Supersymmetry

Yoshiaki Yasui
RIKEN BNL Resech Center
Upton,NY, 11973,USA

1 Supersymmetry[1, 2]

1.1 Introduction

Orthodox SU(N) gauge theory has the symmetry of Lie algebra which is given by the commutation relation,

$$[T^a, T^b] = C^{abc}T^c. \tag{1}$$

In this case, the symmetry provide the relation between particles with same spins;

$$\begin{aligned} \text{boson} &\leftrightarrow \text{boson} \\ \text{fermion} &\leftrightarrow \text{fermion}. \end{aligned} \tag{2}$$

Now we consider the symmetry between particles with different spins;

$$\text{boson} \leftrightarrow \text{fermion}. \tag{3}$$

Coleman and Mandula found that there is no nontrivial symmetry between particle with different spins. It is well known as the "No-go theorem"[4]. However, they assume that the symmetry is written in terms of commutation relations. The nontrivial symmetry is possible if one uses anti-commutation relations. We call the generalized Lie algebra which includes anti-commutation as Graded Lie algebra. Supersymmetry was discovered by Miyazawa[3]. Haag, Lopuszanski and Sohnius pointed out that only the Supersymmetry is the symmetry between particle with different spins in which the S-matrix is consistent with relativistic quantum field theory[5].

1.2 Supersymmetry algebra

Supersymmetry is the symmetry between fermions and bosons. Now we introduce the new Grassmann parameters θ and $\overline{\theta}$ to connect fermions and bosons, and we define the superspace as $(x, \theta, \overline{\theta})$. The Lorentz transformation is,

$$x \to x + \delta x. \tag{4}$$

Now we consider the special case where δx is linear in θ and $\overline{\theta}$,

$$(x^\mu, \theta, \overline{\theta}) \longrightarrow (x^\mu + i\theta\sigma^\mu\overline{\xi} - i\xi\sigma^\mu\overline{\theta}\ ,\ \theta + \xi\ ,\ \overline{\theta} + \overline{\xi}), \tag{5}$$

here ξ and $\overline{\xi}$ are infinitesimal Grassmann numbers. This transformation generated by the differential operators Q and $\overline{Q}$,

$$\xi Q + \overline{\xi}\,\overline{Q} = \xi^\alpha \left(\frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_\mu \right) + \overline{\xi}^{\dot{\alpha}} \left(\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\epsilon^{\dot{\beta}\dot{\alpha}}\partial_\mu \right). \tag{6}$$

Operators Q and $\overline{Q}$ are group generators of the supersymmertry algebra;

$$\begin{aligned} \{Q_\alpha, \overline{Q}_{\dot{\alpha}}\} &= 2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu \\ \{Q_\alpha, Q_\beta\} &= \{\overline{Q}_{\dot{\alpha}}, \overline{Q}_{\dot{\beta}}\} = 0 \\ [P_\mu, Q_\alpha] &= [P_\mu, \overline{Q}_{\dot{\alpha}}] = 0 \\ [P_\mu, P_\nu] &= 0. \end{aligned} \tag{7}$$

¿From this algebra, we notice that Hamiltonian H is,

$$H = \frac{1}{4}(\overline{Q}_1 Q_1 + Q_1\overline{Q}_1 + \overline{Q}_2 Q_2 + Q_2\overline{Q}_2). \tag{8}$$

This means $\langle\Phi|H|\Phi\rangle \geq 0$ for evey state $|\Phi\rangle$. The vacuum expectation value of H vanishes;$\langle 0|H|0\rangle = 0$, since supersymmetry does not change the grand state; $Q|0\rangle = 0$,$\overline{Q}|0\rangle = 0$.

Here we only consider $N = 1$ supersymmetric theory. However, we give some comments on the N-extended supersymmetric algebra which is including the generators Q^L_α $(L = 1, \cdots, N)$.

- $N = 8$ supersymmetry is the maximal in the four dimension and it contain gravition $(\lambda = \pm 2)$ automatically.
- If you require renormalizable theory, $N \leq 4$.
- $N = 4$ supersymmetric theory is all order finite.

1.3 Component fields

Generators Q and $\overline{Q}$ have mass dimension $\frac{1}{2}$. Then, fields of dimension l are transformed into dimension $l+\frac{1}{2}$ fields or derivatives of lower dimension fields.

A set of fields $(A, \psi, \cdots)$ is the component multiplet of SUSY. We consider the infinitesimal supersymmetric transformation δ_ξ as,

$$\begin{aligned} \delta_\xi A &= (\xi Q + \overline{\xi Q}) \times A \\ \delta_\xi \psi &= (\xi Q + \overline{\xi Q}) \times \psi \\ &\vdots \end{aligned} \tag{9}$$

On the other hand, supersymmetric algebra eq.(7) is rewritten by commutators,

$$\begin{aligned} [\theta Q, \overline{\theta Q}] &= 2\theta\sigma^\mu\overline{\theta}P_\mu \\ [\theta Q, \theta Q] &= [\overline{\theta Q}, \overline{\theta Q}] = 0 \\ [P_\mu, \theta Q] &= [P_\mu, \overline{\theta Q}] = 0 \end{aligned} \tag{10}$$

Thus, commutation relation of the δ_ξ is given by,

$$\begin{aligned} (\delta_\eta\delta_\xi - \delta_\xi\delta_\eta)A &= 2(\eta\sigma^\mu\overline{\xi} - \xi\sigma^\mu\overline{\eta})P_\mu A \\ &= -i2(\eta\sigma^\mu\overline{\xi} - \xi\sigma^\mu\overline{\eta})\partial_\mu A \\ &\vdots \quad . \end{aligned} \tag{11}$$

First, we define the fermion field ψ as,

$$\delta_\xi A = \sqrt{2}\xi\psi, \tag{12}$$

where A is the scalar field. ¿From $\delta_\eta\delta_\xi A$ the field ψ trnsforms into,

$$\delta_\xi\psi = i\sqrt{2}\sigma^\mu\overline{\xi}\partial_\mu A + \sqrt{2}\xi F \tag{13}$$

¿From $\delta_\eta\delta_\xi\psi$,

$$\begin{aligned} (\delta_\eta\delta_\xi - \delta_\xi\delta_e ta)\psi &= -2i(\eta\sigma^\nu\overline{\xi})\partial_\nu\psi \\ &\quad -i\sigma^\nu\overline{\sigma}^\mu\partial_\mu[\eta\sigma_\nu\overline{\xi} - \xi\sigma_\nu\overline{\eta}] + \sqrt{2}(\xi\delta_\eta F - \eta\delta_\xi F) \end{aligned} \tag{14}$$

This equation closes if the transformation of F is,

$$\delta_\xi = i\sqrt{2}\overline{\xi}\overline{\sigma}^\mu\partial_\mu\psi. \tag{15}$$

1.4 Superfield

Superfield formalism is very convenient and systematic way to construct the supersymmetric field theory. Superfields are functions in the superspace which can be expanded by the Grassmann number θ,

$$\begin{aligned} F(x,\theta,\overline{\theta}) &= f(x)+\theta\phi(x)+\overline{\theta\xi}(x) \\ &+ \theta\theta m(x)+\overline{\theta\theta}n(x)+\theta\sigma^{\mu}\overline{\theta}v_{\mu}(x) \\ &+ \theta\theta\overline{\theta\lambda}(x)+\overline{\theta\theta}\theta\psi(x)+\theta\theta\overline{\theta\theta}d(x) \end{aligned} \tag{16}$$

The supersymmetric transformation of the superfield is defined by,

$$\begin{aligned} \delta_{\xi}F(x,\theta,\overline{\theta}) &= \delta_{\xi}f(x)+\theta\delta_{\xi}\phi(x)+\overline{\theta}\delta_{\xi}\overline{\xi}(x) \\ &+ \theta\theta\delta_{\xi}m(x)+\overline{\theta\theta}\delta_{\xi}n(x)+\theta\sigma^{\mu}\overline{\delta}_{\xi}\theta v_{\mu}(x) \\ &+ \theta\theta\overline{\theta}\overline{\delta}_{\xi}\lambda(x)+\overline{\theta\theta}\theta\delta_{\xi}\psi(x)+\theta\theta\overline{\theta\theta}\delta_{\xi}d(x) \\ &\equiv (\xi Q+\overline{\xi Q})F. \end{aligned} \tag{17}$$

The general from of the superfield has too many degree of freedom. To obtain smaller number of components, we impose constraints.

1.5 Chiral superfield

The chiral scalar field is defined by,

$$\overline{D}_{\dot{\alpha}}\Phi = 0, \tag{18}$$

the antichiral field is,

$$D_{\dot{\alpha}}\Phi = 0, \tag{19}$$

where,

$$\begin{aligned} D_{\alpha} &= \frac{\partial}{\partial\theta^{\alpha}}+i\sigma^{\mu}_{\alpha\dot{\alpha}}\overline{\theta}^{\dot{\alpha}}\partial_{\mu} \\ \overline{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}}-i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_{\mu}. \end{aligned} \tag{20}$$

It is easy to notice that the solution of $\overline{D}_{\dot{\alpha}}\Phi = 0$ is,

$$\Phi = A(y)+\sqrt{2}\theta\psi(y)+\theta\theta F(y), \tag{21}$$

where $y^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta}$. The supersymmetric transformation eq.(17) satisfies the eq.(12) to eq.(15) automatically.

Lagrangian for the chiral superfields is,

$$\begin{aligned}\mathcal{L} &= \Phi_i^\dagger \Phi_i \mid_{\theta\theta\overline{\theta}\overline{\theta}} \\ &+ \{(\frac{1}{2}m_{ij}\Phi_i\Phi_j + \frac{1}{3}g_{ijk}\Phi_i\Phi_j\Phi_k + \lambda_i\Phi_i) \mid_{\theta\theta} + h.c.\},\end{aligned} \tag{22}$$

in terms of the component fields,

$$\begin{aligned}\mathcal{L} &= i\partial_\mu\overline{\psi}_i\overline{\sigma}^\mu\psi_i + A_i^*\partial^2 A_i - \frac{1}{2}m_{ik}\psi_i\psi_k - \frac{1}{2}m_{ik}^*\overline{\psi}_i\overline{\psi}_k \\ &- g_{ijk}\psi_i\psi_j A_k - g_{ijk}^*\overline{\psi}_i\overline{\psi}_j A_k^* - F^*F.\end{aligned} \tag{23}$$

1.6 Vector superfield

Vector superfield is defined by,

$$V = V^\dagger. \tag{24}$$

With using the component fields,

$$\begin{aligned}V(x,\theta,\overline{\theta}) &= C(x) + i\theta\chi(x) - i\overline{\theta}\overline{\chi}(x) \\ &+ \frac{i}{2}\theta\theta[M(x) + iN(x)] - \frac{i}{2}\overline{\theta\theta}[M(x) - iN(x)] \\ &- \theta\sigma^\mu\overline{\theta}v_\mu(x) + i\theta\theta\overline{\theta}[\overline{\lambda}(x) + \frac{i}{2}\overline{\sigma}^\mu\partial_\mu\chi(x)] \\ &- i\overline{\theta\theta}\theta[\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\overline{\chi}(x)] + \frac{1}{2}\theta\theta\overline{\theta\theta}[D(x) + \frac{1}{2}\partial^2 C(x)].\end{aligned} \tag{25}$$

By the way, $\Phi + \Phi^\dagger$ is also vector superfield,

$$\begin{aligned}\Phi + \Phi^\dagger &= A + A^* + \sqrt{2}(\theta\psi + \overline{\theta\psi}) \\ &+ \theta\theta F + \overline{\theta\theta}F^* \\ &+ i\theta\sigma^\mu\overline{\theta}\partial_\mu(A - A^*) + \frac{i}{\sqrt{2}}\theta\theta\overline{\theta}\overline{\sigma}^\mu\partial_\mu\psi \\ &- \frac{i}{\sqrt{2}}\overline{\theta\theta}\theta\sigma^\mu\partial_\mu\overline{\psi} + \frac{1}{4}\theta\theta\overline{\theta\theta}\partial^2(A + A^*),\end{aligned} \tag{26}$$

where Φ and $\Phi^\dagger$ are chiral fields. This combination has the derivative term $i\partial_\mu(A - A^*)$. From the analogy of orthodox gauge transformation, we define the supersymmetric gauge transformation as,

$$V \to V + \Phi + \Phi^\dagger \tag{27}$$

Lagrangian for the vector superfileds,

$$\mathcal{L} = \frac{1}{4}(W^\alpha W_\alpha \mid_{\theta\theta} + \overline{W}^{\dot{\alpha}} \overline{W}_{\dot{\alpha}} \mid_{\overline{\theta}\overline{\theta}}), \tag{28}$$

where

$$\begin{aligned} W_\alpha &= -\frac{1}{4}\overline{DD}D_\alpha V \\ \overline{W}_{\dot{\alpha}} &= -\frac{1}{4}DD\overline{D}_{\dot{\alpha}} V. \end{aligned} \tag{29}$$

In terms of the component fields,

$$\mathcal{L} = \frac{1}{2}D^2 - \frac{1}{4}v^{\mu\nu}v_{\mu\nu} - i\lambda\sigma^\mu\partial_\mu\overline{\lambda}, \tag{30}$$

where λ is gaugino and $v_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu$.

2 Supersymmetric SU(2) × U(1) model

Yukawa interaction in the standard model[6],

$$\mathcal{L}_y = g_d U_L h \overline{d}_R + g_u U_L h^* \overline{u}_R, \tag{31}$$

where

$$U_L \equiv \begin{pmatrix} u_L \\ d_L \end{pmatrix}. \tag{32}$$

If we consider naive replacement from orthodox fields to superfields, Yukawa interaction becomes,

$$\mathcal{L}_y = g_d \hat{U}_L \hat{h} \hat{\overline{d}}_R + g_u \hat{U}_L \hat{h}^\dagger \hat{\overline{u}}_R, \tag{33}$$

where $\hat{a}$'s denote superfields. But this interaction does not conserve the supersymmetry. Thus, we need additional Higgs doublet in the supersymmetric theory[7, 8, 9].

The minimal supersymmetric extension of the standard model (MSSM) requires two Higgs doublets(table.1). The supersymmetric potential for the Higgs sector is given by,

$$\mathcal{W} = \mu \epsilon_{ij} H_1^i H_2^j + \epsilon_{ij}[g_l^{ab} H_1^i \tilde{L}_a^j \tilde{R}_b + g_d^{ab} H_1^i \tilde{Q}_a^j \tilde{D}_b + g_u^{ab} H_2^i \tilde{Q}_a^j \tilde{U}_b]. \tag{34}$$

It is possible to consider the potentials of the following types, too.

$$\epsilon_{ij} \tilde{L}^i \tilde{L}^j \tilde{R} + \epsilon_{ij} \tilde{Q}^i \tilde{L}^j \tilde{D} + \tilde{U} \tilde{D} \tilde{D}. \tag{35}$$

But these potentials do not conserve the lepton and baryon numbers.

This model includes two charged higgs $H^\pm$, two CP-even neutral Higgs H_1, H_2 and one CP-odd neutral Higgs H_3. Higgs masses are given by,

$$\begin{aligned} m^2_{H^0_3} &= m^2_{12}(\tan\beta + \cot\beta) \\ m^2_{H^\pm} &= m^2_{H^0_3} + m^2_W \\ m_{H^0_1, H^0_2} &= \frac{1}{2}(m^2_{H^0_3} + m^2_Z \pm \sqrt{(m^2_{H^0_3} + m^2_Z)^2 - 4 m^2_Z m^2_{H^0_3} \cos^2 2\beta}\ ), \end{aligned} \tag{36}$$

where β is defined by,

$$\tan\beta \equiv \frac{v_2}{v_1}, \tag{37}$$

Thus, we have following constraint on $m_{H_2^0} \leq m_{H_1^0}$,

$$
\begin{aligned}
& m_{H_2^0} \leq m \mid \cos 2\beta \mid \leq m_Z \qquad \text{with} \qquad m \equiv \min(m_Z, m_{H_3^0}) \\
& m_{H_1^0} \geq m_Z \\
& m_{H^\pm} \geq m_W .
\end{aligned} \tag{38}
$$

	Boson fields	Fermion fields	$SU(2)_L$	Y
Gauge multiplets				
	V^a	$\tilde{\lambda}^a$	triplet	0
	V'	$\tilde{\lambda}'$	singlet	0
Matter multiplets				
Scalar leptons	$\tilde{L}^j = (\tilde{\nu}, \tilde{e}^-)_L$	$(\nu, e^-)_L$	doublet	-1
	$\tilde{R} = \tilde{e}_R^+$	e_L^c	singlet	2
Scalar quarks	$\tilde{Q}^j = (\tilde{u}, \tilde{d})_L$	$(u, d)_L$	doublet	$1/3$
	$\tilde{U} = \tilde{u}_R^*$	u_L^c	singlet	$-4/3$
	$\tilde{D} = \tilde{d}_R^*$	d_L^c	singlet	$2/3$
Higgs bosons	H_1^j	$(\tilde{H}_1^0, \tilde{H}_1^-)_L$	doublet	-1
	H_2^j	$(\tilde{H}_2^+, \tilde{H}_2^0)_L$	doublet	1

Table.1:The gauge and matter multiplets of the supersymmetric SU(2)×U(1) model.

A Wely spinor and Majorana spinor

Metric tensor

$$g_{\mu\nu} = diag(1, -1, -1, -1) \tag{39}$$

We define the Lorentz transformation of the two component spinor ξ_α as,

$$\begin{aligned} \xi'_\alpha &= M_{\alpha\beta}\xi_\beta & \overline{\xi}'_{\dot\alpha} &= M^*_{\dot\alpha\dot\beta}\overline{\xi}_{\dot\beta} \\ \xi'^\alpha &= M^{-1}_{\beta\alpha}\xi^\beta & \overline{\xi}'^{\dot\alpha} &= (M^*)_{\dot\beta\dot\alpha}\overline{\xi}^{\dot\beta} \end{aligned} \tag{40}$$

where,

$$\begin{aligned} M &\in SL(2,C) \\ \overline{\xi}_{\dot\alpha} &= \xi^*_\alpha. \end{aligned} \tag{41}$$

Spinors with upper and lower indeces are related through the ϵ tensor,

$$\xi^\alpha = \epsilon^{\alpha\beta}\xi_\beta \qquad\qquad \xi_\beta = \epsilon_{\beta\alpha}\xi^\alpha, \tag{42}$$

where $\epsilon^{\alpha\beta}$ is,

$$\epsilon^{\alpha\beta} = -\epsilon_{\alpha\beta} = i\sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{43}$$

We use the simple notation

$$\begin{aligned} \eta\xi &= \eta^\alpha\xi_\alpha = \xi\eta \\ \overline{\eta}\overline{\xi} &= \overline{\eta}_{\dot\alpha}\overline{\xi}^{\dot\alpha} = \overline{\xi}\overline{\eta} \\ \overline{\eta}_2\overline{\sigma}^\mu\eta_1 &= \overline{\eta}_{2\dot\alpha}\overline{\sigma}^{\mu\dot\alpha\beta}\eta_{1\beta} = -\eta_1\sigma^\mu\overline{\eta}_2 \end{aligned} \tag{44}$$

Dirac eq. for two component spinors are given by,

$$(\overline{\sigma}_\mu p^\mu)^{\dot\alpha\beta}\xi_\beta = m\overline{\eta}^{\dot\alpha} \qquad\qquad (\sigma_\mu p^\mu)_{\alpha\dot\beta}\overline{\eta}^{\dot\beta} = m\xi_\alpha, \tag{45}$$

where p^μ ia the momentum,

$$p^\mu = (E, \vec{p}), \tag{46}$$

and σ^μ is,

$$\sigma^\mu = (1, \vec{\sigma}) \qquad\qquad \overline{\sigma}^\mu = (1, -\vec{\sigma}) \tag{47}$$

$$\overline{\sigma}^{m\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\sigma^{m}_{\beta\dot{\beta}}. \tag{48}$$

The relation between two component spinor and four component Dirac spinor is,

$$\psi_\alpha = \begin{pmatrix} \chi_\alpha \\ \overline{\xi}^{\dot{\beta}} \end{pmatrix}. \tag{49}$$

Using the chiral representation and eq.(45),

$$\gamma_\nu = \begin{pmatrix} 0 & \sigma_{\mu\alpha\dot{\beta}} \\ \overline{\sigma}^{\dot{\alpha}\beta}_{\mu} & 0 \end{pmatrix} \qquad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{50}$$

we obtain the usual Dirac equation,

$$(\gamma_\mu p^\mu - m)\psi = 0. \tag{51}$$

We also present the useful relations,

$$\begin{aligned}
\overline{\psi}_1\psi_2 &= \eta_1\xi_2 + \overline{\eta}_2\overline{\xi}_1 \\
\overline{\psi}_1\gamma_5\psi_2 &= -\eta_1\xi_2 + \overline{\eta}_2\overline{\xi}_1 \\
\overline{\psi}_1\gamma^\mu\psi_2 &= \overline{\xi}_1\overline{\sigma}^\mu\xi_2 - \overline{\eta}_2\overline{\sigma}^\mu\eta_1 \\
\overline{\psi}_1\gamma^\mu\gamma_5\psi_2 &= -\overline{\xi}_1\overline{\sigma}^\mu\xi_2 - \overline{\eta}_2\overline{\sigma}^\mu\eta_1 \\
-\frac{1}{2}i\overline{\psi}_1\sigma^{\mu\nu}\psi_2 &= \eta_1\sigma^{\mu\nu}\xi_2 - \overline{\eta}_2\overline{\sigma}^{\mu\nu}\overline{\xi}_1
\end{aligned} \tag{52}$$

References

[1] J.Wess and J.Bagger, *Supersymmetry and Supergravity*, 2d ed. (Princeton University Press,1992)

[2] N.Sakai, hep-ph/9010376 (1999).

[3] H.Myazawa, *Prog. Theor. Phys.* **36** (1966)1968;
*Phys. Rev.***170** 1586.

[4] S.Coleman and J.Mandula, *Phys. Rev.***159** (1967) 257.

[5] R.Haag, J.Lopuszanski and M. Sohnius, *Nucl. Phys.***B88** (1975) 257.

[6] K.Aoki, Z.Hioki, R.Kawabe, M.Konuma and T.Muta, *Suppl.Prog.Theor.Phys.***73** (1982).

[7] J.F.Gunion and H.Haber, *Nucl.Phys.***B272** (1986) 1.

[8] H.E.Haber and G.L.Kane, *Phys.Rep.***117**(1985) 75.

[9] J.F.Gunion, H.Haber, G.Kane and S.Dawson, *The Higgs hunter's guide*(Addison - Wesley,Reading,MA,1990).