Resonant Plasma Heating below the Cyclotron Frequency

Roscoe White

Liu Chen[*], Zhihong Lin,
Dave Gates, Nikolai Gorelenkov

Princeton Plasma Physics Lab
Princeton University
[*]University of California, Irvine

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Wave particle interaction in a magnetic field is a fundamental dynamical process in plasma physics.

The adiabatic invariant μ is easily broken with large amplitude waves, even at frequencies well below the cyclotron frequency.

- Simplest textbook problem: a single electrostatic wave propagating perpendicular to a constant $\vec{B}$

  Threshold $\qquad k\Delta x_g = \frac{k^2 e\Phi_0}{m\Omega_c^2} \sim 1$

- Low frequency resonances
- The route to Chaos and heating
- Hamiltonian analysis
- Single large amplitude Alfven wave
- Many mode Spectra
- Solar Corona and NSTX
- Conclusions

Electrostatic Cyclotron Heating

McChesney, Bellan, Stern – Experiment, PRL 1987
Chen, Lin, White, – Phys Plasmas Nov. 2001

Hamiltonian

$$H = \frac{(\vec{p} - \vec{A})^2}{2} + \Phi(x,t)$$

$\vec{B} = B\hat{z}$, vector potential $\vec{A} = -By\hat{x}$

$$\dot{v}_x = Bv_y - \partial_x \Phi, \qquad \dot{v}_y = -Bv_x - \partial_y \Phi.$$

$\Phi = \Phi_0 cos(kx - \omega t)$.

Dimensionless parameters, cyclotron radius $\rho = v/\Omega_c$

- $k\rho =$ ratio of cyclotron radius to wave length,
- Nonlinearity parameter is $k\Delta x_g = k^2 e\Phi_0/m\Omega_c^2 \sim \omega_b^2/\Omega_c^2$ ratio of polarization drift motion to wave length

The equations of motion become
$\dot{v}_x = v_y + k\Phi_0 sin(kx - \omega t), \quad v_y = -x + x_0$, giving

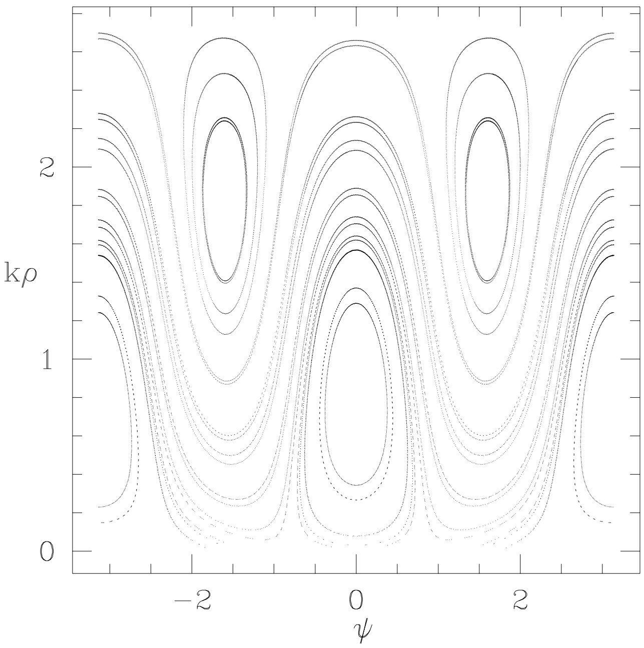
$$\frac{d^2x}{dt^2} + x = x_0 + k\Phi_0 sin(kx - \omega t)$$

Let $s \equiv k(x - x_0) \ll 1 \quad$ and $\quad 2T = kx_0 - \omega t$
lowest order in s

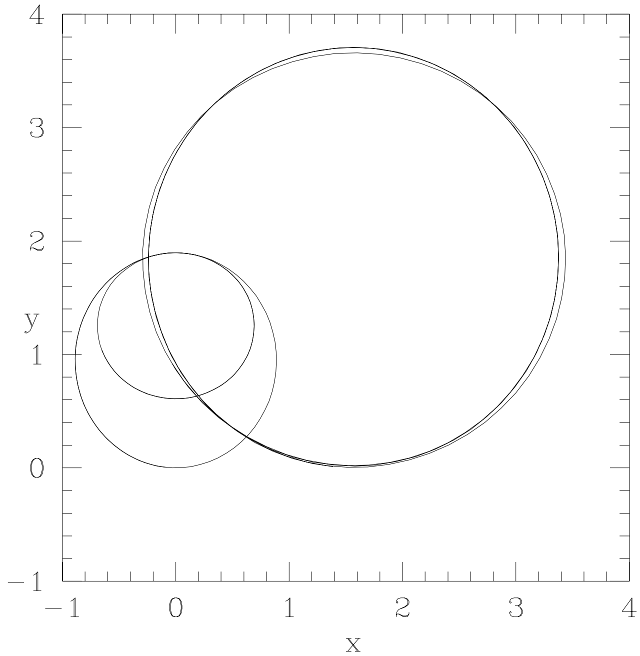
$$\frac{d^2 s}{dT^2} + \left[\frac{4}{\omega^2} - \frac{4k^2\Phi_0}{\omega^2} cos(2T)\right] s = \frac{4k^2\Phi_0}{\omega^2} sin(2T)$$

Mathieu equation - unstable solutions for $\omega \simeq 2/N$

Low frequency secularity $N \gg 1$



Poincare section of $k\rho$, $\psi = kx - \omega t$, at $v_y = 0, \dot{v}_y > 0$.
$k^2\Phi_0 = 0.1$, $\omega = 1/2$



Fixed point orbits for $k^2\Phi_0 = 0.1$, $\omega = 1/2$

At t=0 $v_y = 0$, $\dot{v}_y > 0$, $x = x_0$, $\dot{x} < 0$

Guiding center fixed in space, not moving with wave.

Trapping impossible

Guided by numerical results

Fourier analysis of the fixed point trajectories
consider $\omega = 1/2$ analytically.

Ansatz: $x = x_0 + \lambda cos(t) - \mu sin(t) - \beta sin(\omega t)$

Using $e^{\pm iasin(b)} = \sum_m J_m(a) e^{\pm imb}$,

$$-2\frac{d\mu}{dt}cos(t) - 2\frac{d\lambda}{dt}sin(t) - (1-\omega^2)\beta sin(\omega t) =$$
$$k\Phi_0 \sum_{jlm} J_j(k\lambda) J_l(k\mu) J_m(k\beta) sin[(j-l-m\omega-\omega)t] cos[\psi_0 + j\pi/2]$$
$$+k\Phi_0 \sum_{jlm} J_j(k\lambda) J_l(k\mu) J_m(k\beta) cos[(j-l-m\omega-\omega)t] sin[\psi_0 + j\pi/2]$$

Integrating over the short time scales

$$(1-\omega^2)\beta = k\Phi_0 \sum_{jlm} J_j(k\lambda) J_l(k\mu) J_m(k\beta) cos(\psi_0 + j\pi/2)\Delta_{\omega-}$$

$$2\frac{d\mu}{dt} = -k\Phi_0 \sum_{jlm} J_j(k\lambda) J_l(k\mu) J_m(k\beta) sin(\psi_0 + j\pi/2)\Delta_{1+}$$

$$2\frac{d\lambda}{dt} = -k\Phi_0 \sum_{jlm} J_j(k\lambda) J_l(k\mu) J_m(k\beta) cos(\psi_0 + j\pi/2)\Delta_{1+}$$

with $\Delta_{\zeta\pm} = \delta_{j-l+(-m-1)\omega,\zeta} \pm \delta_{j-l+(-m-1)\omega,-\zeta}$.

For small wave amplitude, $k^2\Phi_0 \ll 1$
with $C_0 = cos(\psi_0)$, $S_0 = sin(\psi_0)$,
$F_{a,b}(\lambda, \mu) = \sum_{n=-\infty}^{\infty}(-1)^n J_{2n+a}(k\lambda)J_{2n+b}(k\mu)$

$$(1-\omega^2)\beta = k\Phi_0[C_0(F_{0,0} - F_{0,-1}) + S_0(F_{1,0} - F_{1,1})],$$

$$2\frac{d\mu}{dt} = k\Phi_0 J_1(k\beta)[C_0(F_{1,0} - F_{1,-1} + F_{1,2} - F_{1,1}) + S_0(F_{0,-1} - F_{0,-2} + F_{0,1} - F_{0,0})],$$

$$2\frac{d\lambda}{dt} = k\Phi_0 J_1(k\beta)[S_0(-F_{1,0} + F_{1,-1} + F_{1,2} - F_{1,1}) + C_0(F_{0,-1} - F_{0,-2} - F_{0,1} + F_{0,0})],$$

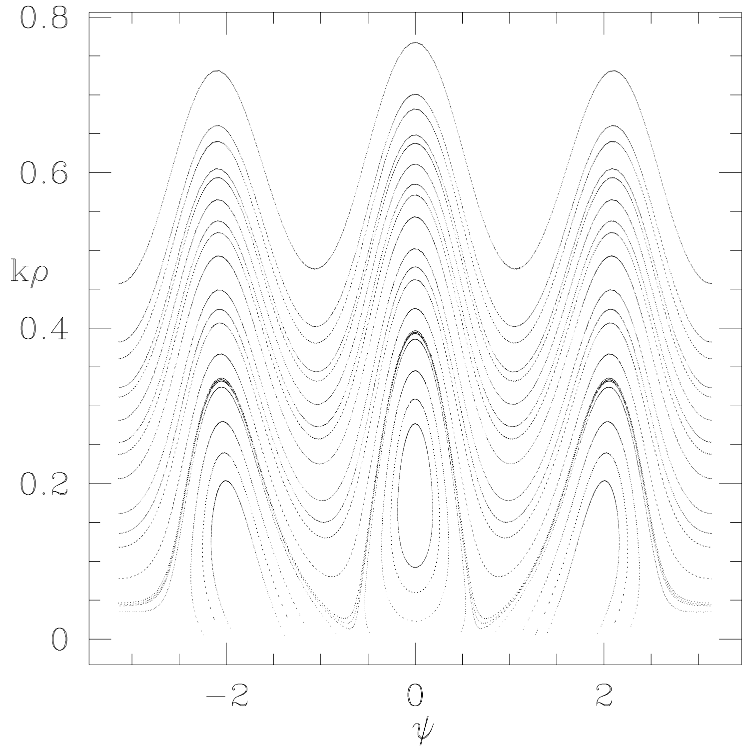
Fixed points of the Poincaré map given by

$$d\lambda/dt = d\mu/dt = 0$$

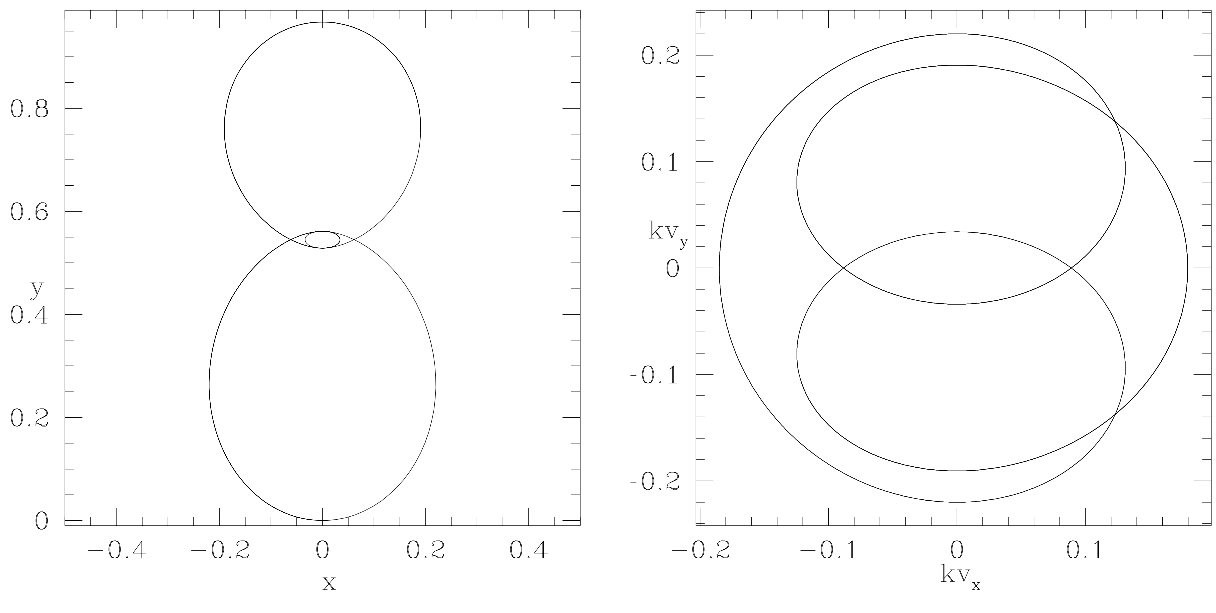
$$0 = -2J_1(k\mu) - J_2(k\mu) + J_0(k\mu), \qquad k\mu = 0.825.$$

$$\lambda = 0, \qquad \beta = -1.62k\Phi_0.$$

Fixed points to lowest order independent of Φ_0
$k\rho = k(\mu - \omega\beta), \qquad \psi = 0$
$k\rho = k(\mu + \omega\beta), \qquad \psi = \pi,$ **agree with numerical**

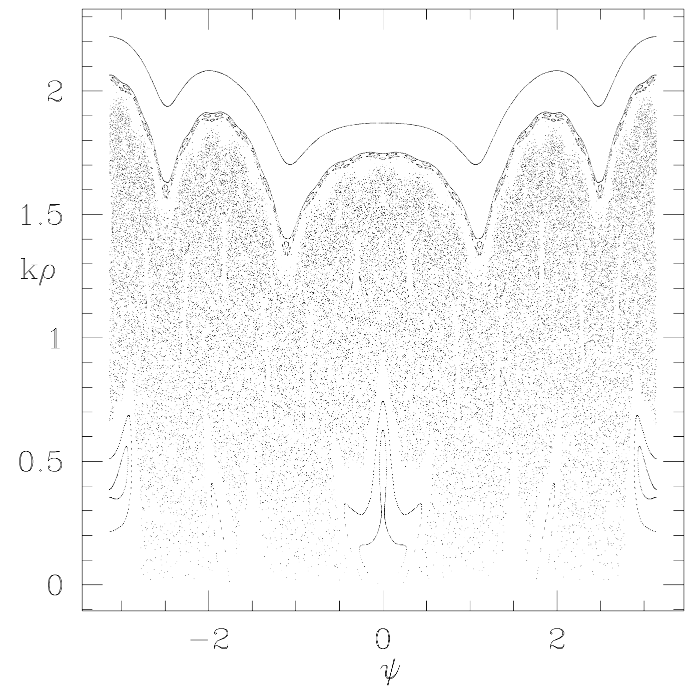


Poincare, $k^2\Phi_0 = 0.1$, $\omega = 1/3$

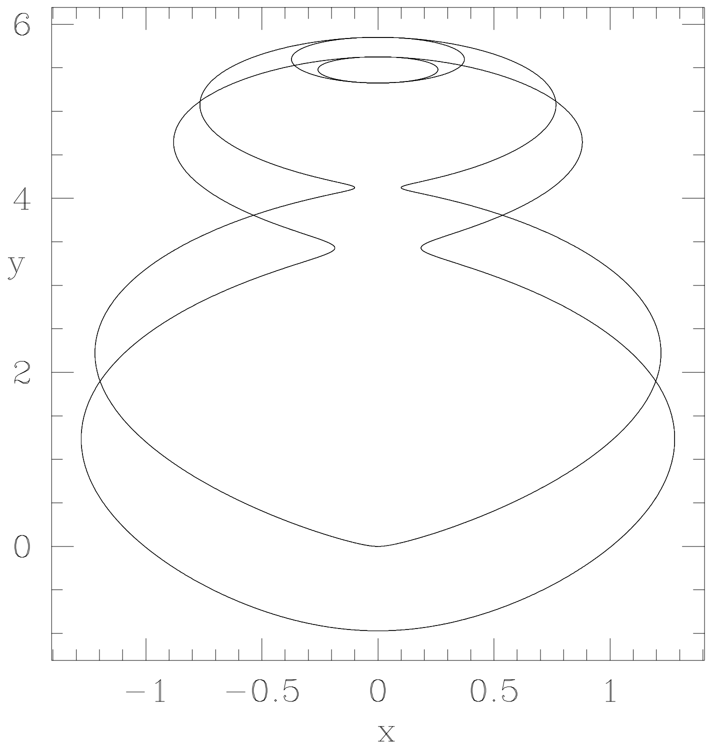


Fixed point orbit, $k^2\Phi_0 = 0.1$, $\omega = 1/3$

Qualitatively different, $k\rho \to 0$ for $\Phi_0 \to 0$



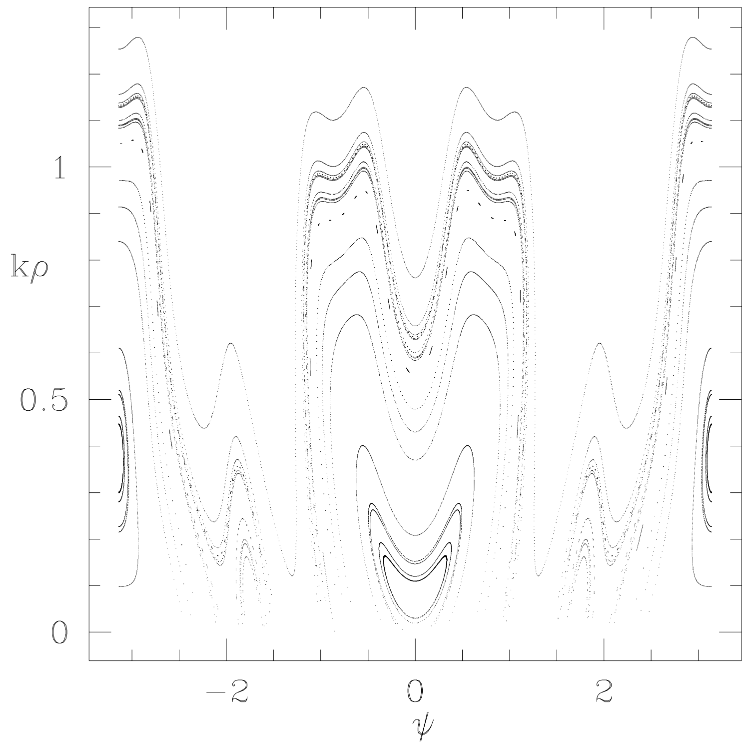
Poincare, $k^2\Phi_0 = 0.77$, $\omega = 1/4$, Good KAM large $k\rho$



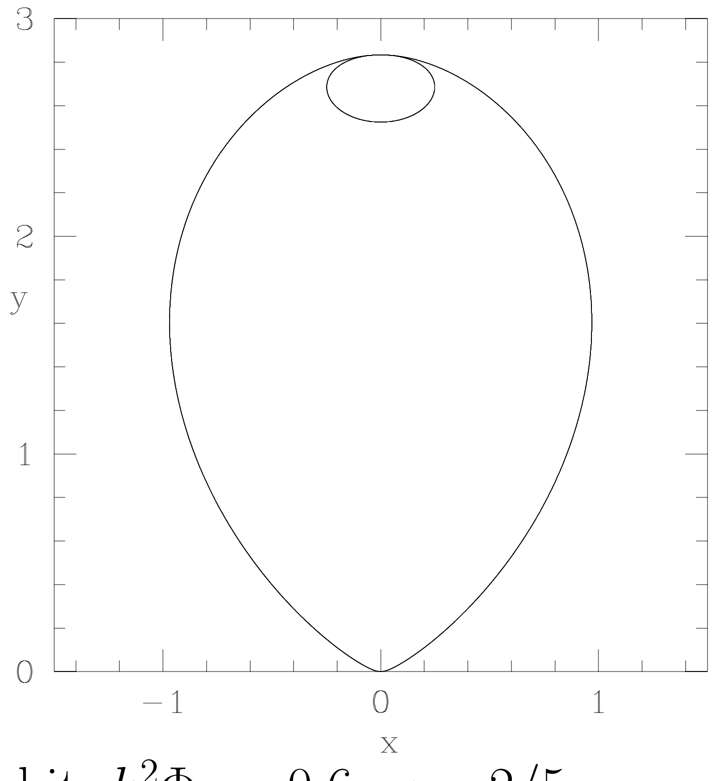
Fixed point orbit, $k^2\Phi_0 = 0.77$, $\omega = 1/4$

With additional perturbation fixed points change position

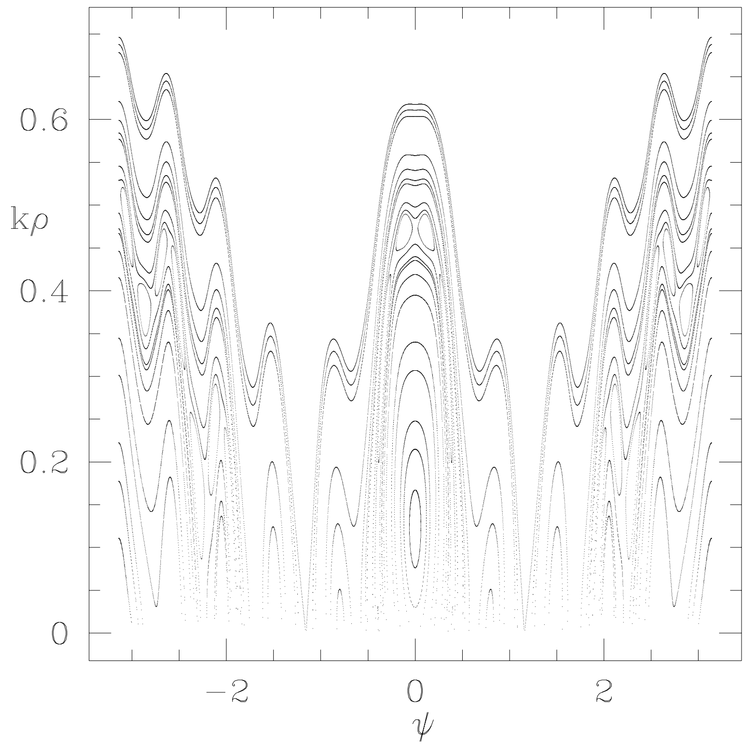
But existence is robust.



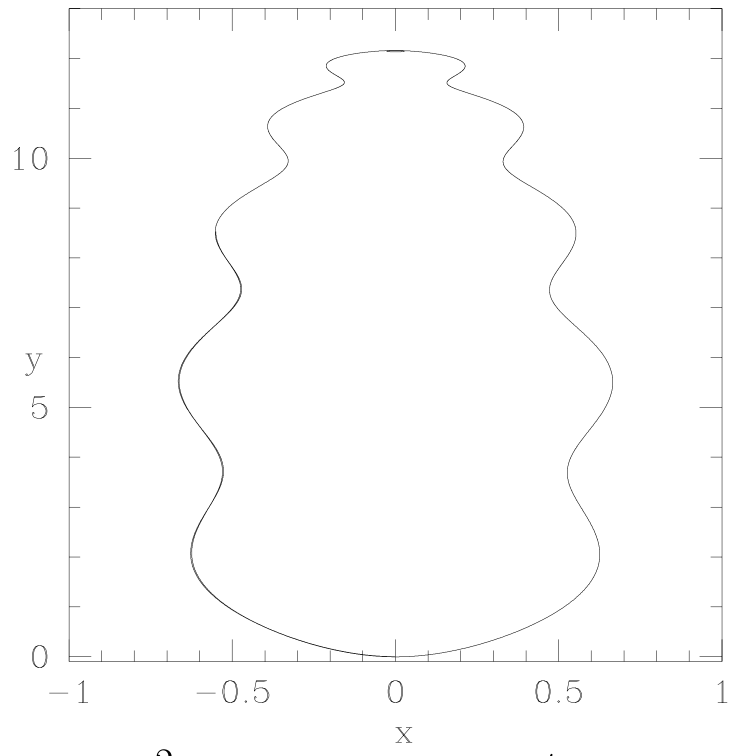
Poincare, $k^2\Phi_0 = 0.6$, $\omega = 2/5$



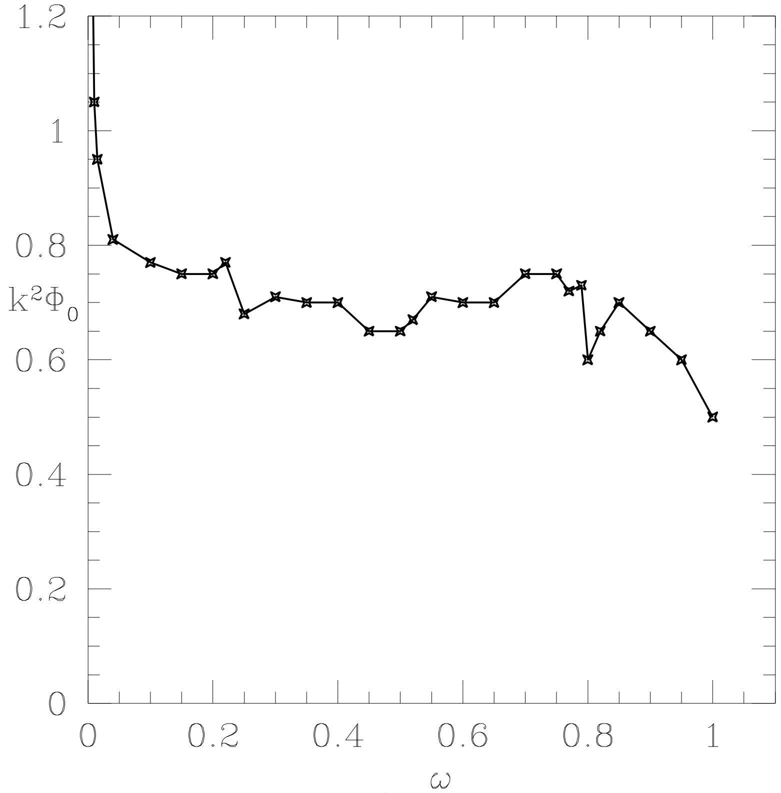
Fixed point orbit, $k^2\Phi_0 = 0.6$, $\omega = 2/5$
Mathiew resonance q = 5, with 2.5 gyro periods per wave



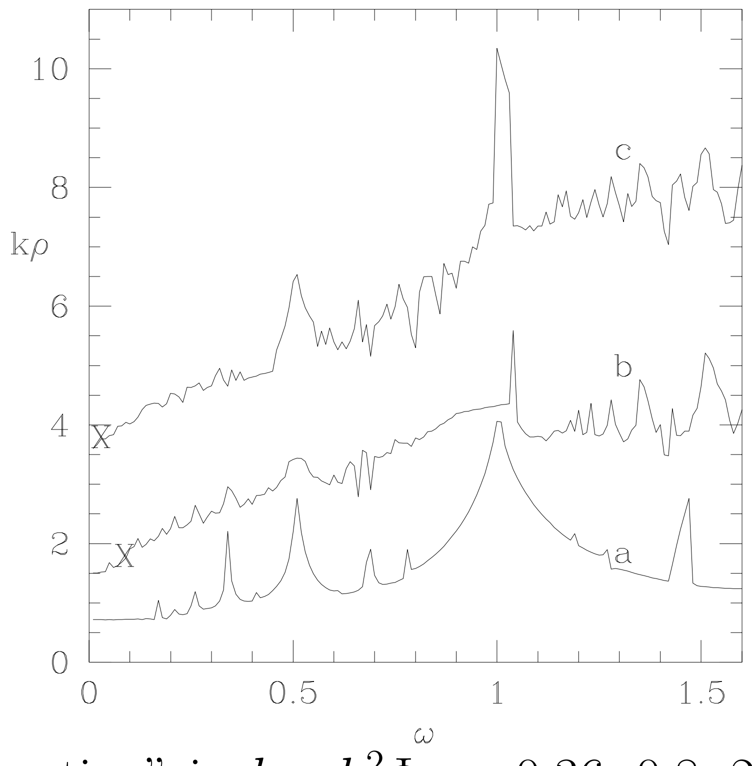
Poincare, $k^2\Phi_0 = 0.6$, $\omega = 1/10$



Fixed point orbit, $k^2\Phi_0 = 0.6$, $\omega = 1/10$



Threshold for chaos in the $k^2\Phi_0$, ω plane.



Extent of "heating" in $k\rho$, $k^2\Phi_0 = 0.36,\ 0.8,\ 2.6$.

Hamiltonian formalism for low frequency Follow Lichtenberg and Lieberman

$$H_0 = \frac{(\vec{p} - \vec{A})^2}{2}$$

$\vec{B} = B\hat{z}$, vector potential $\vec{A} = \Omega_c x \hat{y}$
variables x, y, p_x, p_y

Transform to guiding center variables ϕ, X, P_X, P_ϕ using

$F_1(q, Q, t) = \Omega_c[\frac{1}{2}(x - X)^2 cot\phi + yX]$

$p = \frac{\partial F_1}{\partial q}, \qquad P = -\frac{\partial F_1}{\partial Q}$

$p_x = \Omega_c(x - X)cot\phi, \qquad p_y = \Omega_c X$

$tan\phi = \frac{v_x}{v_y}, \qquad P_\phi = \Omega_c \rho^2/2, \qquad P_X = -\Omega_c Y$

$y = Y + \rho cos\phi, \qquad x = X + \rho sin\phi,$

$$H_0 = P_\phi \Omega_c$$

Now introduce perturbation

$H = H_0 + \Phi_0 sin(kx - \omega t).$

Transform $X, P_X \to \psi, P_\psi$ using

$F_2(q, P, t) = (kX - \omega t)P_\psi + P_\phi \phi$

$p = \frac{\partial F_2}{\partial q}, \qquad Q = \frac{\partial F_2}{\partial P},$

$P_X = kP_\psi, \qquad P_\phi = P_\phi, \qquad \psi = kX - \omega t, \qquad \phi$

$$H = P_\phi \Omega_c - P_\psi \omega + \Phi_0 sin(\psi + k\rho sin\phi)$$

$$H = P_\phi \Omega_c - P_\psi \omega + \Phi_0 \Sigma_m J_m(k\rho) sin(\psi + m\phi)$$

variables $\psi, \phi, P_\psi, P_\phi$

Unperturbed frequencies

$\omega_\phi = \partial_{P_\phi} H_0 = \Omega_c, \qquad \phi \sim \Omega_c t$

$\omega_\psi = \partial_{P_\psi} H_0 = -\omega \qquad \psi \sim -\omega t$

First order perturbation gives resonances,

secularity of the perturbation when $\psi + m\phi = constant$

$\omega - m\Omega_c = 0$

Standard resonant heating analysis, cyclotron harmonics

But if Φ_0 is large, this is incomplete

To obtain higher order in Φ_0, again transform $X, P_X \rightarrow \psi, P_\psi$ using $F_2(q, P, t)$ as before

Equation for ψ

$$\psi = \psi_0 + k^2\Phi_0 \langle sin(\psi + k\rho sin\phi)\rangle$$

$$\psi_0 = kx_0 - \omega t$$

Iterate in powers of $k^2\Phi_0$

$$\begin{aligned} H_2 = H_0 &+ \Phi_0 \Sigma_m J_m(k\rho) \langle sin(\psi_0 + m\phi)\rangle \\ &+ \tfrac{k^2\Phi_0^2}{2}\Sigma_{mn} J_m(k\rho) J_n(k\rho) \langle sin(2\psi_0 + m\phi + n\phi)\rangle \\ &+ \tfrac{k^2\Phi_0^2}{2}\Sigma_{mn} J_m(k\rho) J_n(k\rho) \langle sin(m\phi - n\phi)\rangle \end{aligned}$$

Unperturbed frequencies

$$\omega_\phi = \partial_{P_\phi} H_0 = \Omega_c, \qquad \phi \sim \Omega_c t$$
$$\omega_\psi = \partial_{P_\psi} H_0 = -\omega \qquad \psi_0 \sim -\omega t$$

H_2 terms with $m + n = 1$ give secularity at $2\omega = \Omega_c$
finite kx_g introduces higher harmonics of ω, $N\omega = \omega_c$

To find the secularities at $\omega = \Omega_c/q$ must perform q iterations, the Hamiltonian will become very complicated. Furthermore this is a $\omega \ll \Omega_c$ approximation, cannot reproduce $\omega = \Omega_c/2$ exactly.-

Alfven wave Heating

Chen, Lin, White, – Phys Plasmas Nov. 2001
magnetic shear- J. Johnson and F. Cheng - AGU 1997

Linearly polarized Alfvèn wave laboratory frame X, Y, Z,
$\vec{B}_w = B_w \hat{y} cos(\psi)$ with $\psi = \vec{k} \cdot \vec{X} - \omega t$
$\vec{B} = B_0 \hat{z}$.

Ions cold in the laboratory frame $\omega = k_z v_A = k_z v$

$v =$ velocity in the wave frame x, y, z with $z = Z - v_A t$.

wave frame : $\psi = k_x x + k_z z$ and the velocity $v = v_A = const$

Dimensionless $\quad k_x v, \quad k_z v = \omega/\Omega_c, \quad B_w/B_0.$

Equations of motion

Wave frame $d\vec{v}/dt = \vec{v} \times (\vec{B}_0 + \vec{B}_w)$

$$\dot{v}_x = v_y - v_z B_w cos\psi, \quad v_y = x_0 - x, \quad \dot{v}_z = v_x B_w cos\psi$$

$$\frac{d^2x}{dt^2} + x = x_0 - v_z B_w cos\psi.$$

First order in B_w $\qquad d^2x/dt^2 + x = x_0 - v_z(0) B_w cos\psi.$

$\psi = k_x x + k_z z \qquad$ and $\qquad z = v_z(0)t$

To lowest order the equations are the same as the electrostatic problem

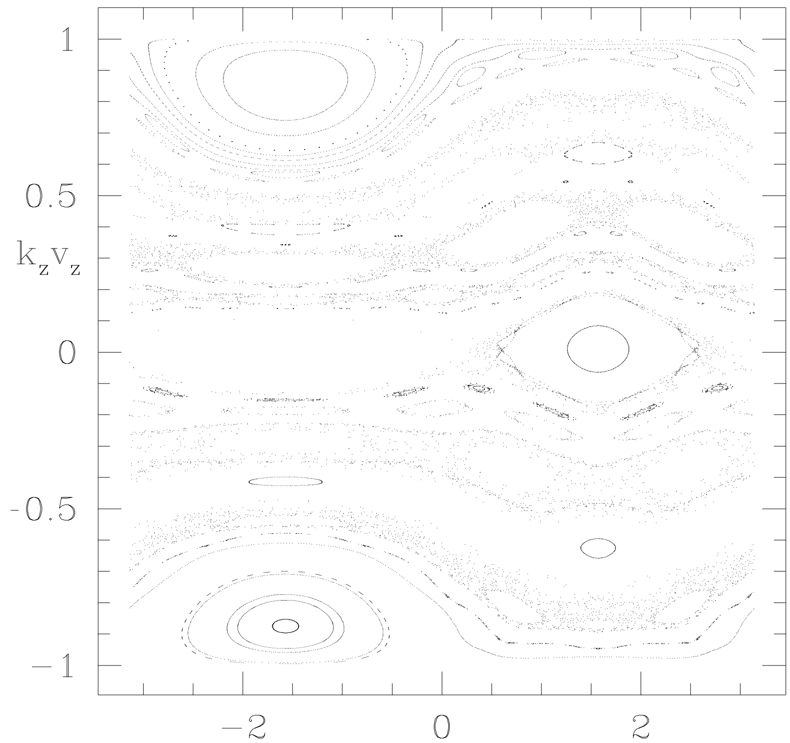
$\omega/\Omega_c = k_z v_z(0)/\Omega_c$ - Wave frequency in lab frame

$k_x v_z(0) B_w/(B_0\Omega_c)$ - nonlinearity parameter. $(k^2\Phi_0)$ related to kx_g wave induced curvature drift

For small excursions again find Mathieu equation- unstable solutions for $\omega \simeq 2/N$, $k_x \neq 0$.

For small B_w -same resonances as found in the analysis of electrostatic heating, $k_x \neq 0$.
But to higher order in B_w more complicated

Poincaré section of λ, ψ, - points when $v_y = 0$ and $\dot{v}_y > 0$



Poincaré, $B_w = 0.22$, $k_x v = 0.1$, $k_z v = 1.0$.
Symmetry, linear polarized wave = left + right.
Chaos produced by Chirikov island overlap

Left hand circularly polarized Alfven wave.

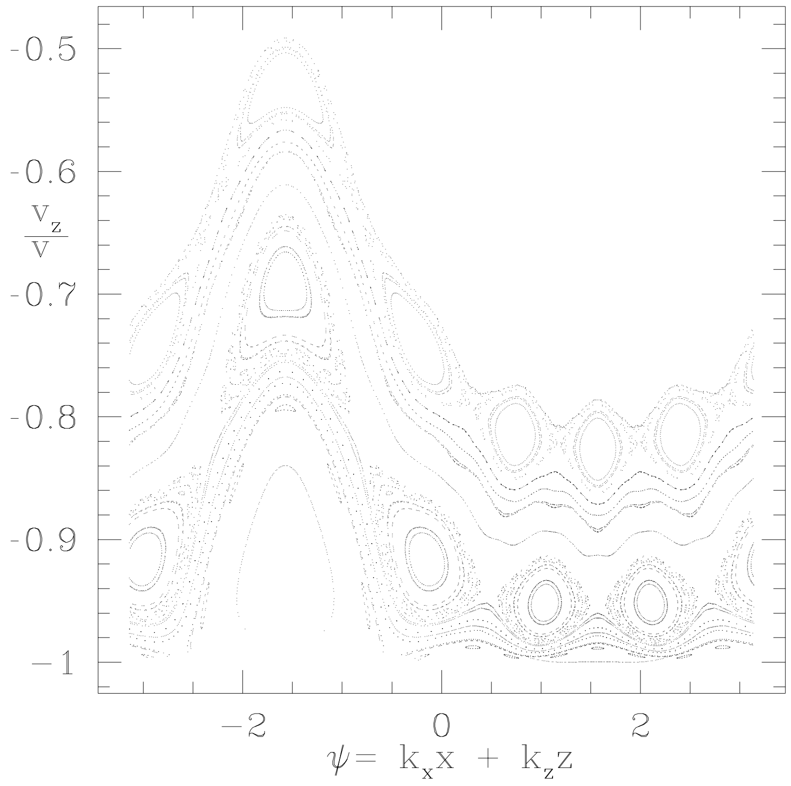
Wave frame

$\vec{B}_w = -B_w \hat{x} cos(\alpha) sin(\psi) + B_w \hat{y} cos\psi + B_w \hat{z} sin(\alpha) sin(\psi)$

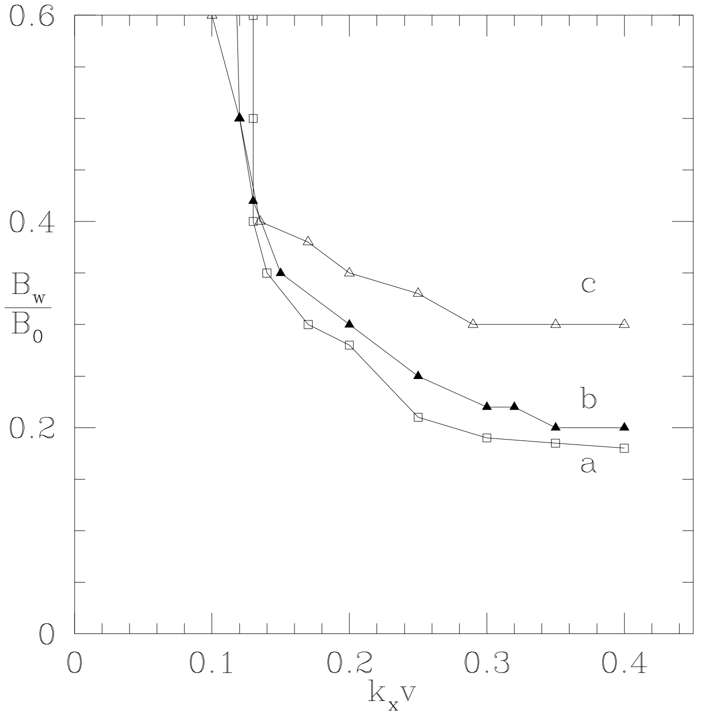
$\psi = k_x x + k_z z$

$tan(\alpha) = k_x / k_z.$

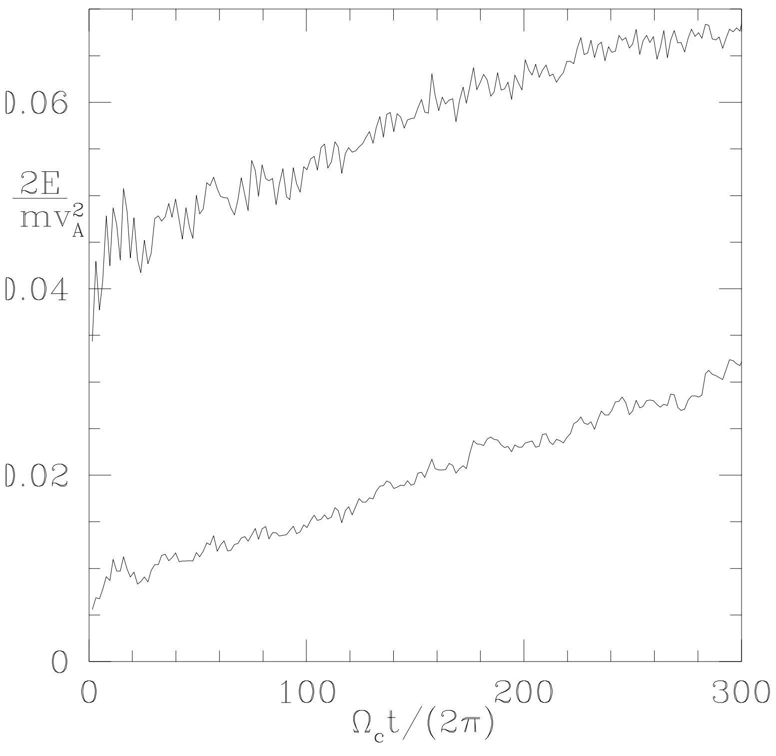
laboratory frame - wave propagates to positive z.
Ions cold in the lab frame are at $v_z/v = -1$ and $v = v_A$



Poincaré, $B_w = 0.25$, $k_x v = 0.16$, $\omega = 0.25$.
$\delta B_x / B = 0.2$, $\delta B_y / B = 0.25$, $\delta B_z / B = 0.13$, $k_z = 1$



Stochastic threshold, (a) $\omega = 0.5$, (b) $\omega = 0.25$, (c) $\omega = 0.1$
Note threshold is at $k_x v B_w \simeq 0.1$, fairly low



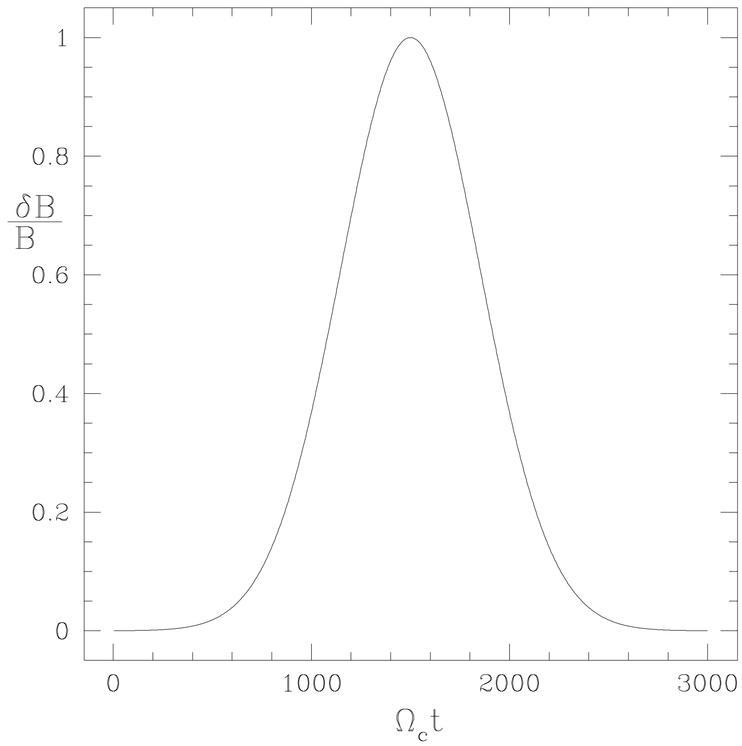
$E_\perp$ (larger) and $E_\parallel$ vs t, $B_w = 0.25$, $k_x v = 0.27$, $\omega = 0.25$.

Wave Spectrum of many modes

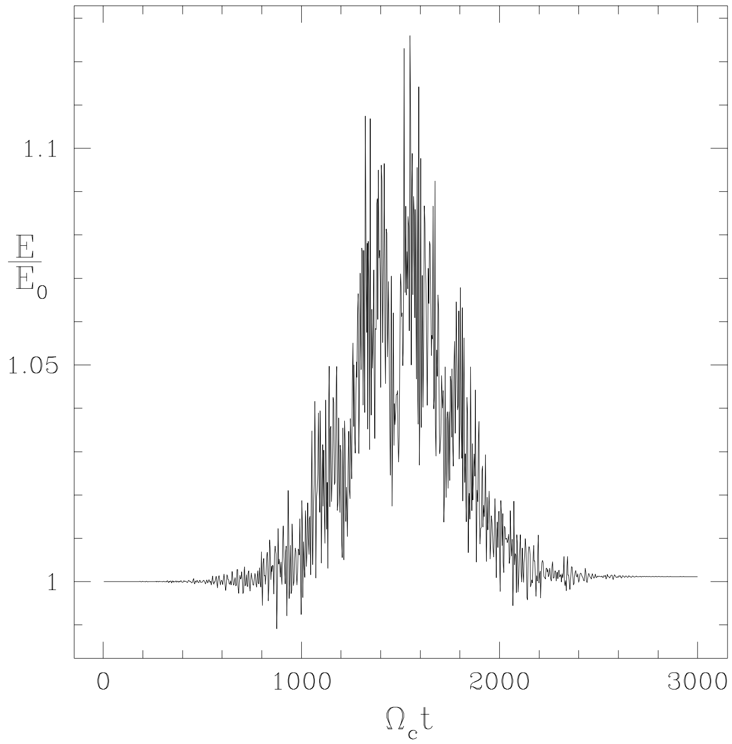
No longer possible to use Poincare plot to examine Chaos

Lyapunov exponent measurement numerically difficult because of cyclotron orbits

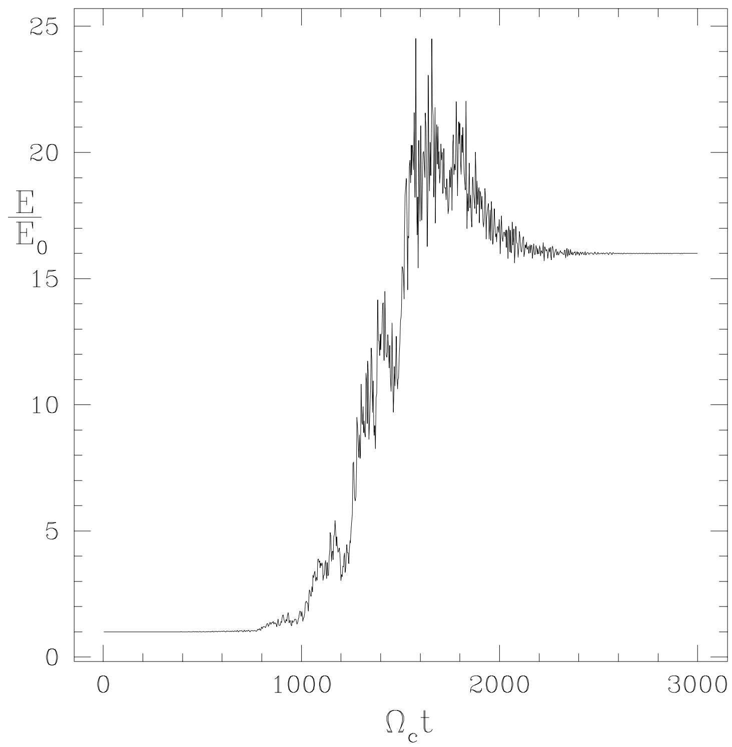
Classical test of adiabaticity of magnetic moment μ
particle passes through a region of field perturbation
emerges with $\mu(t=-\infty)=\mu(t=\infty)$



Time dependence of field perturbation



21 modes, $0.2 < \omega < 0.7$, average $\delta B/B = 4 \times 10^{-3}$.



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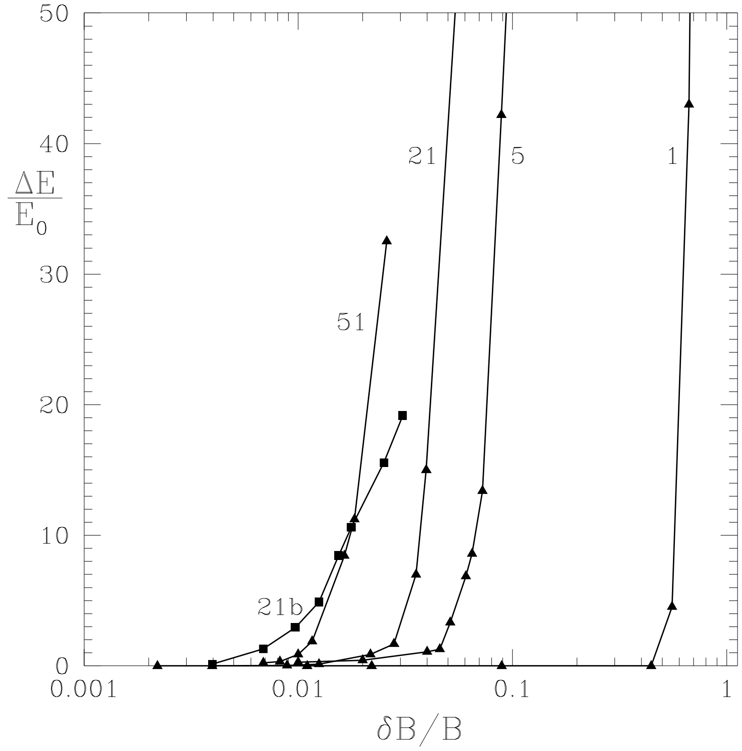


Spectrum of modes, $0.2 < \omega < 0.7$,
$\Delta E/E$ vs $\delta B/B$ using Gaussian time perturbation
For 1, 5, 21, 51 modes with $\delta B \sim A/\omega$ and $k_x v_A = 1$
Plus spectrum with 21 modes with $k_x v_A = 10$
ΔE high power in δB above threshold

Possible Applications

Heating in the Solar Corona

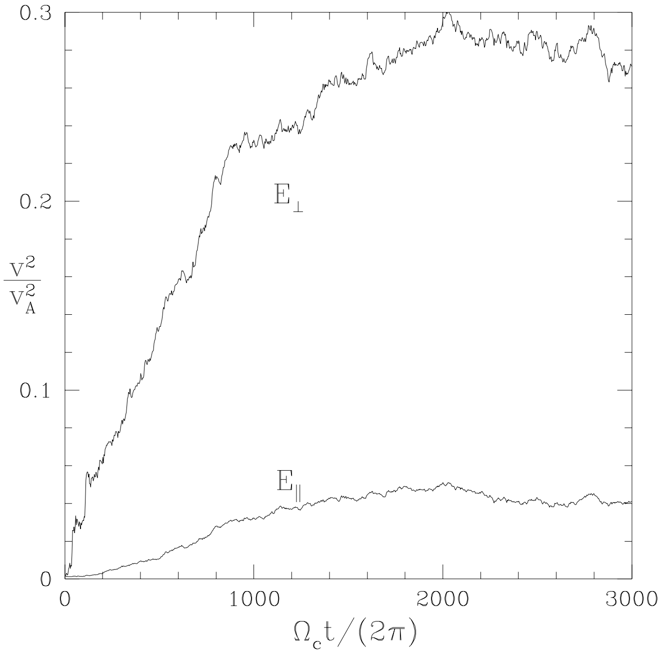
Acceleration region of Solar wind $\sim 3\ R_{\odot}$
Heating to a significant fraction of the Alfven speed
Heat in a few hundred cyclotron periods. Mostly $E_{\perp}$
Preferential heating of heavier mass ions.

Heating by compressional Alfven waves in NSTX

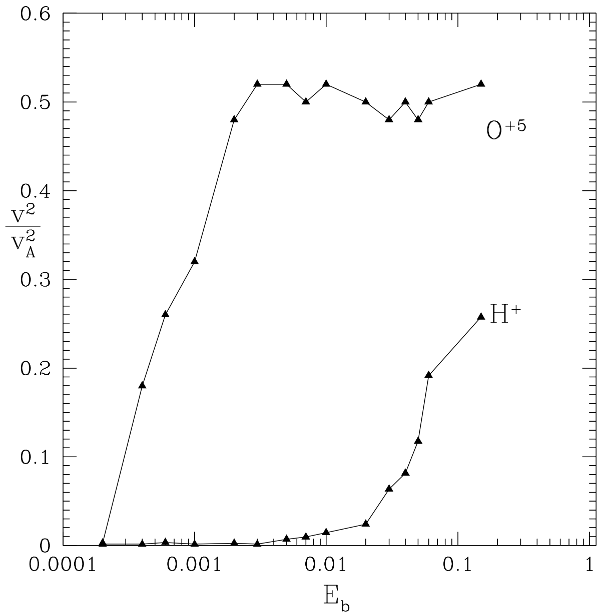
Beam or Alphas $\rightarrow$ Alfven waves $\rightarrow$ ions

Solar Corona, $\delta B_k = A/\omega_k$ **with** $E_b = < \delta B^2 >$

Spectrum given by solar ultraviolet -
Cranmer, Field, Kohl – Astrophys. J. 518, 937 (1999)



O^{+5} heating, $\quad .02 < \omega/\Omega_p < .7, \quad .06 < \omega/\Omega_{O^5} < 2,$
51 modes, equal spacing in ω, $E_b = 10^{-3}$, $k_x v_A = 1$



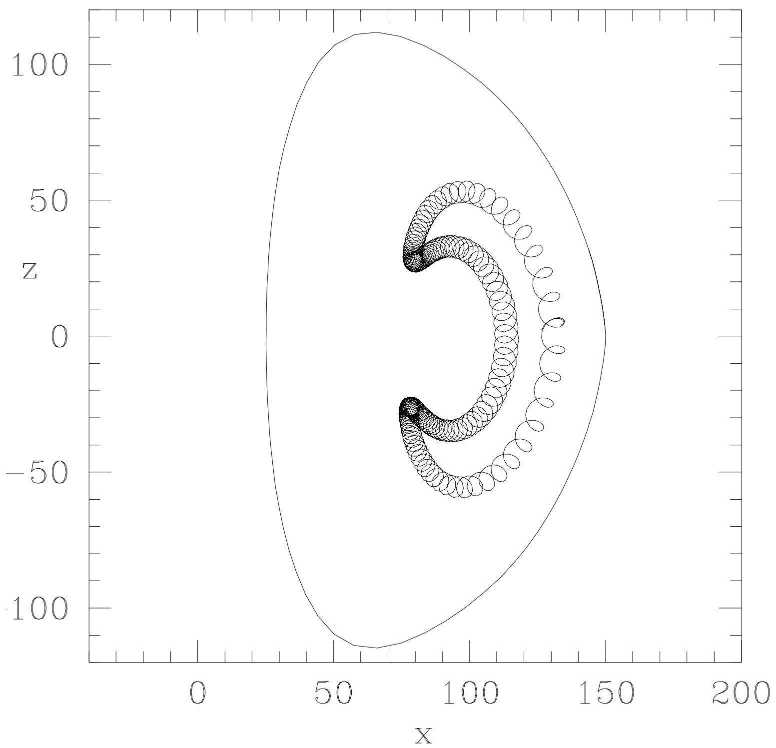
Heating for H^+ and O^{+5}, saturation when $k_x \rho_{O^{+5}} \simeq 1$

NSTX- Toroidal geometry simulation

(Gates, Gorelenkov, White, slab model PRL Nov. 01)

- **Mechanism - channelling Fisch, Hermann 1994**
- Excitation of compressional Alfven waves by super Alfvenic beam particles. (Gorelenkov Cheng Nucl. Fus. 1995)
- Waves are below ω_c because of $k_{||}$ Doppler shift.
- Small Landau damping because of β_e.
- Modes grow to threshold for stochastic heating of bulk ions
- Stochastic domain at low energy, beam not affected

• Numerical simulation in toroidal geometry using NSTX equilibria, Constant mode amplitude, Pitch angle scattering included with $\nu = 0.01\omega_c$



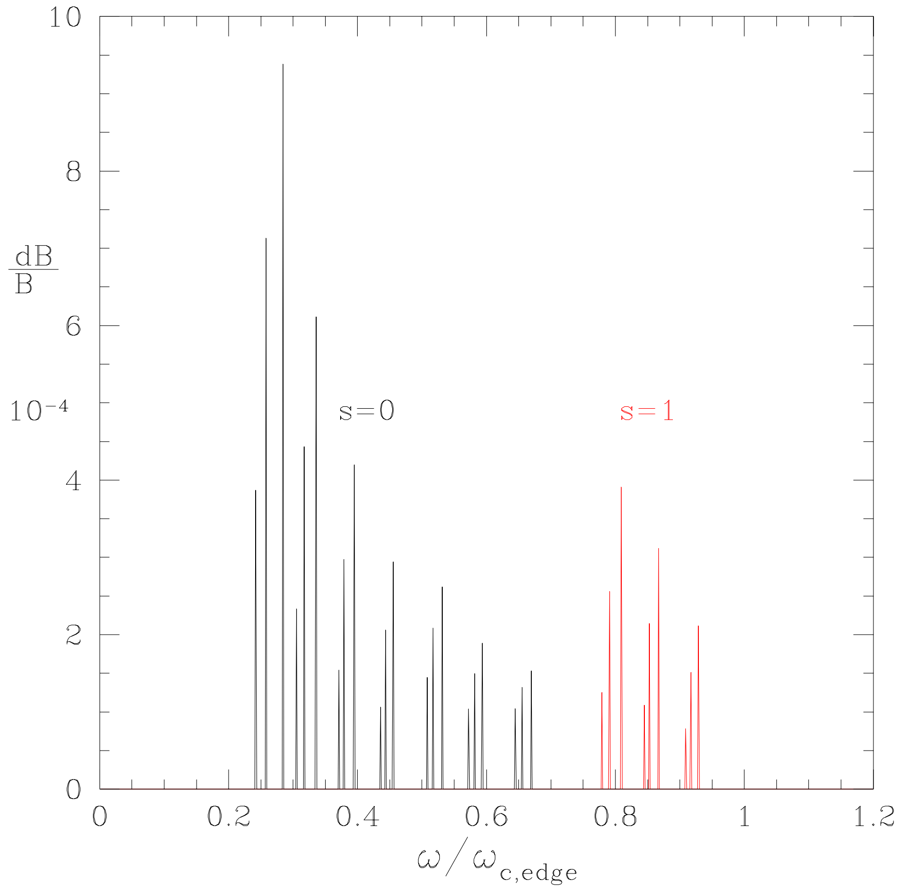
Sample Orbit in NSTX

- The mode spectrum was chosen to match local analytical analysis and the experimental data.

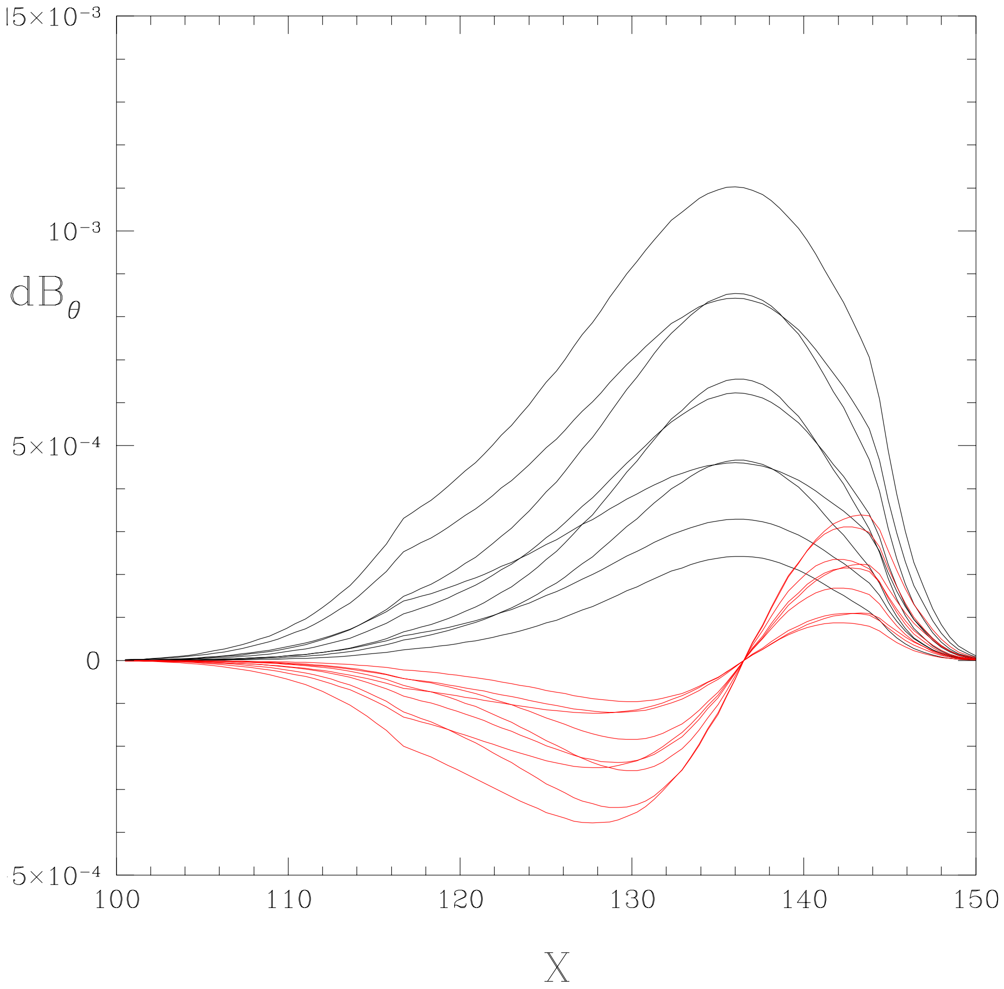
$$E_\theta = \Sigma_{mns} E_0(\omega)\phi_m(\frac{\sqrt{2}\theta}{\Theta})\phi_s(\frac{\sqrt{2}(r-r_0)}{\Delta})e^{i(n\phi-\omega t)}$$
$$E_r = -i\omega_c E_\theta/\omega$$
$$\partial\vec{B}/\partial t = -\nabla\times\vec{E}$$

$$\omega = c_0 + c_1 m + c_2 s + c_3 n^2/(2m+1)$$

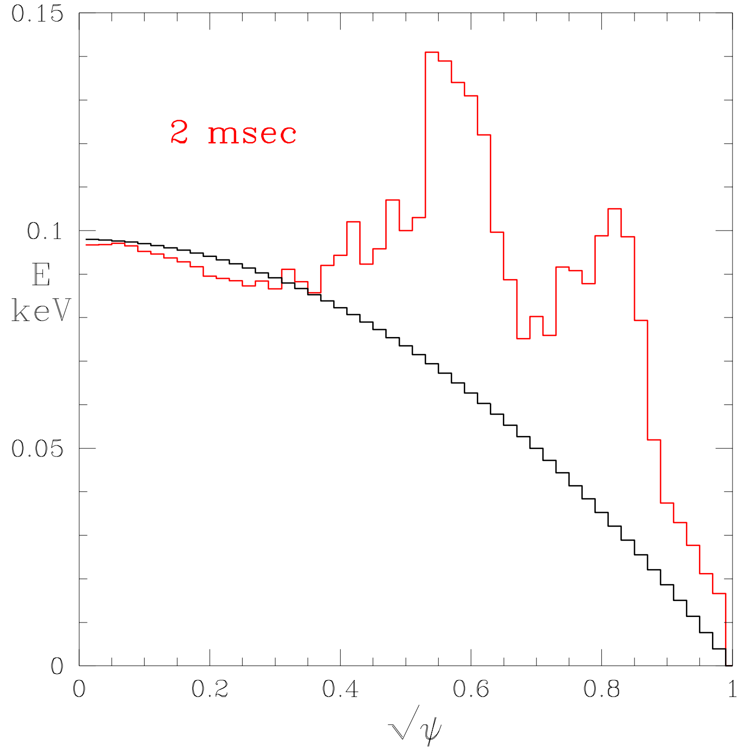
- Constants c_k geometry dependent given approximately by cylindrical model but adjusted to match experiment.



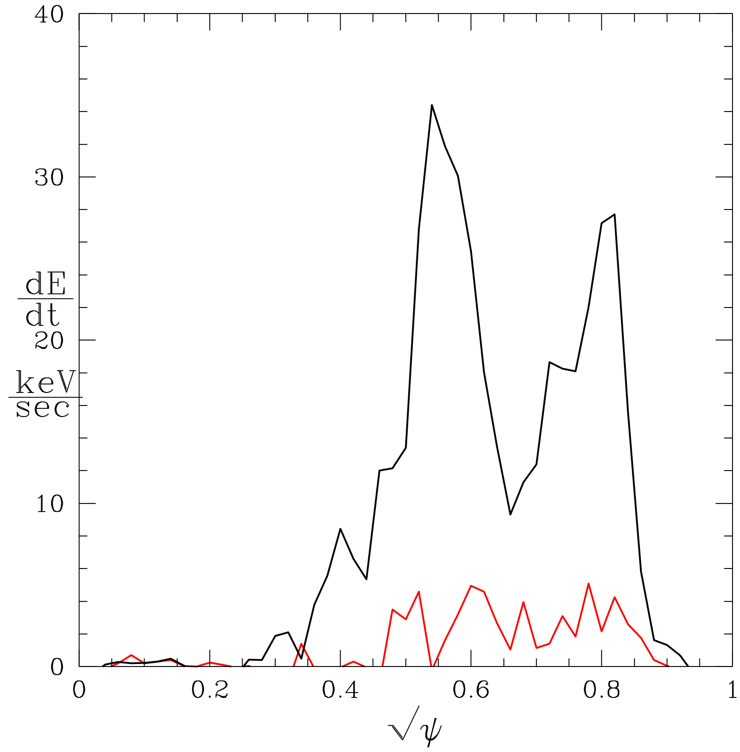
Mode Spectrum



Mode Shape - $\delta B_\theta / B$ at $\theta = 0$.



Energy change at $t = 2msec$, $\delta B/B = 5 \times 10^{-4}$



NSTX heating rate, $\delta B/B = 2 \times 10^{-4}$ and $\delta B/B = 5 \times 10^{-4}$

Conclusion

- Significant perpendicular heating can be obtained with a single wave well below the cyclotron frequency.
- Stochastic threshold almost independent of frequency.
- The case of a simple electrostatic wave propagating perpendicular to $\vec{B}$ provides a textbook example which displays the general features common to all large amplitude low frequency waves.
- Stochastic threshold much lower for a many-mode spectrum Modes above ω_c assist nonlinear coupling.
- Possible astrophysical applications- Alfvèn wave heating in the solar corona
- Possible NSTX ion heating- Direct channelling of beam energy to heat ions.
  - Spectrum
    modes above ω_c lower threshold, mode number
    Amplitude and shape matter for heating profile
  - Energy flow
    Beam energy loss
    Mode growth rate
    stochastic heating rate
- Result appears to be of very general nature