errors not being equally likely. Hence to compute the average bit error probability we must once again compute the probability of a *particular* symbol error for a given transmitted message. Analogous to (17), the probability of choosing m_k when message m_i was sent followed by message m_j, is given by

$$P_{sk}\left(E \middle| m_i, m_j, \theta\right) = \int_0^{\infty} x_k \exp\left\{-\left(\frac{x_i^2}{2} + \frac{E_s}{N_0} f_\lambda^2(i,j,k)\right)\right\} I_0\left(x_i \sqrt{\frac{2E_s}{N_0} f_\lambda^2(i,j,k)}\right)$$
$$\times \prod_{\substack{m=0 \\ m \neq k}}^{M-1} \left[1 - Q\left(\frac{2E_s}{N_0} f_\lambda^2(i,j,m), x_i\right)\right] dx_i \tag{27}$$

with $f_{\xi_\lambda}(\xi)$ as in (22). Finally, the average (over the data) bit error probability y is, analogous to (18),

$$P_b(E|\theta) = \frac{1}{M^2 \log_2 M} \sum_{i=0}^{M-1} \sum_{j=0}^{M-1} \sum_{\substack{k=0 \\ k \neq i}}^{M-1} w(k,i) P_{sk}\left(E \middle| m_i, m_j, \theta\right) \tag{28}$$

Here again the evaluation of (28) will depend on the mapping of the symbols to bits. For a conventional Gray code mapping, Figure 5 depicts average bit error probability versus E_b/N_0 in dB for binary, 4-ary and 8-ary FSK with λ as a parameter and the case of continuous phase M-FSK. The numerical results in this figure are obtained by setting θ = O in (28). Digital computer simulations were used to confirm some of the cases, in particular, the results corresponding to M =4 in Fgure 5b. For purposes of comparison, the corresponding results for the discontinuous phase case with M = 4 are illustrated in Figure 6 and are obtained by averaging (28) over a uniform distribution for θ. We observe that discontinuous phase M-FSK is much more sensitive to timing offset than continuous phase M-FSK is. When the timing is perfect (λ = O), the two performances are, of course, identical. This can be seen by noting that (23) becomes independent of θ when A = O.

4.0 Effect of Timing and Frequency Errors on Orthogonal M-FSK Noncoherent Detection

When both the incoming carrier frequency and symbol epoch are unknown (see Fig. 7), then the received signal is still given by (19) but with f_c replaced by $f_c + \Delta f$. The inphase and quadrature outputs now become

$$z_{c,k} = 2P\left\{\int_{\Delta t}^{T} \cos\left(2\pi(f_c + \Delta f + f_i)t\right)\cos\left(2\pi(f_c + f_k)t\right)dt + \int_{T}^{\Delta t+T} \cos\left(2\pi(f_c + \Delta f + f_j)t + \theta\right)\cos\left(2\pi(f_c + f_k)t\right)dt\right\} + n_{c,k} \tag{29}$$

and

$$z_{s,k} = 2P\left\{\int_{\Delta t}^{T} \cos\left(2\pi(f_c + \Delta f + f_i)t\right)\sin\left(2\pi(f_c + f_k)t\right)dt + \int_{T}^{\Delta t+T} \cos\left(2\pi(f_c + \Delta f + f_j)t + \theta\right)\sin\left(2\pi(f_c + f_k)t\right)d\right\} + n_{s,k} \tag{30}$$

Normalizing as before and following a similar procedure, we obtain the pdf of ξ_k given by

$$f_{\xi_k}(\xi) = \xi \exp\left\{-\left(\frac{\xi^2}{2} + \frac{E_s}{N_0} f_{\rho,\lambda}^2(i,j,k)\right)\right\} I_0\left(\xi\sqrt{\frac{2E_s}{N_0} f_{\rho,\lambda}^2(i,j,k)}\right), \quad 0 \le \xi \le \infty \tag{31}$$

where

$$\begin{aligned} f_{\rho,\lambda}^2(i,j,k) = {} & \frac{\sin^2\left(\pi(i-k+\rho)(1-\lambda)\right)}{\pi^2(i-k+\rho)^2} + \frac{\sin^2\left(\pi(j-k+\rho)\lambda\right)}{\pi^2(j-k+\rho)^2} \\ & - \frac{\sin^2\left(\pi\left[(i-j)-(j-k+\rho)\lambda\right]-\frac{\theta}{2}\right)}{\pi^2(i-k+\rho)(j-k+\rho)} - \frac{\sin^2\left(\pi\left[(i-k+\rho)\lambda-(j-k+\rho)\right]-\frac{\theta}{2}\right)}{\pi^2(i-k+\rho)(j-k+\rho)} \\ & + \frac{\sin^2\left(\pi\left[(k-j-\rho)-(j-i)\lambda\right]-\frac{\theta}{2}\right)}{\pi^2(i-k+\rho)(j-k+\rho)} + \frac{\sin^2\left(\pi(i-j)-\frac{\theta}{2}\right)}{\pi^2(i-k+\rho)(j-k+\rho)}, \quad i \ne k, j \ne k \end{aligned} \tag{32}$$

If ρ = O, then $f_{\rho,\lambda}^2(i,j,k)$ reduces to $f_\lambda^2(i,j,k)$ of (23) as expected. Similarly, if λ = O, then

$f_{\rho,\lambda}^2(i,j,k)$ reduces to $f_\rho^2(i,k)$ of (9). The probability of bit and symbol error are still given by (26) together with (25) and (28) together with (27), respectively, with $f_\lambda^2(i,j,k)$ replaced by $f_{\rho,\lambda}^2(i,j,k)$ of (32). For a conventional Gray code mapping, Fig. 8 depicts average bit error probability versus E_b/NO in dB for binary, 4-ary and 8-ary FSK with ρ and A as parameters and the case of continuous phase M-FSK. The numerical results in this figure are obtained by setting θ = O in (32). Digital computer simulations were again used to confirm some of the cases, in particular, those illustrated in Figs. 8c, and 8d. Note that when the timing and frequency errors occur simultaneously, the losses are not additive. In particular, the interaction of the two types of error results in a degradation larger than the sum of the degradations due to each error acting alone.

5.0 A Bound on the Performance of Orthogonal M-FSK Detection in the Presence of Frequency Error

A number of years back, Jim K. Omura developed a Chernoff-type bound on the error probability performance of certain types of M-ary communicated systems. This unpublished result [7] has particular application in noncoherent M-FSK communications.

In this section, we apply a slightly generalized version of the bound to predicting the error probability performance of noncoherent M-FSK with frequency error. Upper union bounds for this performance have been previously obtained in [3] in terms of the exact result for the performance of binary FSK with frequency error. The latter is expressed in terms of the Marcum Q-function, which in general, is cumbersome to compute. Here, we shall derive a simpler-to-compute bound on this performance that will enable system comparisons to be made. The result is obtained in a form that is similar to the exact error probability performance of noncoherent M-FSK with no frequency error which is exponential in behavior,

Since as mentioned above, Omura's bound was never published but rather privately communicated to the authors, Appendix A presents the derivation of the bound in its generalized form. Assuming that signal n (message m_n) is transmitted, then the detector matched to f_n produces ξ_n with pdf as given by (8) with i = k = n while the remaining M -1 detectors produce independent ξ_i's with pdf as given by (8) with i = n and k = i. As required by the results in Appendix A, we need to evaluate the characteristic functions of the these two pdfs. In particular, letting $\beta_i = \xi_i^2$, then

$$g_1(x) \triangleq \int_{-\infty}^{\infty} e^{x\xi^2} f_{\xi_n}(\xi)d\xi = \frac{1}{1-2x}\exp\left\{\frac{2E_s}{N_0}\frac{f_\rho^2(n,n)x}{1-2x}\right\} \tag{33}$$

and

$$g_0\left(x; f_\rho^2(n,i)\right) \triangleq \int_{-\infty}^{\infty} e^{x\xi^2} f_{\xi_i}(\xi) d\xi = \frac{1}{1-2x} \exp\left\{\frac{2E_s}{N_0} \frac{f_\rho^2(n,i)x}{1-2x}\right\} \qquad (34)$$

Letting x_0 = 2x, then applying (A.8) gives after some manipulations

$$P_s\left(E|m_n\right) \le \min_{x_0 \ge 0} \sum_{k=1}^{M-1} (-1)^{k+1} \left(\frac{1}{1-x_0}\right)^k \left(\frac{1}{1+kx_0}\right) \exp\left(-\frac{E_s}{N_0} \frac{kx_0 f_\rho^2(n,n)}{1+kx_0}\right)$$
$$\times \underbrace{\sum_{i_1=0}^{M-1} \sum_{i_2=0}^{M-1} \cdots \sum_{i_k=0}^{M-1}}_{\substack{i_1 < i_2 < \cdots i_k \\ i_1, i_2, \cdots, i_k \ne n}} \exp\left\{\frac{E_s}{N_0} \left(\frac{x_0}{1-x_0}\right) \sum_{j=1}^{k} f_\rho^2\left(n, i_j\right)\right\} \qquad (35)$$

Finally, the desired upper bound on $P_s(E)$ is given by

$$P_s(E) \le \frac{1}{M} \sum_{n=0}^{M-1} P_s\left(E|m_n\right) \qquad (36)$$

Figure 9 illustrates the upper bound on $P_s(E)$ as given by *(36)* versus E_b / N_0 in dB for M = 4 and various values of normalized frequency error, ρ, assuming minimum frequency spacing for orthogonality. It is to be emphasized that the results in Fig. 8 are upper bounds and thus should not be used to predict the true error probability performance. Rather, their value is for making system comparisons and trade-offs since the relative tightness of the bound to the exact result should be about the same in all cases considered. As is true for most Chernoff-type bounds, they are asymptotically loose by about 1 to 1.5 dB.

6.0 Conclusions

The error probability performance of noncoherent M-FSK is quite sensitive to the presence of timing and frequency offsets (errors) in the system. For a given number of frequencies, M, and fractional offset, the performance is much more sensitive to timing error than it is to frequency error. By studying these errors individually and then combined, we are able to note that the losses due to these errors are not additive. In particular, the interaction of the two types of error results in a degradation larger than the sum of the degradations due to each error acting alone, Furthermore, for the case of timing error (either alone or in combination with frequency error), the performance is much less robust for discontinuous phase M-FSK than it is for continuous phase M-FSK.

References

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Appendix A
Generalized M-ary Symbol Error Probability Bound

Consider an M-ary communication system whose decisions are made based on a relation among M outputs $x_0, x_1, \cdots, x_{M-1}$. Let these outputs be represented by independent random variables with pdfs as follows:

$$
\begin{aligned}
x_n &\longrightarrow f_1(x_n) \\
x_i &\longrightarrow f_0(x_i;\ \zeta_{in}) \text{ for } i = 0, 1, \cdots, n-1, n+1, \cdots, M-1
\end{aligned} \tag{A.1}
$$

That is to say, for some particular n, the random variable x_i has a fixed pdf whereas the remaining $M-1$ r.v.'s x_i, i # n, all have the identical form pdf (perhaps different than that for x_n)[1] which, however, depend on a parameter ζ_{in} that varies with both i and n. Assuming that signal s.(t) is transmitted, then a correct decision is made at the receiver when x_n > xi for all i # n. Then, the conditional probability of a correct decision is

$$
\begin{aligned}
P_s(C/\ m_n) &= \text{Prob}\{\text{Correct decision}/\ m_n\} \\
&= \text{Prob}\{x_n > x_i \text{ for all i + } n/\ m_n\} \\
&= \int_{-\infty}^{\infty} \text{Prob}\{x_n > x_i \text{ for all } i \neq n/m_n,\ x_n = \alpha\} f_1(\alpha)d\alpha \\
&= \int_{-\infty}^{\infty} \text{Prob}\ \{x_i < \alpha \text{ for all i \# n}/\ m_n\} f_1(\alpha)d\alpha \\
&= \int_{-\infty}^{\infty} \prod_{i=0,i\neq n}^{M-1} \text{Prob}\{x_i < \alpha/\ m_n\} f_1(\alpha)d\alpha \\
&= \int_{-\infty}^{\infty} \prod_{i=0,i\neq n}^{M-1} [1 - \text{Prob}\{x_i \geq \alpha/\ m_n\}] f_1(\alpha)d\alpha
\end{aligned} \tag{A.2}
$$

If $\text{Prob}\{x_i \geq \alpha/\ m_n\}$ is hard to evaluate, then use the Chernoff bound

$$
\begin{aligned}
\text{Prob}\{x_i \geq \alpha/\ m_n\} &\leq E\{e^{\lambda(x_i-\alpha)}/\ m_n\} \\
&= e^{-\lambda\alpha}E\{e^{\lambda x_i}/\ m_n\} \text{ for any } \lambda \geq 0
\end{aligned} \tag{A.3}
$$

[1] **If the two pdfs** have identical form, then we shall ignore the "0" and "1" subscripts on them and simply write $f(x)$ or $f(x;\zeta)$, as appropriate, **An example of where the two pdfs are, in principle, different in form would correspond to the case of ideal (zero frequency error) noncoherent detection of M-FSK, in which case $f_1(x)$ would be Rician and $f_0(x)$ would be Rayleigh. Also in this ideal situation, $f_0(x)$ would not be dependent on a parameter ζ which varies with the random variable being characterized.**

Define

$$g_1(\lambda) \triangleq E\{e^{\lambda x_n}/m_n\} = \int_{-\infty}^{\infty} e^{\lambda\beta} f_1(\beta)d\beta$$

$$g_0(\lambda;\zeta_{in}) \triangleq E\{e^{\lambda x_i}/m_n\} = \int_{-\infty}^{\infty} e^{\lambda\beta} f_0(\beta;\zeta_{in})d\beta \tag{A.4}$$

Then

$$\text{Prob}\{x_i \geq \alpha/m_n\} \leq e^{-\lambda\alpha} g_0(\lambda;\zeta_{in}) \tag{A.5}$$

and

$$P_s(C/m_n) \geq \int_{-\infty}^{\infty} \prod_{\substack{i=0\\ i\neq n}}^{M-1} \left[1 - e^{-\lambda\alpha} g_0(\lambda;\zeta_{in})\right] f_1(\alpha)d\alpha \tag{A.6}$$

Finally

$$\begin{aligned}
P_s(E/m_n) &= 1 - P_s(C/m_n)\\
&\leq \int_{-\infty}^{\infty}\left\{1 - \prod_{\substack{i=0\\ i\neq n}}^{M-1}\left[1 - e^{-\lambda\alpha} g_0(\lambda;\zeta_{in})\right]\right\} f_1(\alpha)d\alpha\\
&= \int_{-\infty}^{\infty}\left\{1 - \left[1 + \sum_{k=1}^{M-1}(-1)^k e^{-\lambda k\alpha}\right.\right.\\
&\quad \times \left.\left.\underbrace{\sum_{i_1=0}^{M-1}\sum_{i_2=0}^{M-1}\cdots\sum_{i_k=0}^{M-1}}_{\substack{i_1,i_2,\cdots,i_k\neq n \text{ and}\\ i_1<i_2<\cdots<i_k}}\prod_{j=1}^{k} g_0(\lambda;\zeta_{i_j n})\right]\right\} f_1(\alpha)d\alpha
\end{aligned} \tag{A.7}$$

or using (A.4), simplifying and minimizing over the Chernoff parameter, we get

$$\begin{aligned}
P_s(E/m_n) &\leq \min_{\lambda_n \geq 0} \sum_{k=1}^{M-1}(-1)^{k+1} g_1(-\lambda_n k)\\
&\quad \times \underbrace{\sum_{i_1=0}^{M-1}\sum_{i_2=0}^{M-1}\cdots\sum_{i_k=0}^{M-1}}_{\substack{i_1,i_2,\cdots,i_k\neq n \text{ and}\\ i_1<i_2<\cdots<i_k}}\prod_{j=1}^{k} g_0(\lambda_n;\zeta_{i_j n})
\end{aligned} \tag{A.8}$$

Assuming equiprobable signals, then the average probability of symbol error is given by

$$P_s(E) = \frac{1}{M}\sum_{n=0}^{M-1} P_s(E/m_n) \tag{A.9}$$

Note that if the parameter ζ_{in} is independent of i, that is, all x_i, $i \neq n$, have identical pdfs $f_0(x)$, then

$$\underbrace{\sum_{i_1=0}^{M-1}\sum_{i_2=0}^{M-1}\cdots\sum_{i_k=0}^{M-1}}_{\substack{i_1,i_2,\cdots,i_k\neq n \text{ and}\\ i_1<i_2<\cdots<i_k}}\prod_{j=1}^{k} g_0(\lambda;\zeta_{i_j n}) = \binom{M-1}{k} g_0^k(\lambda) \tag{A.10}$$

where

$$g_0(\lambda) = \int_{-\infty}^{\infty} e^{\beta\lambda} f_0(\beta) d\beta \tag{A11}$$

Thus, (A.9) together with (A.8) simplify to

$$P_s(E) \leq \min_{\lambda} \sum_{k=1}^{M-1} (-1)^{k+1} \binom{M-1}{k} g_1(-\lambda k) g_0^k(\lambda) \tag{A.12}$$

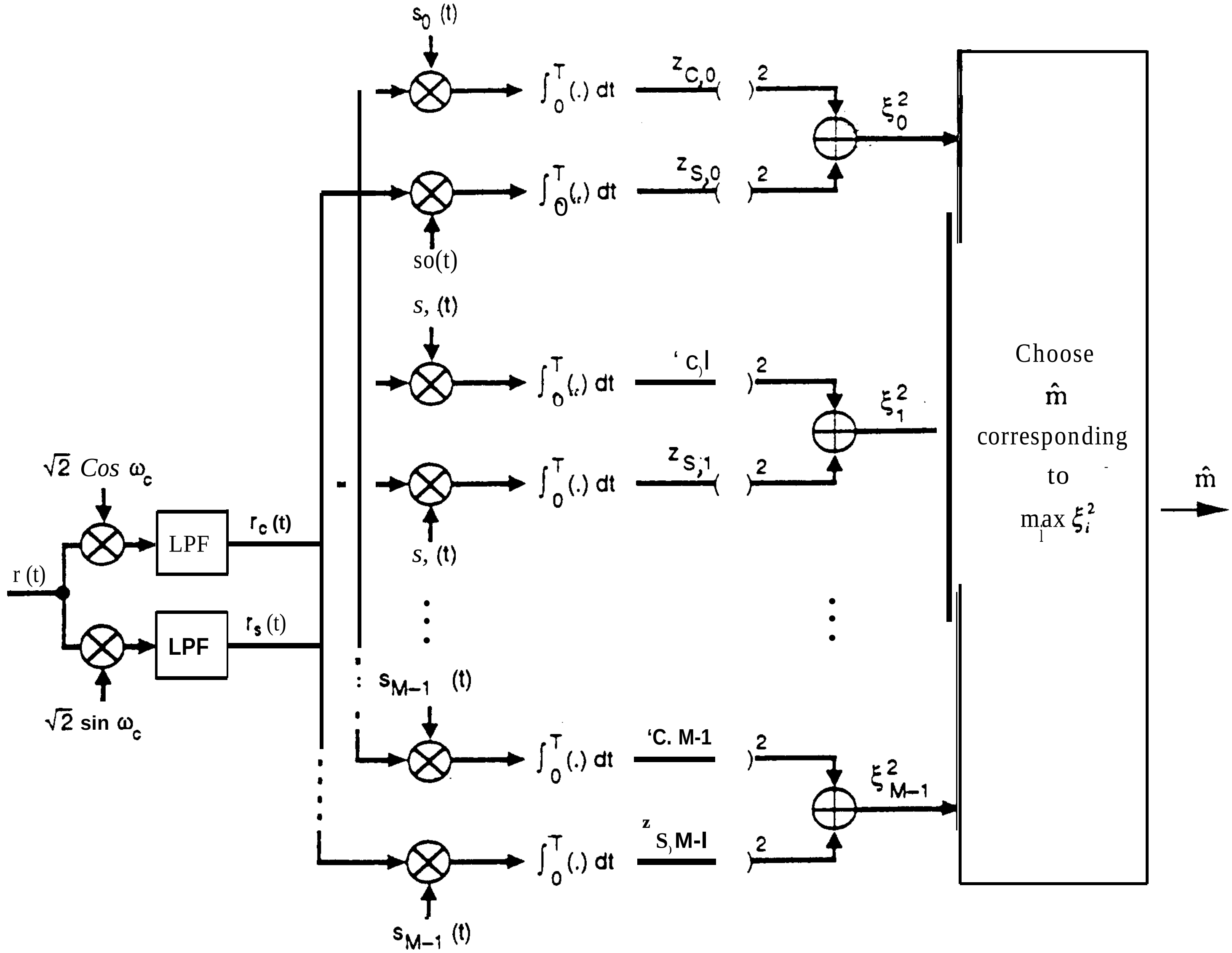


Figure 1. M-ary Noncoherent Receiver for Equal Energy Signals

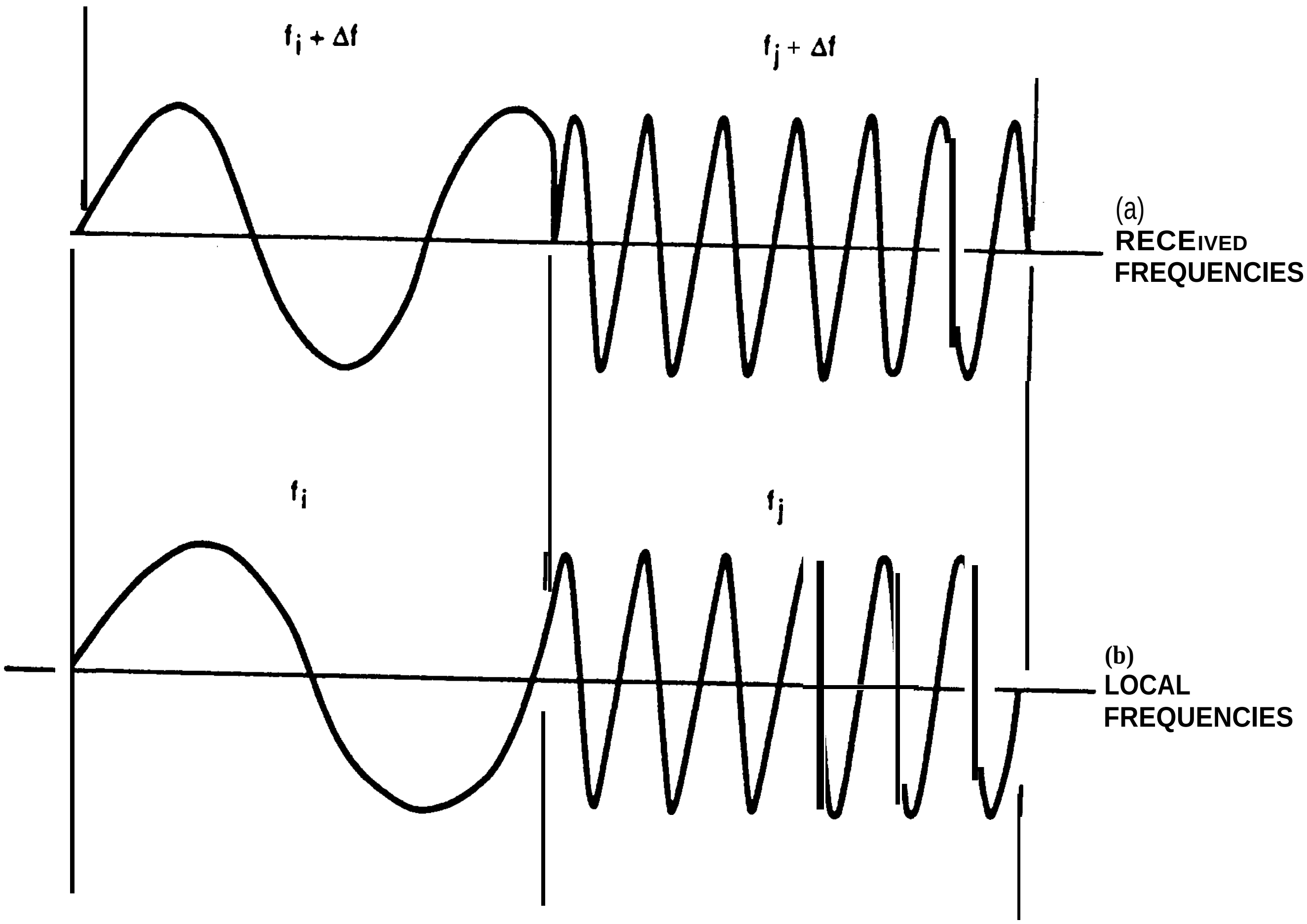
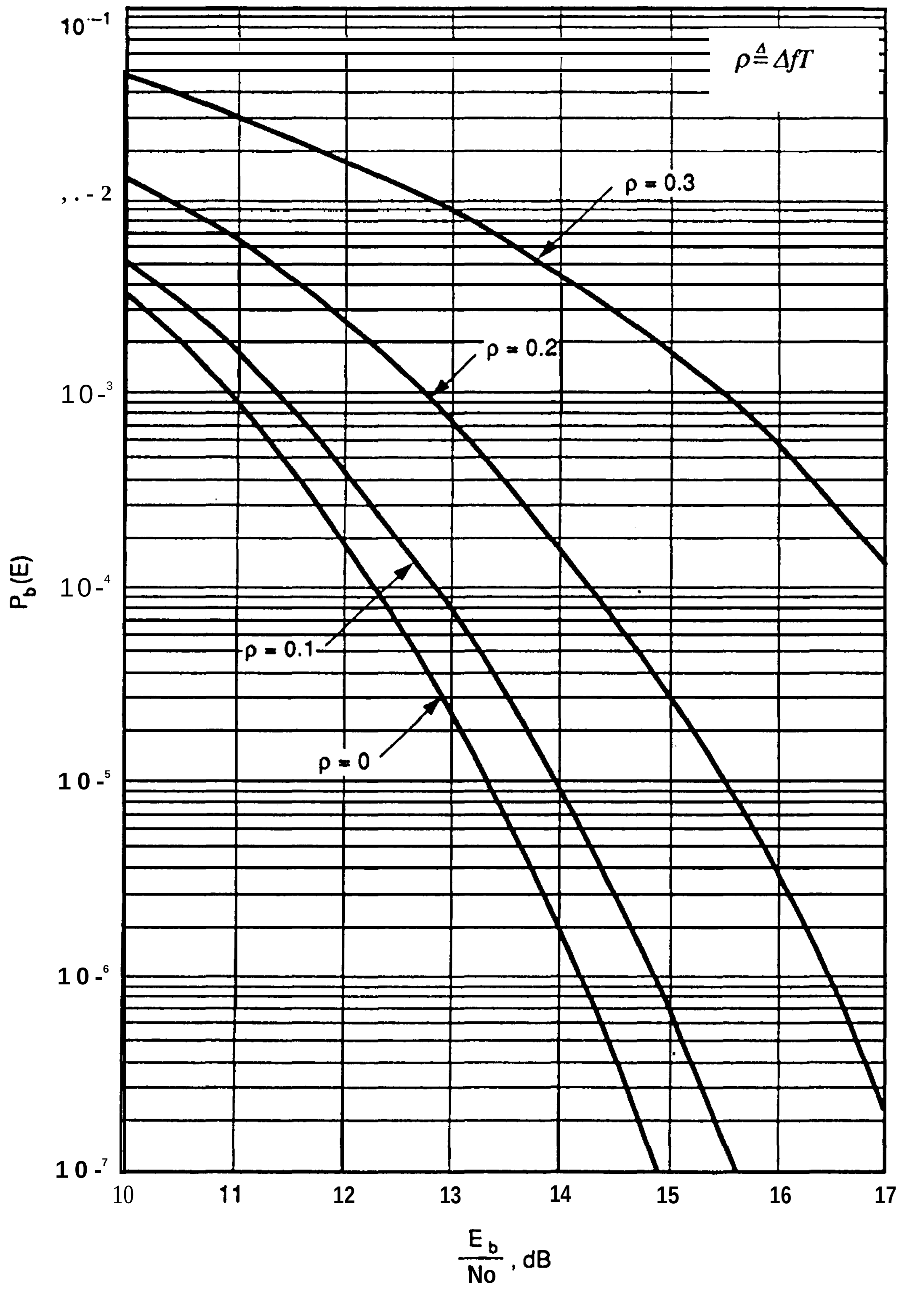
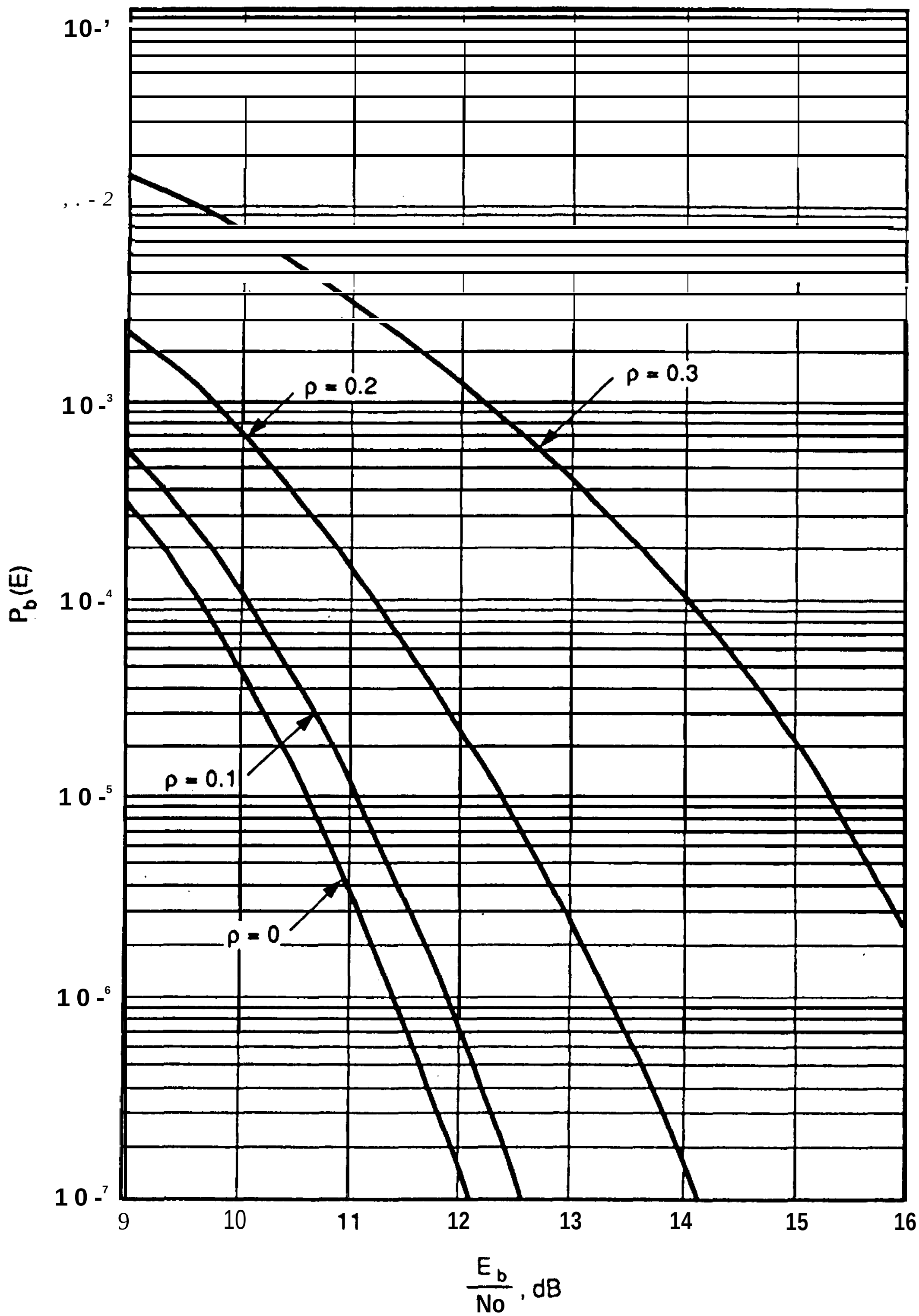


Figure z. Effect of Frequency Error



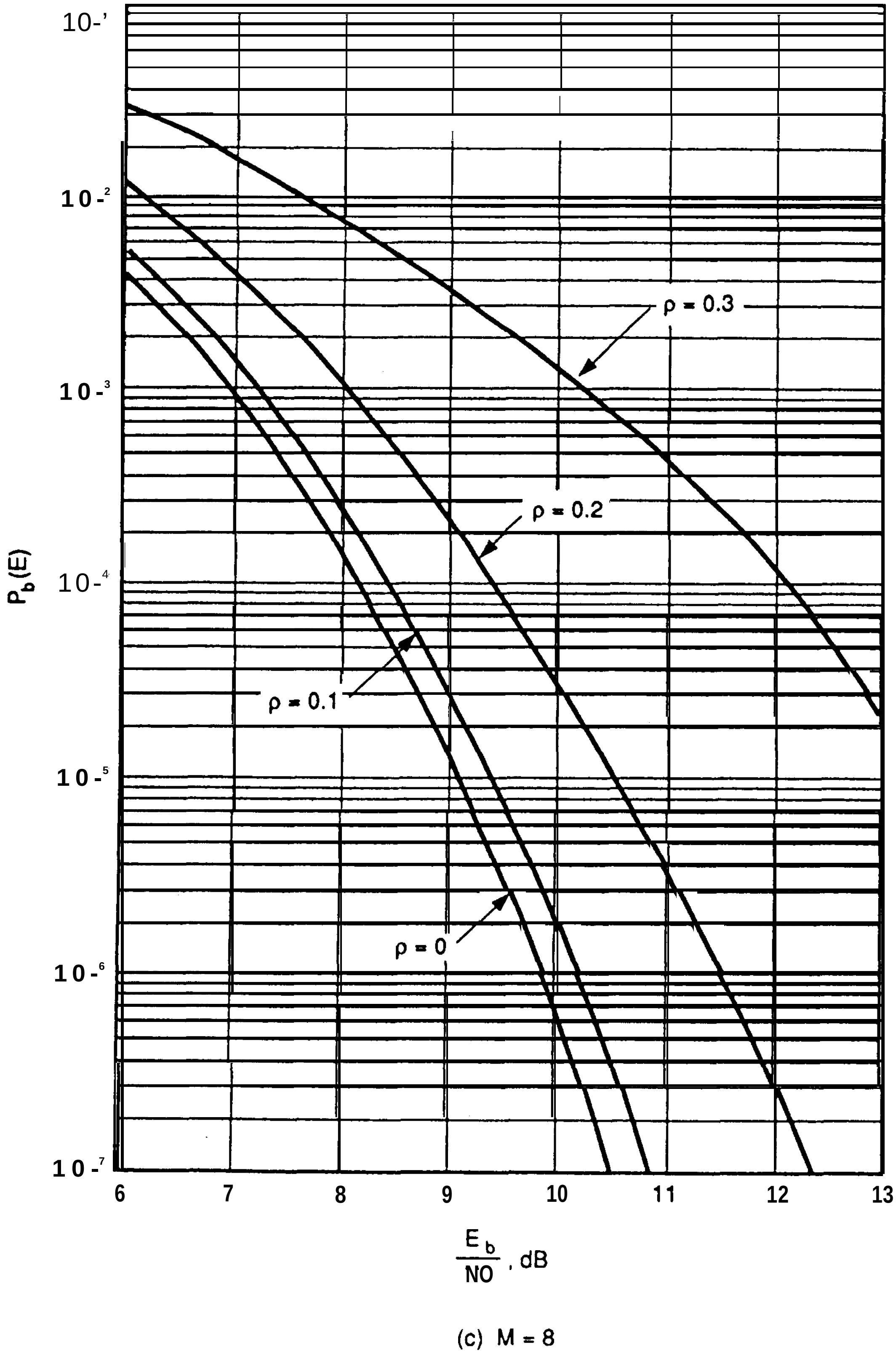
(a) M = 2

Figure 3. Bit Error Probability for *M*-FSK With Frequency Error



10-'
, . - 2
10-3
10-4
10-5
10-6
10-7
Pb(E)
ρ = 0.2
ρ = 0.3
ρ = 0.1
ρ = 0
9
10
11
12
13
14
15
16
Eb/No, dB

(b) M = 4



10-'
10-2
10-3
10-4
10-5
10-6
10-7
$P_b(E)$
ρ = 0.3
ρ = 0.2
ρ = 0.1
ρ = 0
6
7
8
9
10
11
12
13
$\frac{E_b}{NO}$, dB

(c) M = 8

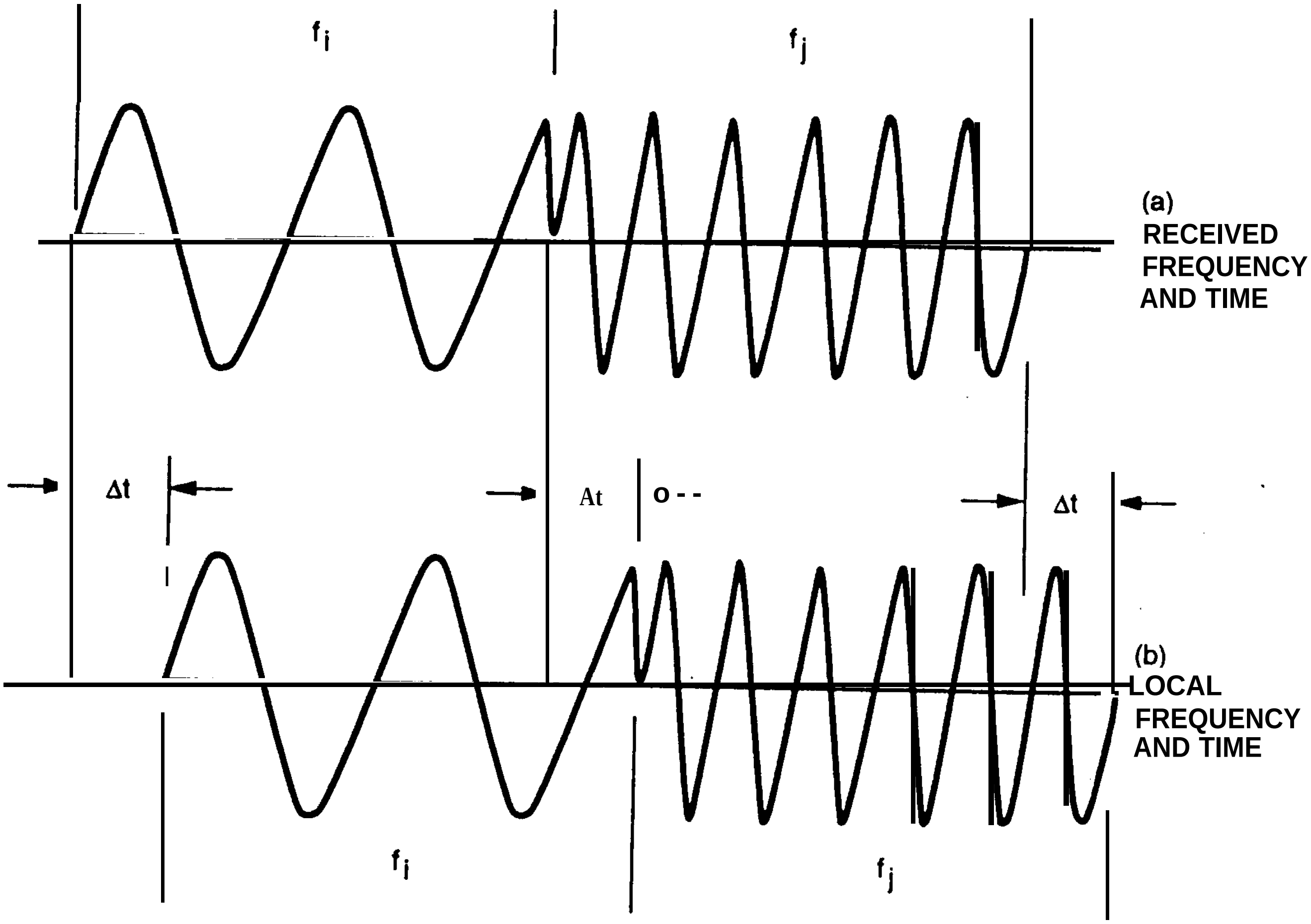
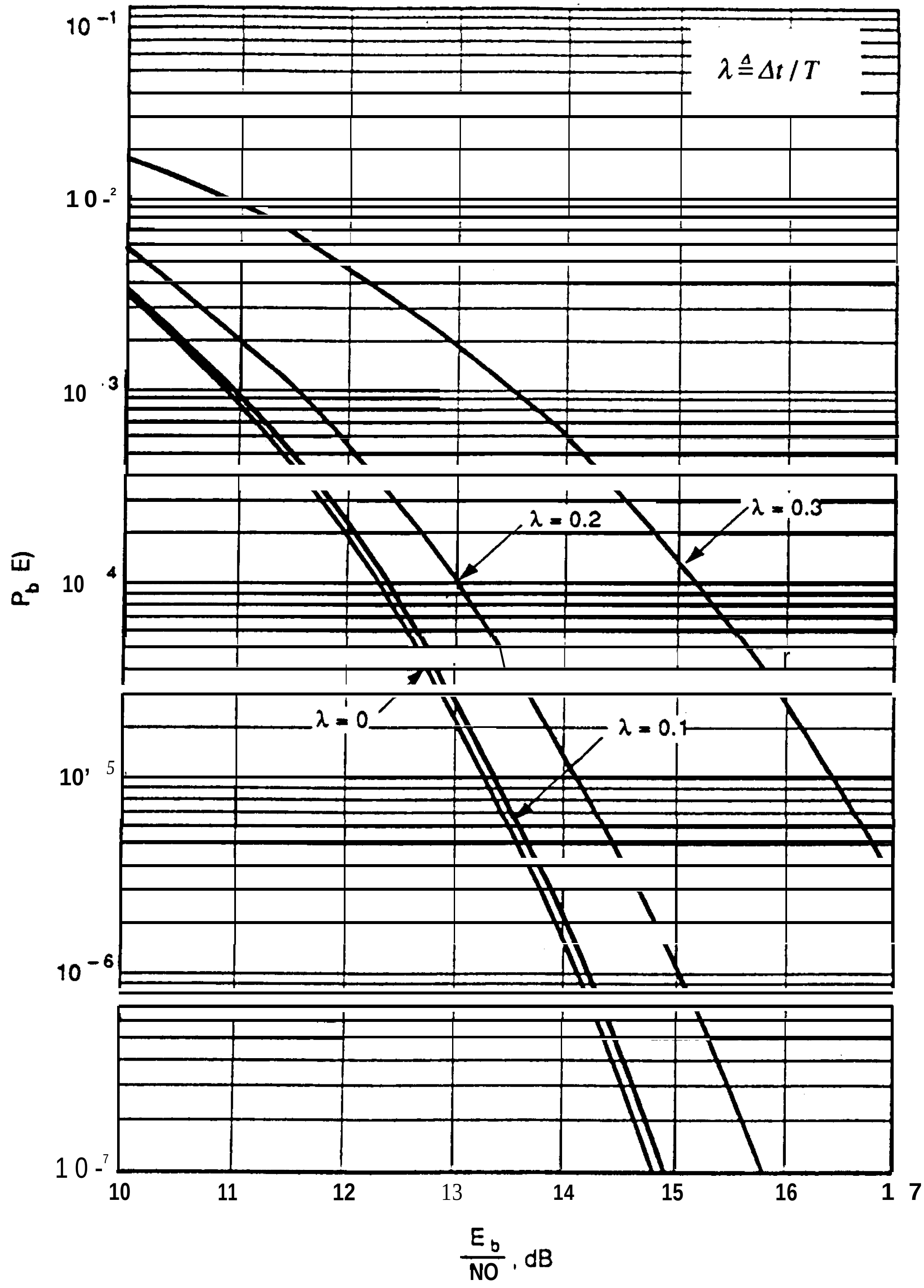
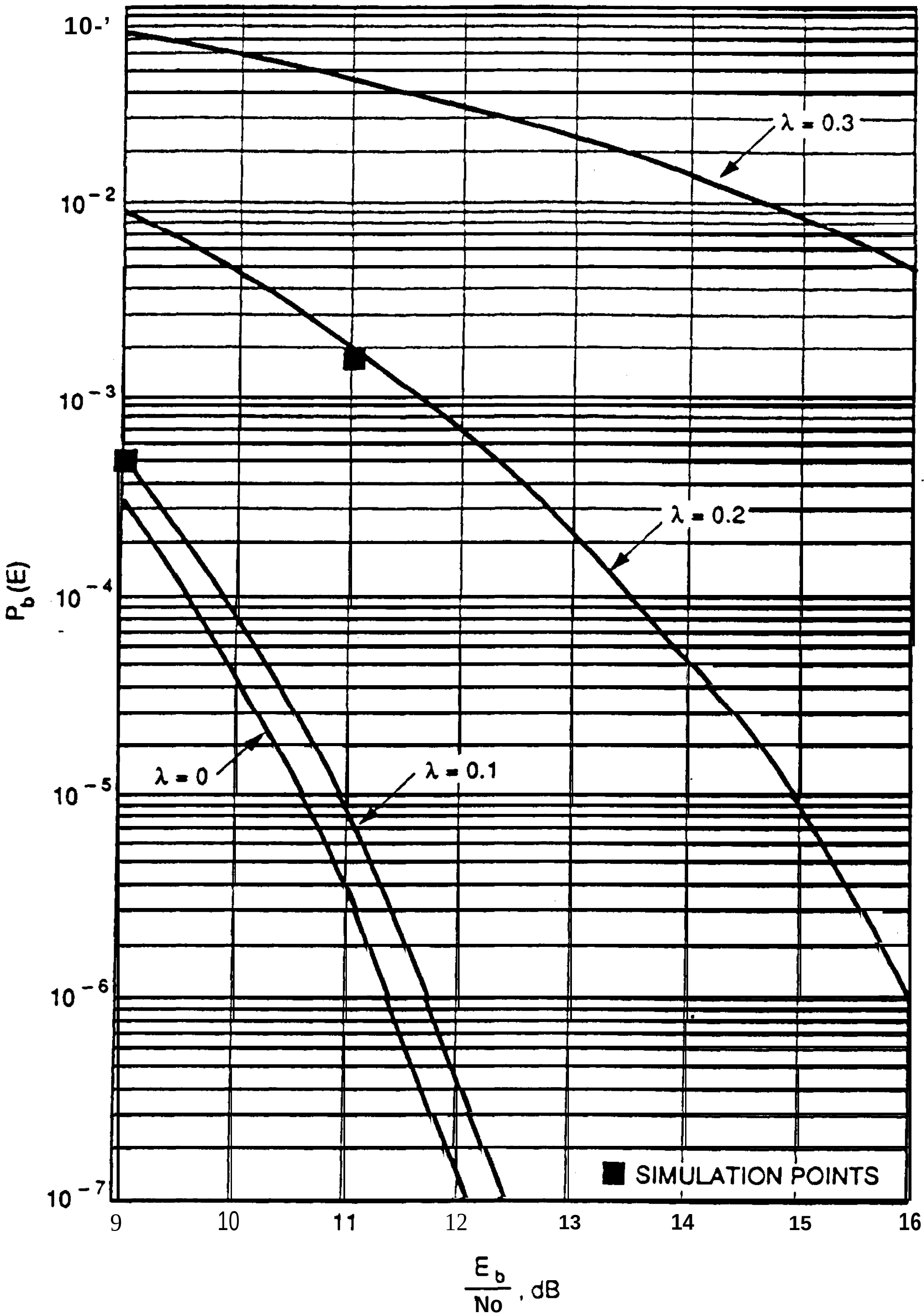


Figure 4. Effect of Timing Error



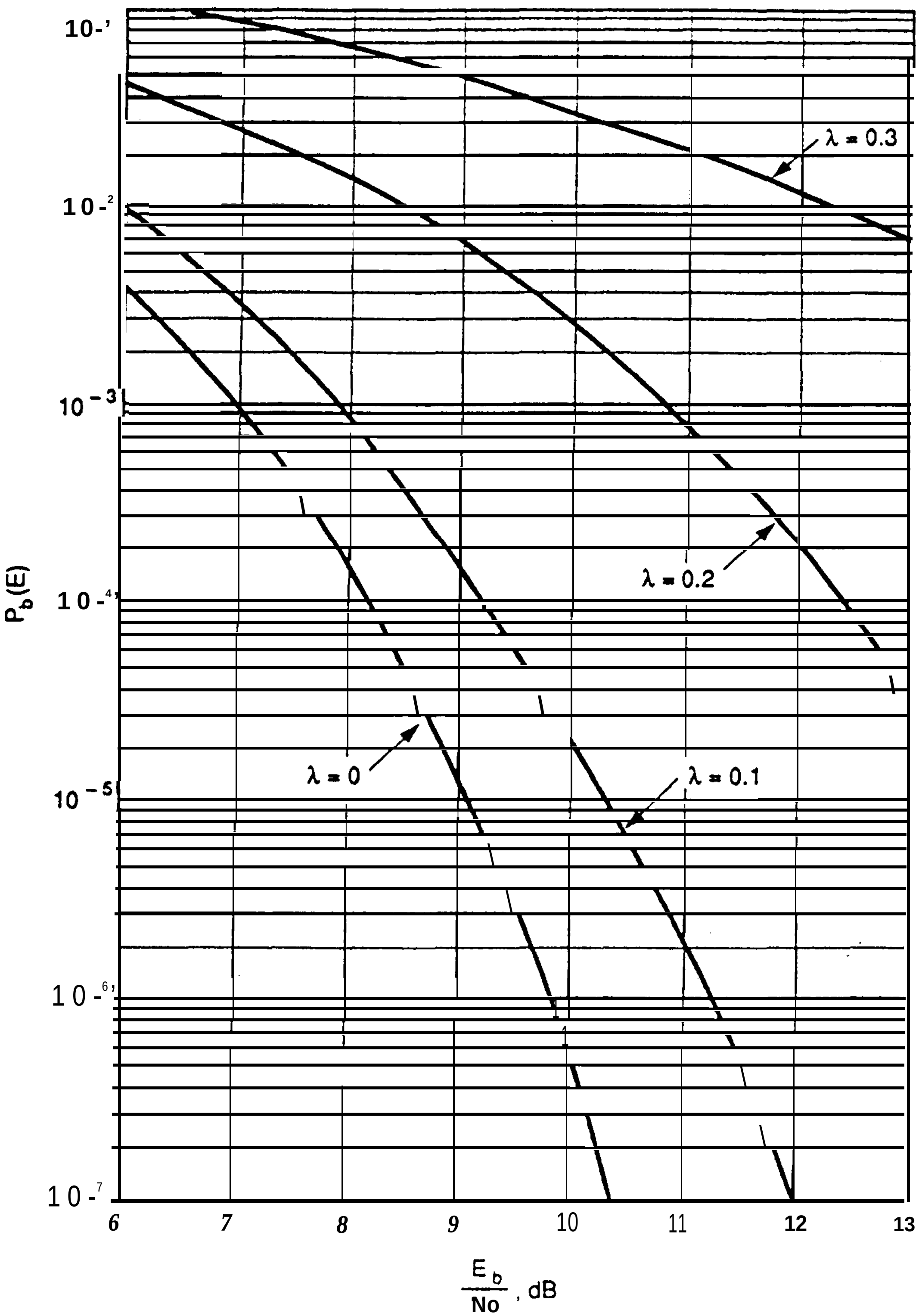
(a) M = 2

Figure 5. Bit Error Probability for Continuous Phase M-FSK with Timing Error.



10-'
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
$P_b(E)$
$\lambda = 0.3$
$\lambda = 0.2$
$\lambda = 0$
$\lambda = 0.1$
SIMULATION POINTS
9
10
11
12
13
14
15
16
$\frac{E_b}{No}$, dB

(b) M = 4



10-'
10-²
10⁻³
10-⁴'
10⁻⁵
10-⁶'
10-⁷
$P_b(E)$
λ = 0.3
λ = 0.2
λ = 0
λ = 0.1
6
7
8
9
10
11
12
13
$\frac{E_b}{N_o}$, dB

(c) M = 8

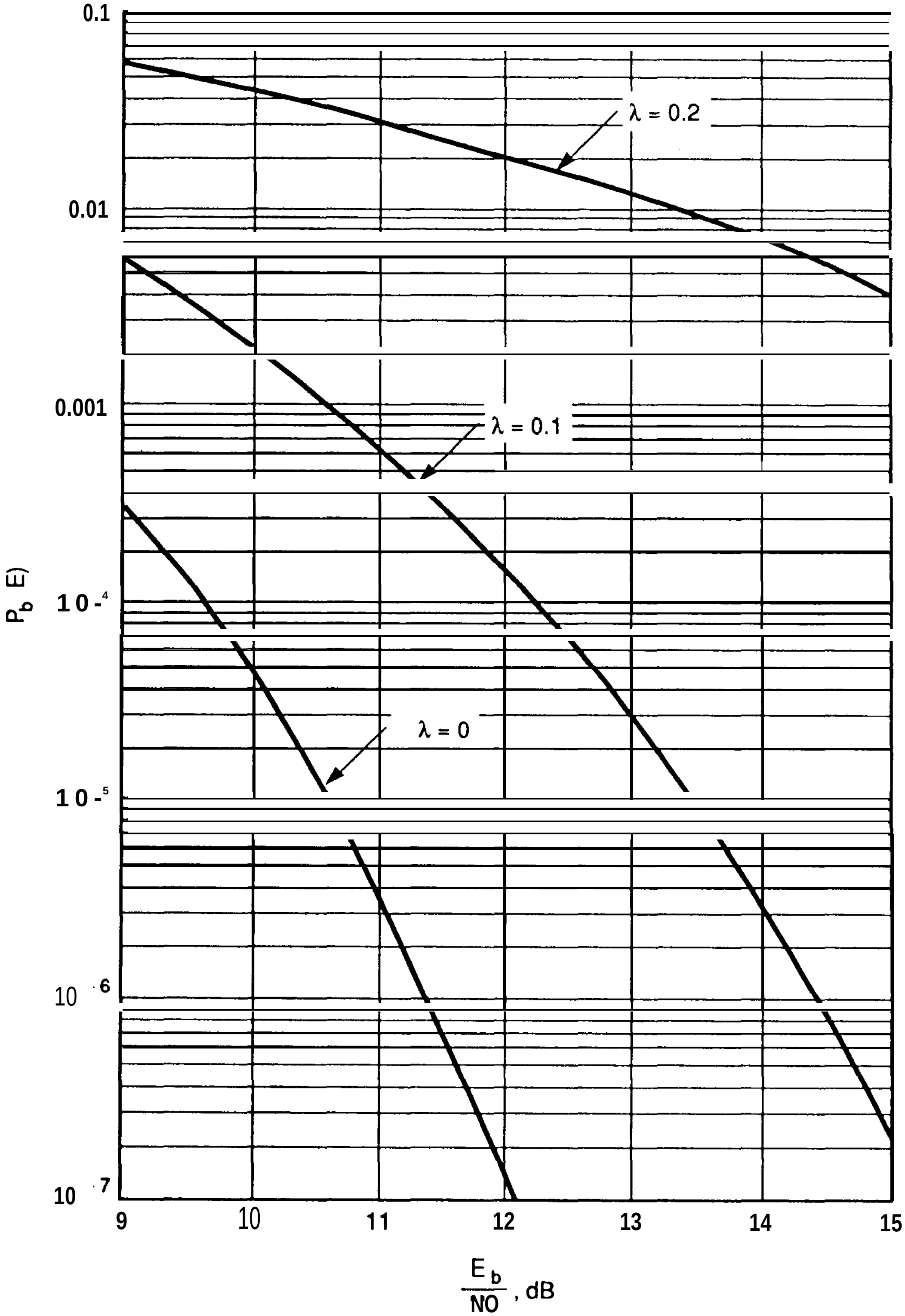


Figure 6. Bit Error Probability for Discontinuous Phase M-FSK with Timing Error; M = 4.

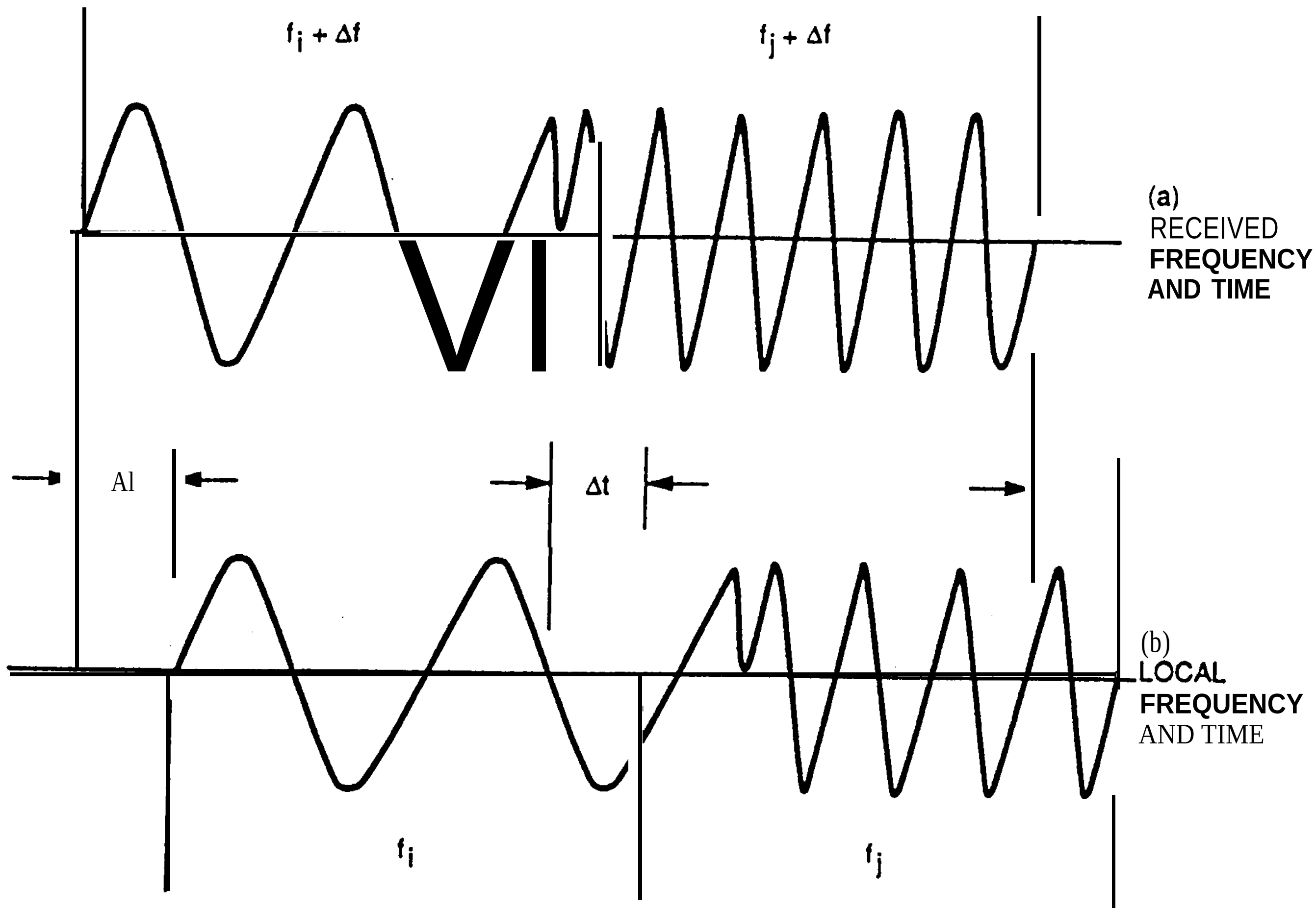
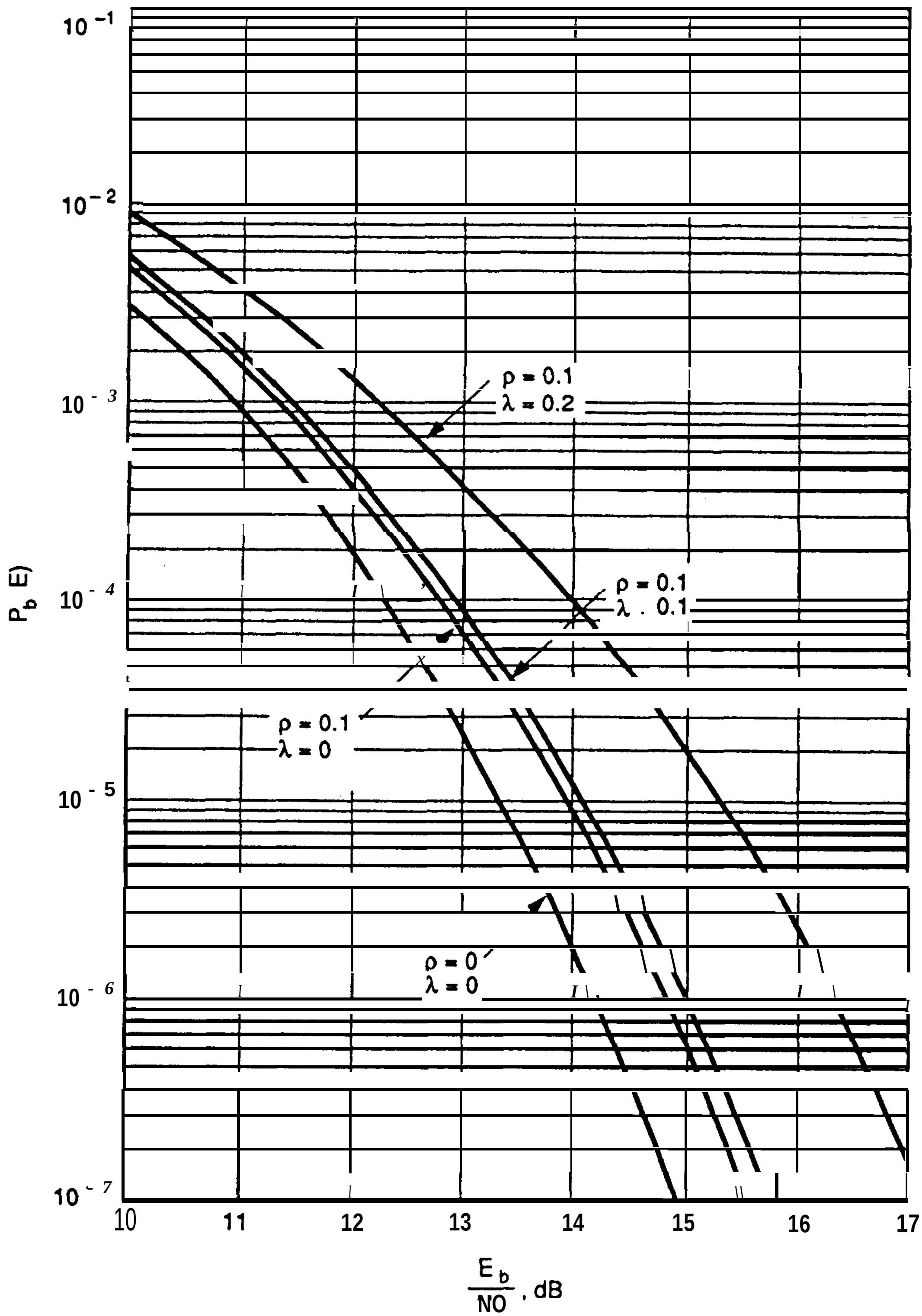
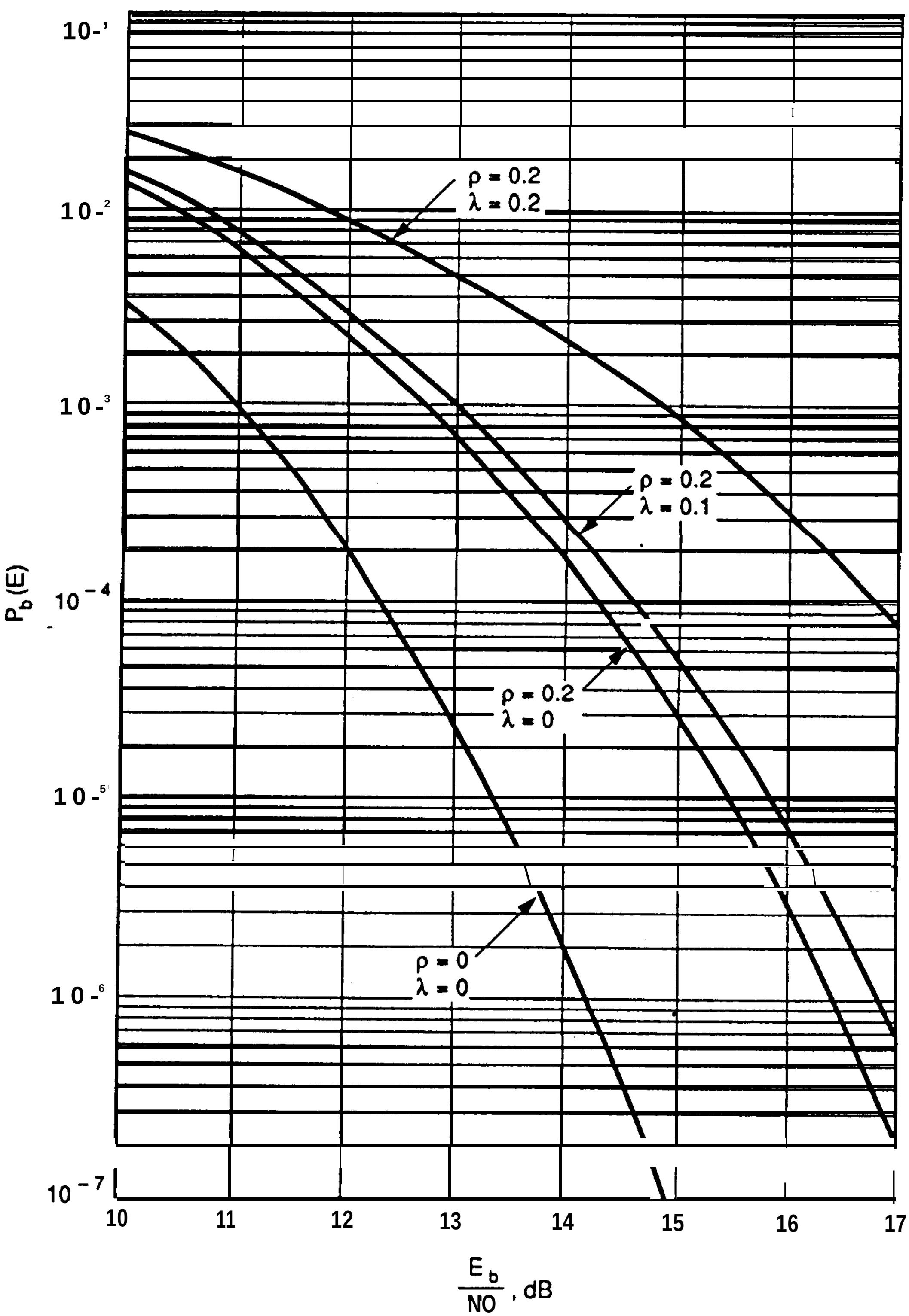


Figure 7. Effect of **Frequency and Timing Errors**



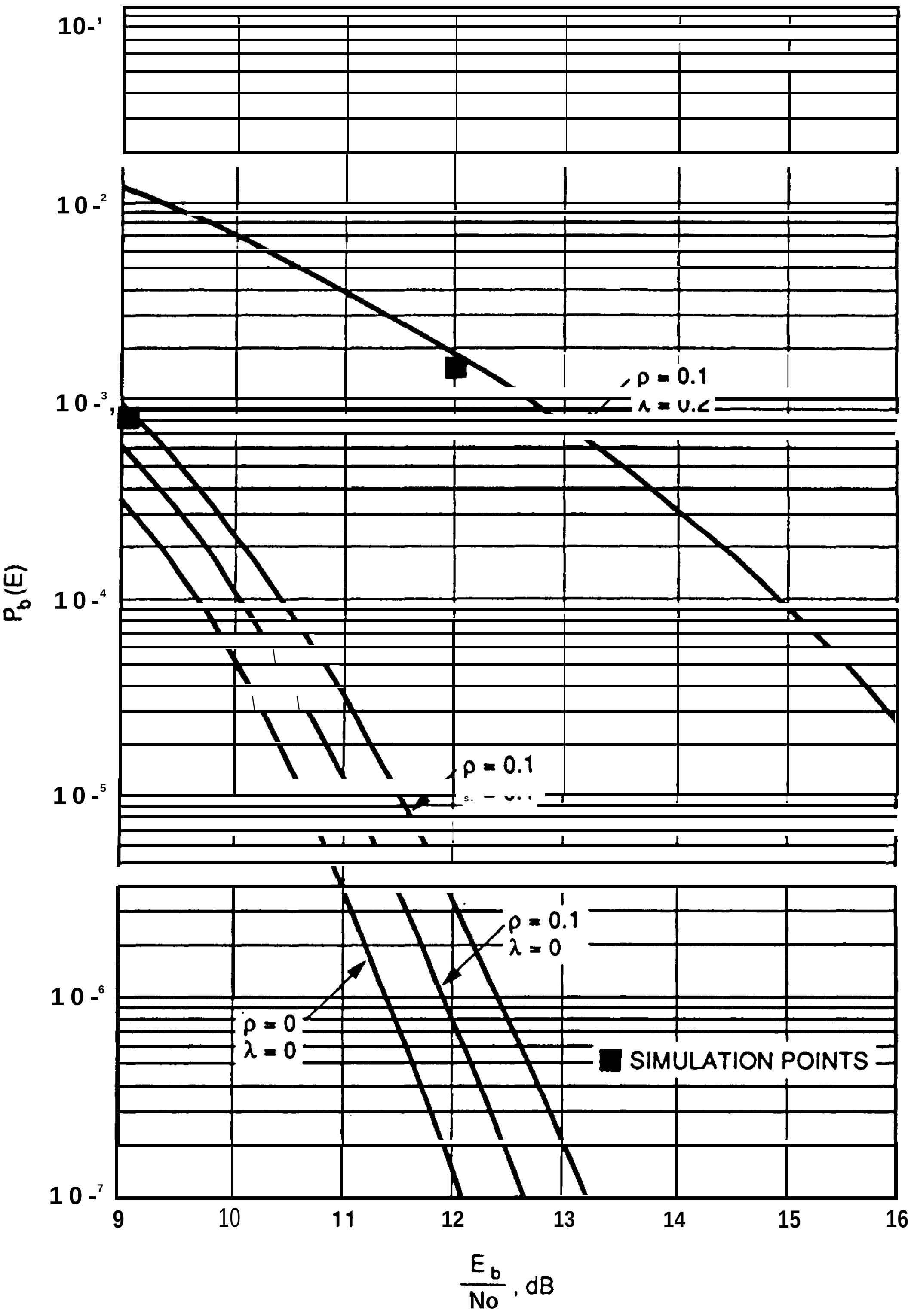
(a) M = 2 and ρ = 0.1

Figure 8. Bit Error Probability for *M*-FSK With Both Frequency and **Timing Errors**



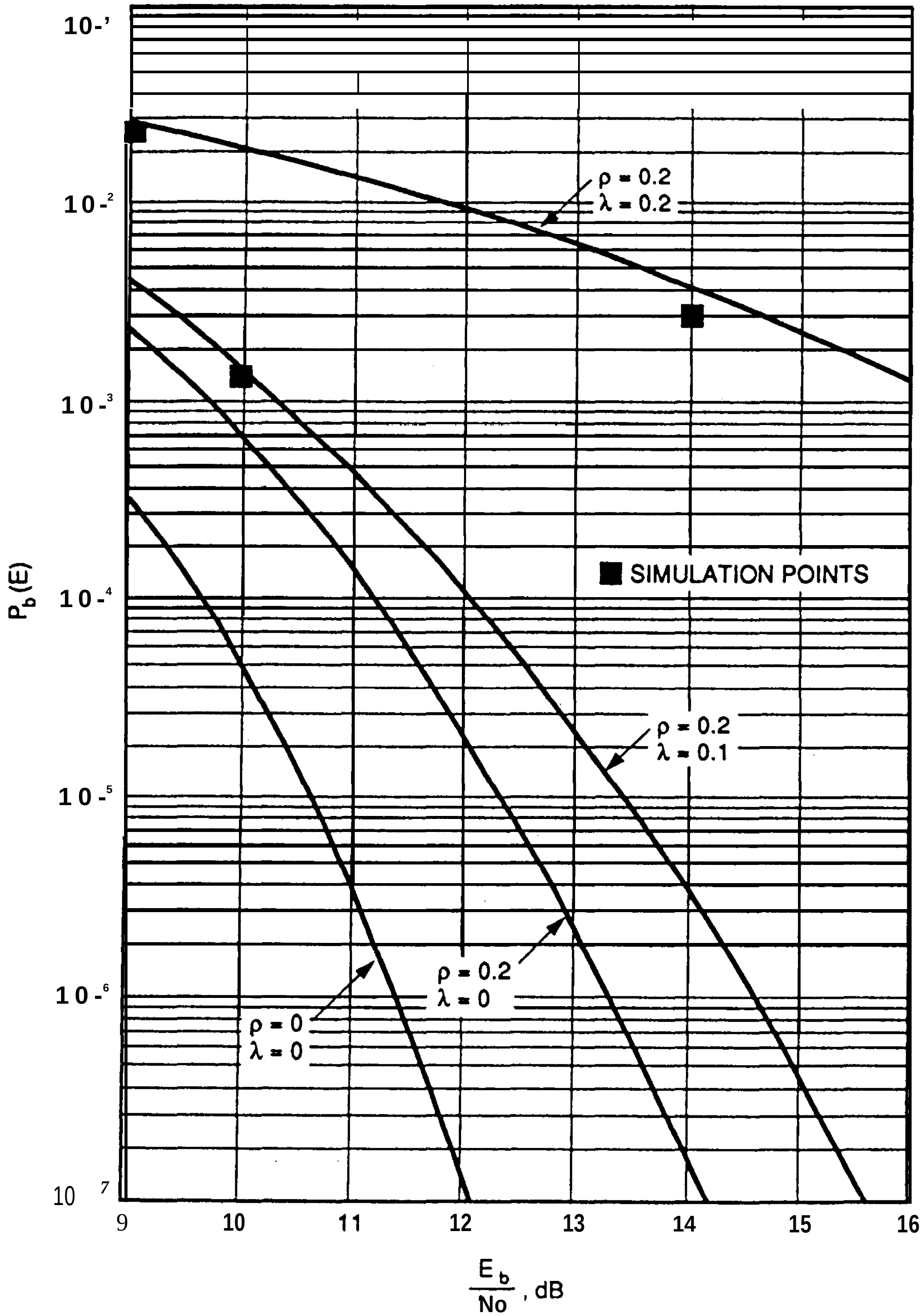
10-'
10-2
10-3
10-4
10-5
10-6
10-7
$P_b(E)$
ρ = 0.2
λ = 0.2
ρ = 0.2
λ = 0.1
ρ = 0.2
λ = 0
ρ = 0
λ = 0
10
11
12
13
14
15
16
17
$\frac{E_b}{NO}$, dB

(b) M = 2 and ρ = 0.2

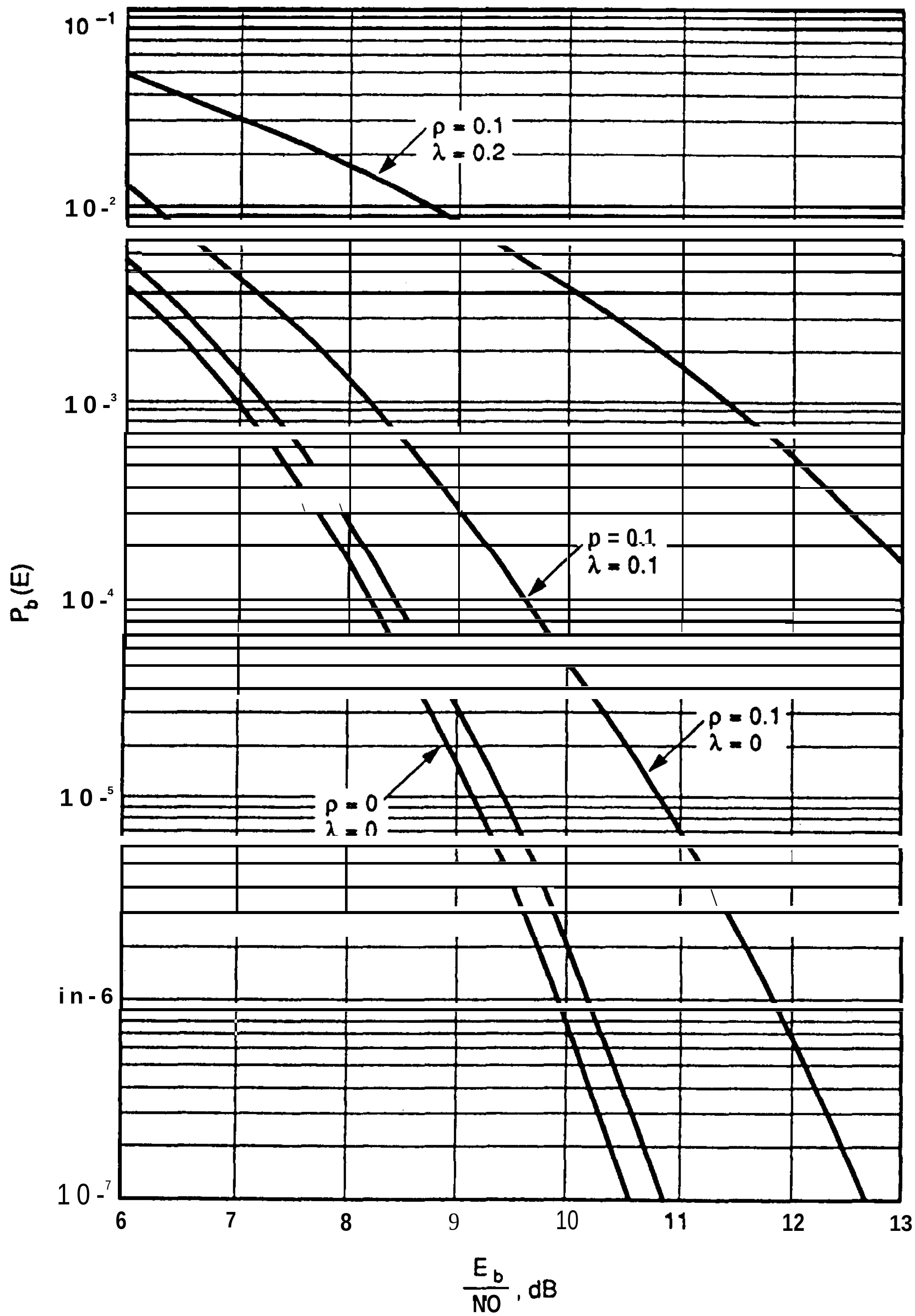


10-'
10-2
10-3
10-4
10-5
10-6
10-7
$P_b(E)$
ρ = 0.1
λ = 0.2
ρ = 0.1
ρ = 0.1
λ = 0
ρ = 0
λ = 0
SIMULATION POINTS
9
10
11
12
13
14
15
16
$\frac{E_b}{No}$, dB

(c) M = 4 and ρ = 0.1

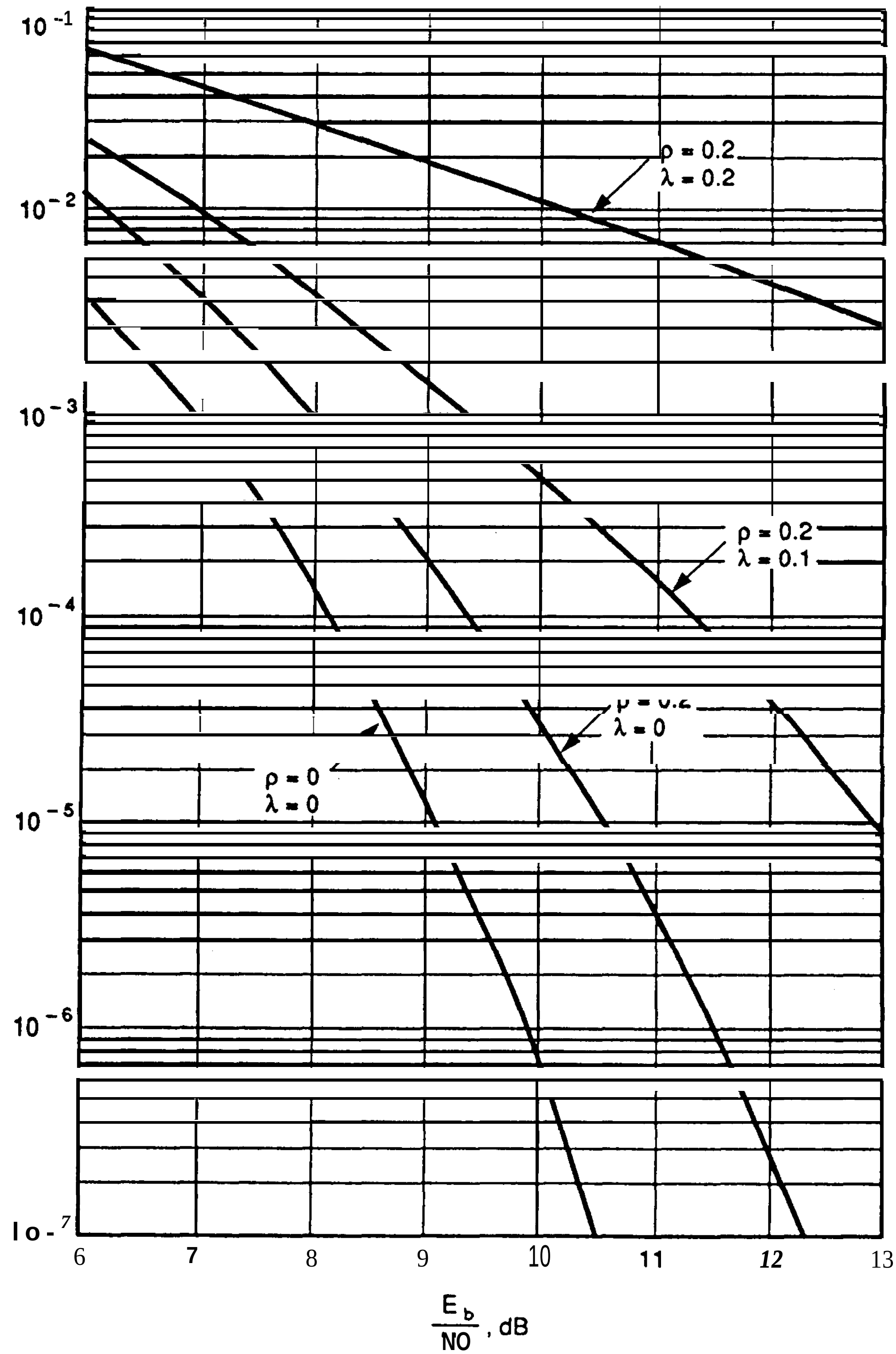

10-'
10-2
10-3
10-4
10-5
10-6
10 7
$P_b(E)$
9
10
11
12
13
14
15
16
$\frac{E_b}{No}$, dB
ρ = 0.2
λ = 0.2
SIMULATION POINTS
ρ = 0.2
λ = 0.1
ρ = 0.2
λ = 0
ρ = 0
λ = 0

(d) M = 4 and ρ = 0.2



10 ⁻¹
10-²
10-³
10-⁴
10-⁵
in-6
10-⁷
$P_b(E)$
ρ = 0.1
λ = 0.2
ρ = 0.1
λ = 0.1
ρ = 0.1
λ = 0
ρ = 0
λ = 0
6
7
8
9
10
11
12
13
$\frac{E_b}{NO}$, dB

(e) M = 8 and ρ = 0.1



10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^{-6}
10^{-7}
$P_b(E)$
ρ = 0.2
λ = 0.2
ρ = 0.2
λ = 0.1
λ = 0
ρ = 0
λ = 0
6
7
8
9
10
11
12
13
$\frac{E_b}{NO}$, dB

(f) M = 8 and ρ = 0.2

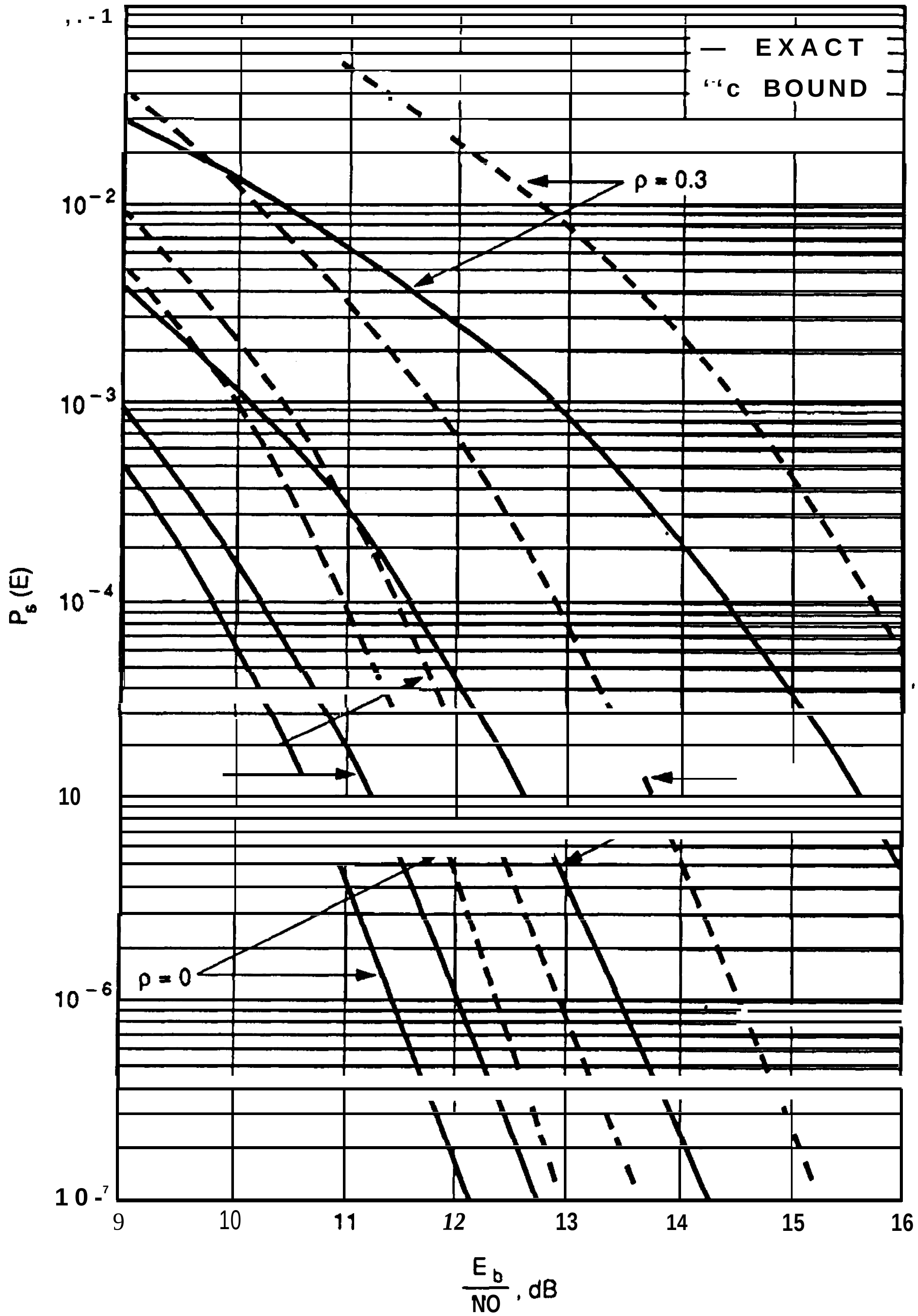


Figure 9. A Bound on the Symbol Error Probability of 4-FSK in the Presence of Frequency Error.