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IMAGE METHOD FOR THE DERIVATION OF POINT SOURCES IN ELASTOSTATIC PROBLEMS WITH PLANE INTERFACES

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Abstract

This report presents an image method algorithm for the derivation of point sources of elastostatics in multi-layered media assuming the infinite space point source is known. Specific cases have been worked out and shown to coincide with well known solutions in the literature.

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Introduction

The importance of point sources (Green's functions are point sources) for some given (linear) governing differential equations and boundary conditions lies in two main reasons. First, any localized process when viewed from a sufficient distance can be modelled as some suitably chosen point sources. Second, Green's functions can be used to reframe the governing differential equations and boundary conditions in an integral equation form; the integral equation form can, for example, be used as the basis for numerically analyzing a large class of problems using the boundary element method.

This paper presents an algorithm for the derivation of point sources of elastostatics in multi-layered media assuming the point source in infinite space is known. The method is similar to the image method that is familiar when deriving Green's functions in plane layered media where there is only one unknown scalar field in the governing equations such as in temperature conduction, potential flow and electrostatics problems. The algorithm is then used to derive the Green's functions for any point source in a region consisting of an elastic layer perfectly bonded to two elastic halfspaces.

Background

There are many known Green's functions for halfspace problems in elastostatics. Most of the known Green's functions are specialized for a single halfspace having a stress free surface (a special case of bonded elastic halfspaces when one of the regions has zero rigidity). We will first briefly survey some of these known solutions with occasional comments on the method of derivation, then we will discuss some of the analytic methods used to derive Green's functions in multilayered halfspaces, and mention a few known results for layered systems with some comments.

The most used point source solutions are the point force, the dislocation and the nuclei of strain (or double couple) solutions. The point force solution for 2-D plane problems in a halfspace with a free surface (Mellan 1932), 3-D problem in a halfspace with a free surface (Mindlin 1936) and 3-D problem in bonded elastic halfspaces (Rongved 1955) are known. Rongved obtained the Green's function through the use of the Papkovich-Neuber potentials and arguments from harmonic analysis; the resulting solution is in the form of the sum of a point force solution in infinite space and some point sources at the image point with respect to the interface plane.

The screw dislocation (e.g. Rybicki 1971) and edge dislocation (e.g. Freund and Barnett (1976a,b) presented it in a convenient form) for a halfspace with a free surface and a general dislocation line intersecting a free surface (Yoffe 1961) are also known. The screw dislocation problem is obtained by the method of images (since there is only one field variable), while Freund and Barnett solved the edge dislocation problem through the use of complex analysis and the Mushkelishvili potentials.

There are six nucleii of strain sources. The solution to the first (double couple in a plane parallel to the free surface) was given by Steketee (1958), the remaining five sources were given by Maruyama (1964). Maruyama used image nucleii of strain sources to cancel the tangential component of the surface traction on the free surface. He then used the Boussinesq solution (in Galerkin vector representation) and the remaining normal tractions on the free surface in a Hankel/Fourier transformed space to obtain the rest of the fields afterwhich he transformed the solution back to real space. This procedure is highly specific to half space problems with a free surface and cannot be generalized to multiple layered systems. We will next consider the known techniques to systematically treat sources in multiple layered media.

A systematic formulation for the derivation of the fields due to 3-D sources in a layered halfspace was presented by Ben-Menahem and Singh (1968) and later refined by Singh (1970) and independently by Sato (1971). The formulation makes use of the analogue of Hansen's eigenvector expansion for electromagnetic problems (1935) applied to elastostatic and dynamic problems, combined with the Haskell-Thompson transfer matrix technique and the Pekeris (1955) 'source condition' at the level of the discontinuity. The formulation leads to a solution for the field variables of the form:

$$\sigma_{ij},\ u_i \approx \int_0^{\infty} \frac{\sum_{s=0}^{m} \alpha_s k^s \exp(\beta_s kz)}{\sum_{q=0}^{p} \gamma_q k^q \exp(\delta_q kz)} J_p(kr)\, dk \qquad (1)$$

where:

α_s, β_s, γ_q, δ_q	are constants dependent on the indices s and q
J_p	is Bessel's function of the first kind and of order p
r	is the radial (cylindrical) coordinate
z	is the z cylindrical coordinate
k	is an integration variable

In the above integrals, α_i's and γ_i's may also depend on both the elastic properties of the half space and the layer.

Sato and Matsu'ura (1973) and Jovanovich et al. (1974 a,b) made use of the above mentioned formulation to calculate surface deformations by numerically integrating the ensuing expressions. Such evaluations require special numerical techniques and significant amounts of computational effort. Furthermore, no field values were computed inside the layer or halfspace. To accomplish such evaluations requires significantly more effort (both in further algebraic manipulations and in computations and special numerical treatments); this observation is especially true for field points close to the source point. The importance of having the field variables being available everywhere occurs when a boundary element/integral equation formulation for processes occuring in a region consisting of such multi-layered media is required.

A formulation in the same spirit as the Ben-Menahem and Singh formulation for multi-layered 2-D problems using the Airy stress function was presented by Singh and Garg (1985). Although the original formulation is applicable to 3-D, 2-D and antiplane problems, the specialized 2-D formulation is less complex.

Simpler but more specialized point source solutions in multi-layered media are available. For example, Rybicki (1971) presented the solution to a screw dislocation (can be specialized from his expressions) in a region consisting of an elastic layer with a free surface and perfectly bonded to an elastic halfspace. Rybicki used the method of images to derive his solution. Rundle and Jackson (1977) presented an approximate solution for a 3-D double couple source parallel to the free surface in a region consisting of an elastic layer perfectly bonded to an elastic halfspace. The approximate solution was obtained by using Steketee's 1958 solution combined with the use of an image method (using Rybicki's technique) on the "antiplane" part of the point source being considered.

Rundle and Jackson compared their solution for the surface deformations to that obtained by the direct integration of the improper integrals of Jovannovich et al. and found errors up to 14% while varying the thickness of the plate and the location of the source; however, the errors are less than 5% for the parameteric values they need in their specific application. We note that they take the rigidity contrast between the halfspace and elastic layer to be 10 to 1, a choice that would favor their approximation.

Rundle and Jackson also obtained the (approximate) response of the point source they considered with a viscoelastic instead of an elastic halfspace through the use of the correspondence principle. The direct application of the correspondence principle to their elastic solution is possible because the material parameters are kept seperate from the geometric coordinates (in the form of a sum of material parameters multiplied by a function of the coordinates). This procedure is not directly applicable to the solution presented in the form of improper integrals (1) because the material parameters and the geometric coordinates are intermixed in the denominator of the integrand.

Finally, we mention the existence of an image method for perfectly bonded elastic halfspaces in terms of the Papkovich-Neuber potentials (Aderogba 1977). Aderogba presents the algorithm for obtaining the four image potentials which involves multiple integrations with respect to the coordinate perpendicular to the interface plane and differentiation with respect to all three coordinates. The algorithm we present in this paper involves 3 potentials only, and only differentiation of the potentials with respect to the coordinate perpendicular to the interface plane (as well as multiplication by scalars) is required. This distinction is especially important while contemplating the repeated use of the image algorithm to obtain the fields due to point sources in regions consisting of an elastic layer perfectly bonded to two elastic halfspaces.

Preliminary Considerations

The image method we consider here is dependent upon expressing the displacements in terms of potentials. The specific potentials we use are the analogue to Hansen's potentials for elastostatics and dynamics. Unlike Ben-Menahem and Singh (1968) we do not expand the potentials in terms of eigenfunctions, instead the algorithm operates directly on the potentials. Note however, that the derivation of the algorithm makes use of the eigenfunction expansion technique (see Appendix 3).

Specifically, we express the displacement field in terms of the Hansen potentials φ_1, φ_2 and φ_3 in the following way:

$$\underline{u}(h,\delta,\underline{\varphi}_R,\underline{\varphi}_L) = \underline{N}(h,\varphi_1) + \underline{F}(\delta,h,\varphi_2) + \underline{M}(h,\varphi_3)$$

$$\underline{\varphi}_R = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} \qquad \underline{\varphi}_L = \begin{bmatrix} \varphi_3 \end{bmatrix}$$

$$\underline{N}(h,\varphi_1) = \nabla\varphi_1(x,y,z-h) \tag{2}$$

$$\begin{aligned} \underline{F}(\delta,h,\varphi_2) = &+ 2\cdot\hat{e}_z\cdot\frac{\partial}{\partial z}\varphi_2(x,y,z-h) \\ &- \nabla\varphi_2(x,y,z-h) \\ &- 2\cdot\delta\cdot(z-h)\ \nabla\frac{\partial}{\partial z}\varphi_2(x,y,z-h) \end{aligned}$$

$$\underline{M}(h,\varphi_3) = \nabla \times [\hat{e}_z\varphi_3(x,y,z-h)]$$

where: ∇ is the gradient operator

$\nabla \times$ is the curl operator

$\nabla^2 \varphi_1 = \nabla^2 \varphi_2 = \nabla^2 \varphi_3 = 0$

$\delta = \dfrac{\lambda + \mu}{\lambda + 3\mu}$

λ is the Lame constant

μ is the shear modulus

h is a scalar for shifting the z-coordinate

Note that the potentials φ_1, φ_2 and φ_3 have to be harmonic in order for $\underline{N}$, $\underline{F}$ and $\underline{M}$ to satisfy equilibrium. The Cartesian components for the displacements, strains and stresses are given in Appendix 1.

In order for these potentials to be useful for our purpose, we describe how to obtain these potentials given an elastic field satisfying equilibrium. We note the following:

$$
\begin{aligned}
&\nabla \cdot \underline{N} = \nabla \cdot \underline{M} = 0 \qquad \nabla \cdot \underline{F} = 2 \cdot (1-\delta) \cdot \frac{\partial^2 \varphi_2}{\partial z^2} \\
&\nabla \times \underline{N} = 0 \\
&\nabla \times \underline{F} = 2 \cdot (1+\delta) \cdot \frac{\partial^2 \varphi_2}{\partial y \partial z} \cdot \hat{e}_x - 2 \cdot (1+\delta) \cdot \frac{\partial^2 \varphi_2}{\partial x \partial z} \cdot \hat{e}_y \\
&\nabla \times \underline{M} = \frac{\partial^2 \varphi_3}{\partial x \partial z} \cdot \hat{e}_x + \frac{\partial^2 \varphi_3}{\partial y \partial z} \cdot \hat{e}_y + \frac{\partial^2 \varphi_3}{\partial z^2} \cdot \hat{e}_z
\end{aligned}
\tag{3}
$$

Therefore, if we have a given displacement field $\underline{u}$, we calculate the following:

$$\nabla \cdot \underline{u} = 2 \cdot (1-\delta) \cdot \frac{\partial^2 \varphi_2}{\partial z^2} \tag{4}$$

and

$$(\nabla \times \underline{u}) \cdot \hat{e}_z = \frac{\partial^2 \varphi_3}{\partial z^2} \tag{5}$$

From the above relations we find that:

$$\varphi_2 = \int dz \int dz \frac{\nabla \cdot \underline{u}}{2 \cdot (1-\delta)} + z \cdot F_2(x,y) + G_2(x,y)$$

$$\varphi_3 = \int dz \int dz [(\nabla \times \underline{u}) \cdot \hat{e}_z + z \cdot F_3(x,y) + G_3(x,y) \tag{6}$$

The F_i's and G_i's are chosen such that φ_2 and φ_3 are harmonic in the required region. Note, for the image method we should choose all the singularities of the potentials to occur in the region where the source occurs. This is made clearer in appendix 5 when we consider examples of the use of the algorithm. Finally, once φ_2 and φ_3 are determined, whatever remains is ascribed to φ_1. If the given displacement field does satisfy equilibrium, the field should be expressible in terms of these three potentials (see Ben-Menahem and Singh 1968, and Morse and Feshbach 1953).

The Hansen potentials for a point force, a line force perpendicular to the z-direction, and an edge dislocation perpendicular to the z-direction are given in Appendix 2. We will show later (as is already known) that there is no need to obtain the potentials for a purely antiplane deformation field, since the image field involves a field of a similiar nature as the source.

The Algorithm

Now we describe the algorithm and the notation associated with it; consider two elastic halfspaces perfectly bonded along an interface plane at z=0 (see figure 1). The material properties of region 1 are described by μ_1 and δ_1, and of region 2 by μ_2 and δ_2. Next we define the following:

$$\overline{\varphi}(x,y,z) \equiv \varphi(x,y,-z)$$

$$\gamma \equiv \mu_2/\mu_1 \tag{7}$$

$$a \equiv \frac{(\delta_1+1)}{(\delta_1+\gamma)} \qquad\qquad b \equiv \frac{(\delta_1+1)}{(\gamma\cdot\delta_2+1)}$$

Note that if φ is harmonic then $\overline{\varphi}$ is also harmonic and hence can be used as a Hansen potential for $\underline{N}$, $\underline{F}$ and $\underline{M}$.

The algorithm states that if we have the representation for a point source in infinite space of elastic constants similar to those of region 1 at the location x=y=0 and z=h described by the displacement field:

$$\underline{u}^0 \equiv \underline{u}^0(h,\delta_1,\underline{\varphi}_R^0,\underline{\varphi}_L^0) \tag{8}$$

then the displacement fields in regions 1 and 2 for a similiar point source in region 1 at x=y=0 and z=h are given by:

$$\underline{u}^1 = \underline{u}^0 + \underline{u}(-h,\delta_1,\underline{\overline{\varphi}}_R^1,\underline{\overline{\varphi}}_L^1)$$

$$\tag{9}$$

$$\underline{u}^2 = \underline{u}(h,\delta_2,\underline{\varphi}_R^2,\underline{\varphi}_L^2)$$

where:

$$\underline{\boldsymbol{\varphi}}_R^1 = \underline{\underline{R}}_R(-h,a,b,\delta_1)\cdot\underline{\boldsymbol{\varphi}}_R^0$$

$$\underline{\boldsymbol{\varphi}}_L^1 = \underline{\underline{R}}_L(\gamma)\cdot\underline{\boldsymbol{\varphi}}_L^0$$

$$\underline{\boldsymbol{\varphi}}_R^2 = \underline{\underline{T}}_R(h,a,b,\delta_2,\delta_1)\cdot\underline{\boldsymbol{\varphi}}_R^0$$

$$\underline{\boldsymbol{\varphi}}_L^2 = \underline{\underline{T}}_L(\gamma)\cdot\underline{\boldsymbol{\varphi}}_L^0$$

$$\underline{\underline{R}}_R(-h,a,b,\delta_1) \equiv \left[\begin{array}{c|c} -2\delta_1(1-a)h\cdot\frac{\partial}{\partial z} & +(1-b) - 4\delta_1^2(1-a)h^2\cdot\frac{\partial^2}{\partial z^2} \\ \hline +(1-a) & +2\delta_1(1-a)h\cdot\frac{\partial}{\partial z} \end{array}\right]$$

$$\underline{\underline{T}}_R(h,a,b,\delta_2,\delta_1) \equiv \left[\begin{array}{c|c} +a & -2(\delta_2 b-\delta_1 a)h\cdot\frac{\partial}{\partial z} \\ \hline 0 & +b \end{array}\right]$$

$$\underline{\underline{R}}_L(\gamma) \equiv \left[\frac{1-\gamma}{1+\gamma}\right] \qquad \underline{\underline{T}}_L(\gamma) \equiv \left[\frac{+2}{1+\gamma}\right] \tag{10}$$

We note that:

$$\bar{\underline{\boldsymbol{\varphi}}}_R^1 = \overline{\underline{\underline{R}}_R(-h,a,b,\delta_1)\cdot\underline{\boldsymbol{\varphi}}_R^0} = \bar{\underline{\underline{R}}}_R(-h,a,b,\delta_1)\cdot\bar{\underline{\boldsymbol{\varphi}}}_R^0$$

and: (11)

$$\bar{\underline{\underline{R}}}_R(-h,a,b,\delta_1) = \underline{\underline{R}}_R(+h,a,b,\delta_1)$$

Note that the $\underline{\boldsymbol{\varphi}}_L^{1,2}$ are simple multiplicatives of $\underline{\boldsymbol{\varphi}}_L^0$. The case when $\underline{\boldsymbol{\varphi}}_R^0 = 0$ corresponds to the purely anti-plane problem, and thus, the algorithm reduces to the scalar image method for that case.

The derivation of the above algorithm is given in Appendix 3, an analytic check of the algorithm (making sure the displacement and traction interface conditions are satisfied) using the Cartesian components is shown in Appendix 4, and finally some sample known solutions are rederived in appendix 5; namely screw dislocation in a half space with a free surface, the Boussinesq and Cerruti point force normal and tangential (respectively) to a free surface, Flamant's line force normal to a free surface, a line force tangential to a free surface, and finally Mindlin's solution of a point force interior to a halfspace.

In anticipation of applying the above algorithm to the derivation of point sources for a region consisting of an elastic layer perfectly bonded to two elastic halfspaces, we consider the effect of shifting the interface plane from $z=0$ to $z=H$ on the form of the terms in the algorithm.

Assume the interface is at $z=H$. Define a new coordinate $z' = z-H$. If $z = h$ is the location of the source point ($h > H$) then $z' = h-H$ is the location of the source point in terms of the new coordinate, and $z' = H-h$ ($z = 2H-h$) is the location of the image of the source point with respect to the interface. In terms of z', the algorithm is applicable as shown above with $h_{effective} = h-H$. We then reexpress z' in terms of z. Therefore, for the case when $z=H$ is the interface plane we get:

given: $$\underline{u}^0 = \underline{u}^0(h, \boldsymbol{\delta}_1, \underline{\boldsymbol{\varphi}}_R^0, \underline{\boldsymbol{\varphi}}_L^0)$$

then: $$\underline{u}^1 = \underline{u}^0 + \underline{u}(2H-h, \boldsymbol{\delta}_1, \overline{\underline{\boldsymbol{\varphi}}}_R^1, \overline{\underline{\boldsymbol{\varphi}}}_L^1)$$

$$\underline{u}^2 = \underline{u}(h, \boldsymbol{\delta}_2, \underline{\boldsymbol{\varphi}}_R^2, \underline{\boldsymbol{\varphi}}_L^2)$$

where:

$$\underline{\boldsymbol{\varphi}}_R^1 = \underline{\underline{R}}_R(H-h,a,b,\boldsymbol{\delta}_1)\cdot\underline{\boldsymbol{\varphi}}_R^0$$

$$\underline{\boldsymbol{\varphi}}_L^1 = \underline{\underline{R}}_L(\gamma)\cdot\underline{\boldsymbol{\varphi}}_L^0$$

$$\underline{\boldsymbol{\varphi}}_R^2 = \underline{\underline{T}}_R(-H+h,a,b,\boldsymbol{\delta}_2,\boldsymbol{\delta}_1)\cdot\underline{\boldsymbol{\varphi}}_R^0$$

$$\underline{\boldsymbol{\varphi}}_L^2 = \underline{\underline{T}}_L(\gamma)\cdot\underline{\boldsymbol{\varphi}}_L^0 \tag{12}$$

<u>Point sources in a region consisting of an elastic layer perfectly bonded to two elastic halfspaces.</u>

In this section, we consider the method of derivation of the displacement field due to a point source in a region consisting of an elastic plate ($0<z<H$) perfectly bonded to two elastic halfspaces (see figure 2). The location of the point source is allowed to be either in the plate or in one of the halfspaces. The case of the source being in the elastic plate and the case of the source being in one of the halfspaces have to be treated seperately. The elastic parameters used to characterize each region are chosen to be μ_i and δ_i, where δ_i is defined as $\delta_i = (\lambda+\mu)/(\lambda+3\mu)$.

In order to simplify the presentation, we define the following terms (this notation is suggested from private notes by Rice 1985 of an outline of using the image method combined with the Papkovich-Neuber potentials to solve the same problem, although a detailed description of the implementation is not performed):

$$\mu^+ \equiv \mu_3 \qquad \mu^- \equiv \mu_2 \qquad \gamma^\pm \equiv \mu^\pm/\mu_1$$

$$\delta^+ \equiv \delta_3 \qquad \delta^- \equiv \delta_2$$

$$a^{\pm} \equiv (\delta_1+1)/(\delta_1+\gamma^{\pm}) \qquad b^{\pm} \equiv (\delta_1+1)/(\gamma^{\pm}\delta^{\pm}+1)$$

$$\underline{\underline{R}}_R^{\pm}(h) \equiv \underline{\underline{R}}_R(h,\delta_1,a^{\pm},b^{\pm}) \qquad \underline{\underline{R}}_L^{\pm} \equiv \underline{\underline{R}}_L(\gamma^{\pm})$$

$$\underline{\underline{T}}_R^{\pm}(h) \equiv \underline{\underline{T}}_R(h,a^{\pm},b^{\pm},\delta^{\pm},\delta_1) \qquad \underline{\underline{T}}_L^{\pm} \equiv \underline{\underline{T}}_L(\gamma^{\pm})$$

$$\gamma^{*} \equiv \mu_1/\mu_2$$

$$a^{*} \equiv (\delta_2+1)/(\delta_2+\gamma^{*}) \qquad b^{*} \equiv (\delta_2+1)/(\gamma^{*}\delta_1+1)$$

$$\underline{\underline{R}}_R^{*}(h) \equiv \underline{\underline{R}}_R(h,\delta_2,a^{*},b^{*}) \qquad \underline{\underline{R}}_L^{*} \equiv \underline{\underline{R}}_L(\gamma^{*})$$

$$\underline{\underline{T}}_R^{*}(h) \equiv \underline{\underline{T}}_R(h,a^{*},b^{*},\delta_1,\delta_2) \qquad \underline{\underline{T}}_L^{*} \equiv \underline{\underline{T}}_L(\gamma^{*}) \tag{13}$$

<u>Case I: source is in region 1 (the elastic plate), and h<H</u>

Referring to figure 3, a point source located in an elastic plate requires additional fields (derived using the algorithm) to conform with the boundary conditions on the upper interface. These additional generated fields have to conform with boundary conditions (derived using the algorithm at a shifted interface of location z=H) on the lower interface; hence each field generated by the algorithm requires a further image. The same argument applies when we start satisfying boundary conditions at the lower interface first. Hence, we can deduce the following:

Given a point source in region 1 (h<H) in the form:

$$\underline{u}^0 = \underline{u}(h,\delta_1,\underline{\gamma}_R^0,\underline{\gamma}_L^0)$$

Then:

$$
\begin{aligned}
\underline{u}^1 = \underline{u}^0 &+ \underline{u}(-h,\boldsymbol{\delta}_1,\underline{\bar{\boldsymbol{\varphi}}}_R^{1+}(0),\underline{\bar{\boldsymbol{\varphi}}}_L^{1+}(0)) \\
&+ \sum_{m=1}^{\infty} \underline{u}(-2mH-h,\boldsymbol{\delta}_1,\underline{\bar{\boldsymbol{\varphi}}}_R^{1+}(m),\underline{\bar{\boldsymbol{\varphi}}}_L^{1+}(m)) \\
&+ \sum_{m=1}^{\infty} \underline{u}(2mH+h,\boldsymbol{\delta}_1,\underline{\bar{\boldsymbol{\varphi}}}_R^{1-+}(m),\underline{\bar{\boldsymbol{\varphi}}}_L^{1-+}(m)) \\
&+ \underline{u}(2H-h,\boldsymbol{\delta}_1,\underline{\bar{\boldsymbol{\varphi}}}_R^{1-}(0),\underline{\bar{\boldsymbol{\varphi}}}_L^{1-}(0)) \\
&+ \sum_{m=1}^{\infty} \underline{u}(2(m+1)H-h,\boldsymbol{\delta}_1,\underline{\bar{\boldsymbol{\varphi}}}_R^{1-}(m),\underline{\bar{\boldsymbol{\varphi}}}_L^{1-}(m)) \\
&+ \sum_{m=1}^{\infty} \underline{u}(-2mH+h,\boldsymbol{\delta}_1,\underline{\bar{\boldsymbol{\varphi}}}_R^{1+-}(m),\underline{\bar{\boldsymbol{\varphi}}}_L^{1+-}(m))
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
\underline{u}^2 = \quad &\sum_{m=1}^{\infty} \underline{u}(-2(m-1)H-h,\boldsymbol{\delta}_2,\underline{\boldsymbol{\varphi}}_R^{2-+}(m),\underline{\boldsymbol{\varphi}}_L^{2-+}(m)) \\
&+ \underline{u}(h,\boldsymbol{\delta}_2,\underline{\boldsymbol{\varphi}}_R^{2-}(0),\underline{\boldsymbol{\varphi}}_L^{2-}(0)) \\
&+ \sum_{m=1}^{\infty} \underline{u}(-2mH+h,\boldsymbol{\delta}_2,\underline{\boldsymbol{\varphi}}_R^{2-}(m),\underline{\boldsymbol{\varphi}}_L^{2-}(m))
\end{aligned}
\tag{15}
$$

where:

$$\underline{\boldsymbol{\varphi}}_R^{1+}(0) = \underline{\underline{R}}_R^{+}(-h)\cdot\underline{\boldsymbol{\varphi}}_R^{0} \qquad \underline{\boldsymbol{\varphi}}_L^{1+}(0) = \underline{\underline{R}}_L^{+}\cdot\underline{\boldsymbol{\varphi}}_L^{0}$$

$$\underline{\boldsymbol{\varphi}}_R^{1-+}(m) = \underline{\underline{R}}_R^{-}((2m-1)H+h)\cdot\underline{\bar{\boldsymbol{\varphi}}}_R^{1+}(m-1) \qquad \underline{\boldsymbol{\varphi}}_L^{1-+}(m) = \underline{\underline{R}}_L^{-}\cdot\underline{\bar{\boldsymbol{\varphi}}}_L^{1+}(m-1)$$

$$\underline{\boldsymbol{\varphi}}_R^{1+}(m) = \underline{\underline{R}}_R^{+}(-2mH-h)\cdot\underline{\bar{\boldsymbol{\varphi}}}_R^{1-+}(m) \qquad \underline{\boldsymbol{\varphi}}_L^{1+}(m) = \underline{\underline{R}}_L^{+}\cdot\underline{\bar{\boldsymbol{\varphi}}}_L^{1-+}(m)$$

$$\underline{\boldsymbol{\varphi}}_R^{1-}(0) = \underline{\underline{R}}_R^{-}(H-h)\cdot\underline{\boldsymbol{\varphi}}_R^{0} \qquad \underline{\boldsymbol{\varphi}}_L^{1-}(0) = \underline{\underline{R}}_L^{-}\cdot\underline{\boldsymbol{\varphi}}_L^{0}$$

$$\underline{\boldsymbol{\varphi}}_R^{1+-}(m) = \underline{\underline{R}}_R^{+}(-2mH+h)\cdot\underline{\bar{\boldsymbol{\varphi}}}_R^{1-}(m-1) \qquad \underline{\boldsymbol{\varphi}}_L^{1+-}(m) = \underline{\underline{R}}_L^{+}\cdot\underline{\bar{\boldsymbol{\varphi}}}_L^{1-}(m-1)$$

$$\underline{\boldsymbol{\varphi}}_R^{1-}(m) = \underline{\underline{R}}_R^{-}((2m+1)H-h)\cdot\underline{\bar{\boldsymbol{\varphi}}}_R^{1+-}(m) \qquad \underline{\boldsymbol{\varphi}}_L^{1-}(m) = \underline{\underline{R}}_L^{-}\cdot\underline{\bar{\boldsymbol{\varphi}}}_L^{1+-}(m)$$

$$\underline{\boldsymbol{\varphi}}_R^{2-}(0) = \underline{\underline{T}}_R^{-}(-H+h)\cdot\underline{\boldsymbol{\varphi}}_R^{0} \qquad \underline{\boldsymbol{\varphi}}_L^{2-}(0) = \underline{\underline{T}}_L^{-}\cdot\underline{\boldsymbol{\varphi}}_L^{0}$$

$$\underline{\boldsymbol{\varphi}}_R^{2-}(m) = \underline{\underline{T}}_R^-(-(2m+1)H+h)\cdot\bar{\underline{\boldsymbol{\varphi}}}_R^{1+-}(m) \qquad \underline{\boldsymbol{\varphi}}_L^{2-}(m) = \underline{\underline{T}}_L^-\cdot\bar{\underline{\boldsymbol{\varphi}}}_L^{1+-}(m)$$
$$\underline{\boldsymbol{\varphi}}_R^{2-+}(m) = \underline{\underline{T}}_R^-(-(2m-1)H-h)\cdot\bar{\underline{\boldsymbol{\varphi}}}_R^{1+}(m-1) \qquad \underline{\boldsymbol{\varphi}}_L^{2-+}(m) = \underline{\underline{T}}_L^-\cdot\bar{\underline{\boldsymbol{\varphi}}}_L^{1+}(m-1) \tag{16}$$

From the above recursive relations for the "image" potentials we would like to obtain their direct relation to the infinite space potentials $\underline{\boldsymbol{\varphi}}_R^0$. and $\underline{\boldsymbol{\varphi}}_L^0$. This can be done by induction and the final results are:

$$\bar{\underline{\boldsymbol{\varphi}}}_R^{1+}(0) = \underline{\underline{R}}_R^+(h)\cdot\bar{\underline{\boldsymbol{\varphi}}}_R^0$$
$$\bar{\underline{\boldsymbol{\varphi}}}_R^{1+}(m) = \left[\prod_{k=1}^{m}\left[\underline{\underline{R}}_R^+(2kH+h)\cdot\underline{\underline{R}}_R^-((2k-1)H+h)\right]_k\right]\cdot\underline{\underline{R}}_R^+(h)\cdot\bar{\underline{\boldsymbol{\varphi}}}_R^0$$
$$\bar{\underline{\boldsymbol{\varphi}}}_L^{1+}(0) = \underline{\underline{R}}_L^+\cdot\bar{\underline{\boldsymbol{\varphi}}}_L^0$$
$$\bar{\underline{\boldsymbol{\varphi}}}_L^{1+}(m) = \left[\prod_{k=1}^{m}\left[\underline{\underline{R}}_L^+\cdot\underline{\underline{R}}_L^-\right]_k\right]\cdot\underline{\underline{R}}_L^+\cdot\bar{\underline{\boldsymbol{\varphi}}}_L^0$$
$$\bar{\underline{\boldsymbol{\varphi}}}_R^{1-+}(m) = \left[\prod_{k=1}^{m}\left[\underline{\underline{R}}_R^-(-(2k-1)H-h)\cdot\underline{\underline{R}}_R^+(-(2k-2)H-h)\right]_k\right]\cdot\underline{\boldsymbol{\varphi}}_R^0$$
$$\bar{\underline{\boldsymbol{\varphi}}}_L^{1-+}(m) = \left[\prod_{k=1}^{m}\left[\underline{\underline{R}}_L^-\cdot\underline{\underline{R}}_L^+\right]_k\right]\cdot\underline{\boldsymbol{\varphi}}_L^0$$

$$\bar{\underline{\boldsymbol{\varphi}}}_R^{1-}(0) = \underline{\underline{R}}_R^-(-H+h)\cdot\bar{\underline{\boldsymbol{\varphi}}}_R^0$$
$$\bar{\underline{\boldsymbol{\varphi}}}_R^{1-}(m) = \left[\prod_{k=1}^{m}\left[\underline{\underline{R}}_R^-(-(2k+1)H+h)\cdot\underline{\underline{R}}_R^+(-2kH+h)\right]_k\right]\cdot\underline{\underline{R}}_R^-(-H+h)\cdot\bar{\underline{\boldsymbol{\varphi}}}_R^0$$
$$\bar{\underline{\boldsymbol{\varphi}}}_L^{1-}(0) = \underline{\underline{R}}_L^-\cdot\bar{\underline{\boldsymbol{\varphi}}}_L^0$$
$$\bar{\underline{\boldsymbol{\varphi}}}_L^{1-}(m) = \left[\prod_{k=1}^{m}\left[\underline{\underline{R}}_L^-\cdot\underline{\underline{R}}_L^+\right]_k\right]\cdot\underline{\underline{R}}_L^-\cdot\bar{\underline{\boldsymbol{\varphi}}}_L^0$$
$$\bar{\underline{\boldsymbol{\varphi}}}_R^{1+-}(m) = \left[\prod_{k=1}^{m}\left[\underline{\underline{R}}_R^+(2kH-h)\cdot\underline{\underline{R}}_R^-((2k-1)H-h)\right]_k\right]\cdot\underline{\boldsymbol{\varphi}}_R^0$$

$$\underline{\bar{\varphi}}_L^{1+-}(m) = \left[\prod_{k=1}^{m}\left[\underline{\underline{R}}_L^{+}\cdot\underline{\underline{R}}_L^{-}\right]_k\right]\cdot\underline{\varphi}_L^{0} \tag{17}$$

Once the φ^1 potentials are known in terms of the φ^0 potentials, the φ^2 potentials are obtained by one further matrix operation as given previously (16).

Note that whereas the $\underline{\underline{R}}_L$ and $\underline{\underline{T}}_L$ matrices consist of one scalar per matrix, the $\underline{\underline{R}}_R$ and $\underline{\underline{T}}_R$ matices are 4x4 matrix operators involving "$\frac{\partial}{\partial z}$" and "$\frac{\partial^2}{\partial z^2}$" operators as well as constants.

One subtle point when deriving any given point source for this case (i.e. when the point source is in the elastic layer) is that the potentials have to have all their singularities in the region $z>0$ for the series of "image" potentials generated with the first reflection being with respect to the upper interface, and the potentials have to have all their singularities in the region $z<H$ for the series of "image" potentials generated with the first reflection being with respect to the lower interface. These conditions are imposed on the choice of the potentials defining the original source, in order not to introduce any further singularities in any given region thru the use of the algorithm. This condition can be satisfied due to the flexibility in choosing the potentials.

<u>Case II: source is in region 2 (the lower halfspace) and h>H</u>

Referring to figure 4, a point source located in an elastic half-space requires additional fields (derived using the algorithm) to conform with the boundary conditions on the first upper interface (i.e. the elastic plate and lower halfspace interface).

Fields in the elastic plate are thus generated, and these have to conform with boundary conditions on the upper interface (z=0) via the algorithm. From thereon each new field generated influencing the elastic plate have to have a further "image". However, there is only one "series" of sources generated by this process since the first elastic field generated in the elastic plate already satisfies the boundary conditions with respect to the lower halfspace. Hence, we can deduce the following:

Given a point source in region 2 (h>H) in the form:

$$\underline{u}^0 = \underline{u}(h,\boldsymbol{\delta}_2,\underline{\boldsymbol{\varphi}}_R^0,\underline{\boldsymbol{\varphi}}_L^0)$$

Then:

$$\begin{aligned}\underline{u}^1 = \quad & \underline{u}(h,\boldsymbol{\delta}_1,\underline{\boldsymbol{\varphi}}_R^{0*},\underline{\boldsymbol{\varphi}}_L^{0*}) \\ & + \underline{u}(-h,\boldsymbol{\delta}_1,\underline{\overline{\boldsymbol{\varphi}}}_R^{1+}(0),\underline{\overline{\boldsymbol{\varphi}}}_L^{1+}(0)) \\ & + \sum_{m=1}^{\infty}\underline{u}(-2mH-h,\boldsymbol{\delta}_1,\underline{\overline{\boldsymbol{\varphi}}}_R^{1+}(m),\underline{\overline{\boldsymbol{\varphi}}}_L^{1+}(m)) \\ & + \sum_{m=1}^{\infty}\underline{u}(2mH+h,\boldsymbol{\delta}_1,\underline{\overline{\boldsymbol{\varphi}}}_R^{1-+}(m),\underline{\overline{\boldsymbol{\varphi}}}_L^{1-+}(m))\end{aligned} \tag{18}$$

$$\begin{aligned}\underline{u}^2 = \underline{u}^0 & + \underline{u}(2H-h,\boldsymbol{\delta}_2,\underline{\overline{\boldsymbol{\varphi}}}_R^{2-}(0),\underline{\overline{\boldsymbol{\varphi}}}_L^{2-}(0)) \\ & + \sum_{m=1}^{\infty}\underline{u}(-2(m-1)H-h,\boldsymbol{\delta}_2,\underline{\boldsymbol{\varphi}}_R^{2-+}(m),\underline{\boldsymbol{\varphi}}_L^{2-+}(m))\end{aligned} \tag{19}$$

where:

$$\underline{\boldsymbol{\varphi}}_R^{0*} = \underline{\underline{T}}_R^*(h-H)\cdot\underline{\boldsymbol{\varphi}}_R^0 \qquad \underline{\boldsymbol{\varphi}}_L^{0*} = \underline{\underline{T}}_L^*\cdot\boldsymbol{\varphi}_L^0$$

$$\underline{\boldsymbol{\varphi}}_R^{1+}(0) = \underline{\underline{R}}_R^+(-h)\cdot\underline{\underline{T}}_R^*(h-H)\cdot\underline{\boldsymbol{\varphi}}_R^0 \qquad \underline{\boldsymbol{\varphi}}_L^{1+}(0) = \underline{\underline{R}}_L^+\cdot\underline{\underline{T}}_L^*\cdot\underline{\boldsymbol{\varphi}}_L^0$$

$$\underline{\boldsymbol{\varphi}}_R^{1-+}(m) = \underline{\underline{R}}_R^-((2m-1)H+h)\cdot\underline{\overline{\boldsymbol{\varphi}}}_R^{1+}(m-1) \qquad \underline{\boldsymbol{\varphi}}_L^{1-+}(m) = \underline{\underline{R}}_L^-\cdot\underline{\overline{\boldsymbol{\varphi}}}_L^{1+}(m-1)$$

$$\underline{\boldsymbol{\varphi}}_R^{1+}(m) = \underline{\underline{R}}_R^+(-2mH-h)\cdot\underline{\overline{\boldsymbol{\varphi}}}_R^{1-+}(m) \qquad \underline{\boldsymbol{\varphi}}_L^{1+}(m) = \underline{\underline{R}}_L^+\cdot\underline{\overline{\boldsymbol{\varphi}}}_L^{1-+}(m)$$

$$\underline{\boldsymbol{\varphi}}_R^{2-}(0) = \underline{\underline{R}}_R^{*}(H-h)\cdot\underline{\boldsymbol{\varphi}}_R^{0} \qquad \underline{\boldsymbol{\varphi}}_L^{2-}(0) = \underline{\underline{R}}_L^{*}\cdot\underline{\boldsymbol{\varphi}}_L^{0}$$

$$\underline{\boldsymbol{\varphi}}_R^{2-+}(m) = \underline{\underline{T}}_R^{-}(-(2m-1)H-h)\cdot\underline{\overline{\boldsymbol{\varphi}}}_R^{1+}(m-1) \qquad \underline{\boldsymbol{\varphi}}_L^{2-+}(m) = \underline{\underline{T}}_L^{-}\cdot\underline{\overline{\boldsymbol{\varphi}}}_L^{1+}(m-1) \tag{20}$$

From the above recursive relations for the "image" potentials we would like to obtain their direct relation to the infinite space potentials $\underline{\boldsymbol{\varphi}}_R^{0}$. and $\underline{\boldsymbol{\varphi}}_L^{0}$. This can be done by induction and the final results are:

$$\underline{\overline{\boldsymbol{\varphi}}}_R^{1+}(0) = \underline{\underline{R}}_R^{+}(h)\cdot\underline{\underline{T}}_R^{*}(-h+H)\cdot\underline{\overline{\boldsymbol{\varphi}}}_R^{0}$$

$$\underline{\overline{\boldsymbol{\varphi}}}_R^{1+}(m) = \left[\prod_{k=1}^{m}\left[\underline{\underline{R}}_R^{+}(2kH+h)\cdot\underline{\underline{R}}_R^{-}((2k-1)H+h)\right]_k\right]\cdot\underline{\underline{R}}_R^{+}(h)\cdot\underline{\underline{T}}_R^{*}(-h+H)\cdot\underline{\overline{\boldsymbol{\varphi}}}_R^{0}$$

$$\underline{\overline{\boldsymbol{\varphi}}}_L^{1+}(0) = \underline{\underline{R}}_L^{+}\cdot\underline{\underline{T}}_L^{*}\cdot\underline{\overline{\boldsymbol{\varphi}}}_L^{0}$$

$$\underline{\overline{\boldsymbol{\varphi}}}_L^{1+}(m) = \left[\prod_{k=1}^{m}\left[\underline{\underline{R}}_L^{+}\cdot\underline{\underline{R}}_L^{-}\right]_k\right]\cdot\underline{\underline{R}}_L^{+}\cdot\underline{\underline{T}}_L^{*}\cdot\underline{\overline{\boldsymbol{\varphi}}}_L^{0}$$

$$\underline{\overline{\boldsymbol{\varphi}}}_R^{1-+}(m) = \left[\prod_{k=1}^{m}\left[\underline{\underline{R}}_R^{-}(-(2k-1)H-h)\cdot\underline{\underline{R}}_R^{+}(-(2k-2)H-h)\right]_k\right]\cdot\underline{\underline{T}}_R^{*}(h-H)\cdot\underline{\boldsymbol{\varphi}}_R^{0}$$

$$\underline{\overline{\boldsymbol{\varphi}}}_L^{1-+}(m) = \left[\prod_{k=1}^{m}\left[\underline{\underline{R}}_L^{-}\cdot\underline{\underline{R}}_L^{+}\right]_k\right]\cdot\underline{\underline{T}}_L^{*}\cdot\underline{\boldsymbol{\varphi}}_L^{0} \tag{21}$$

Again, once the $\boldsymbol{\varphi}^1$ potentials are known in terms of the $\boldsymbol{\varphi}^0$ potentials, the $\boldsymbol{\varphi}^2$ potentials are obtained by one further matrix operation as given previously (20).

When deriving any given point source for this case (i.e. when the point source is in the lower halfspace) the potentials have to have all their singularities in the region z>H, in order for the reflected images not to introduce any further singularities within any given region.

Conclusions and further recommendations

A vector image method has been presented, for elastic problems with planar layering. An algorithm has been presented on how to derive point source solutions for two bonded elastic halfspaces, and then extended to the case of an elastic layer perfectly bonded to two halfspaces. Specific cases have been worked out and shown to coincide with well known solutions in the literature. A study of the number of image potentials required to obtain a good approximation for the layered medium problem has not been investigated, but is of considerable practical interest.

A feature of this image method is that solutions are obtained in either closed form or as infinite series of relatively simple expressions. The form of the expressions allow the use of term by term application of the correspondence principle for obtaining the viscoelastic response of the solution of the corresponding point source in a viscoelastic layered medium.

The method of deriving the algorithm suggests that an analogous algorithm can be obtained for spherical interface problems, and 2-D (but not 3-D) cylindrical interface problems in elastostatics. This suggestion is supported by the existence of a scalar image method and Hansen potential representations for both these geometries.

Finally, it would also be of interest to investigate equivalent algorithms for other governing equations. For example, elastodynamics and poroelasticity could be potential candidates for such an investigation. Elastodynamic problems, in particular, do have Hansen potential representations that have been well established and used and could be investigated first without the considerable preliminary formulations that are needed for poroelastic problems.

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Appendix 1: Cartesian components of the displacements, gradients of the displacements, strains and stresses in terms of Hansen's potentials

Displacements:

$$\begin{aligned}\underline{u}(0,\delta,\underline{\varphi}_R,\underline{\varphi}_L) = {} & \hat{\underline{e}}_x\left[\frac{\partial}{\partial x}\varphi_1 - \frac{\partial}{\partial x}\varphi_2 - 2\cdot\delta\cdot z\cdot\frac{\partial^2}{\partial x\partial z}\varphi_2 + \frac{\partial}{\partial y}\varphi_3\right] \\ & + \hat{\underline{e}}_y\left[\frac{\partial}{\partial y}\varphi_1 - \frac{\partial}{\partial y}\varphi_2 - 2\cdot\delta\cdot z\cdot\frac{\partial^2}{\partial y\partial z}\varphi_2 - \frac{\partial}{\partial x}\varphi_3\right] \\ & + \hat{\underline{e}}_z\left[\frac{\partial}{\partial z}\varphi_1 + \frac{\partial}{\partial z}\varphi_2 - 2\cdot\delta\cdot z\cdot\frac{\partial^2}{\partial z^2}\varphi_2\right]\end{aligned} \tag{1.1}$$

Gradients of the displacements:

$$\frac{\partial}{\partial x}u_x = \frac{\partial^2}{\partial x^2}\varphi_1 - \frac{\partial^2}{\partial x^2}\varphi_2 - 2\cdot\delta\cdot z\cdot\frac{\partial^3}{\partial x^2\partial z}\varphi_2 + \frac{\partial^2}{\partial x\partial y}\varphi_3$$

$$\frac{\partial}{\partial y}u_x = \frac{\partial^2}{\partial x\partial y}\varphi_1 - \frac{\partial^2}{\partial x\partial y}\varphi_2 - 2\cdot\delta\cdot z\cdot\frac{\partial^3}{\partial x\partial y\partial z}\varphi_2 + \frac{\partial^2}{\partial y^2}\varphi_3$$

$$\frac{\partial}{\partial z}u_x = \frac{\partial^2}{\partial x\partial z}\varphi_1 - (1+2\delta)\cdot\frac{\partial^2}{\partial x\partial z}\varphi_2 - 2\cdot\delta\cdot z\cdot\frac{\partial^3}{\partial z^2\partial x}\varphi_2 + \frac{\partial^2}{\partial y\partial z}\varphi_3$$

$$\frac{\partial}{\partial x}u_y = \frac{\partial^2}{\partial x\partial y}\varphi_1 - \frac{\partial^2}{\partial x\partial y}\varphi_2 - 2\cdot\delta\cdot z\cdot\frac{\partial^3}{\partial x\partial y\partial z}\varphi_2 - \frac{\partial^2}{\partial x^2}\varphi_3$$

$$\frac{\partial}{\partial y}u_y = \frac{\partial^2}{\partial y^2}\varphi_1 - \frac{\partial^2}{\partial y^2}\varphi_2 - 2\cdot\delta\cdot z\cdot\frac{\partial^3}{\partial y^2\partial z}\varphi_2 - \frac{\partial^2}{\partial x\partial y}\varphi_3$$

$$\frac{\partial}{\partial z}u_y = \frac{\partial^2}{\partial y\partial z}\varphi_1 - (1+2\delta)\cdot\frac{\partial^2}{\partial y\partial z}\varphi_2 - 2\cdot\delta\cdot z\cdot\frac{\partial^3}{\partial z^2\partial y}\varphi_2 - \frac{\partial^2}{\partial x\partial z}\varphi_3$$

$$\frac{\partial}{\partial x}u_z = \frac{\partial^2}{\partial x\partial z}\varphi_1 + \frac{\partial^2}{\partial x\partial z}\varphi_2 - 2\cdot\delta\cdot z\cdot\frac{\partial^3}{\partial z^2\partial x}\varphi_2$$

$$\frac{\partial}{\partial y}u_z = \frac{\partial^2}{\partial y\partial z}\varphi_1 + \frac{\partial^2}{\partial y\partial z}\varphi_2 - 2\cdot\delta\cdot z\cdot\frac{\partial^3}{\partial z^2\partial y}\varphi_2$$

$$\frac{\partial}{\partial z}u_z = \frac{\partial^2}{\partial z^2}\varphi_1 + (1-2\delta)\cdot\frac{\partial^2}{\partial z^2}\varphi_2 - 2\cdot\delta\cdot z\cdot\frac{\partial^3}{\partial z^3}\varphi_2$$

(1.2)

Strains:

$$\epsilon_{xx} = \frac{\partial^2}{\partial x^2}\varphi_1 - \frac{\partial^2}{\partial x^2}\varphi_2 - 2\cdot\delta\cdot z\cdot\frac{\partial^3}{\partial x^2\partial z}\varphi_2 + \frac{\partial^2}{\partial x\partial y}\varphi_3$$

$$\epsilon_{xy} = \frac{\partial^2}{\partial x\partial y}\varphi_1 - \frac{\partial^2}{\partial x\partial y}\varphi_2 - 2\cdot\delta\cdot z\cdot\frac{\partial^3}{\partial x\partial y\partial z}\varphi_2 + \frac{1}{2}\cdot\left(\frac{\partial^2}{\partial y^2}\varphi_3 - \frac{\partial^2}{\partial x^2}\varphi_3\right)$$

$$\epsilon_{xz} = \frac{\partial^2}{\partial x\partial z}\varphi_1 - \delta\cdot\frac{\partial^2}{\partial x\partial z}\varphi_2 - 2\cdot\delta\cdot z\cdot\frac{\partial^3}{\partial z^2\partial x}\varphi_2 + \frac{1}{2}\cdot\frac{\partial^2}{\partial y\partial z}\varphi_3$$

$$\epsilon_{yy} = \frac{\partial^2}{\partial y^2}\varphi_1 - \frac{\partial^2}{\partial y^2}\varphi_2 - 2\cdot\delta\cdot z\cdot\frac{\partial^3}{\partial y^2\partial z}\varphi_2 - \frac{\partial^2}{\partial x\partial y}\varphi_3$$

$$\epsilon_{yz} = \frac{\partial^2}{\partial y\partial z}\varphi_1 - \delta\cdot\frac{\partial^2}{\partial y\partial z}\varphi_2 - 2\cdot\delta\cdot z\cdot\frac{\partial^3}{\partial z^2\partial y}\varphi_2 - \frac{1}{2}\cdot\frac{\partial^2}{\partial x\partial z}\varphi_3$$

$$\epsilon_{zz} = \frac{\partial^2}{\partial z^2}\varphi_1 + (1-2\delta)\cdot\frac{\partial^2}{\partial z^2}\varphi_2 - 2\cdot\delta\cdot z\cdot\frac{\partial^3}{\partial z^3}\varphi_2$$

$$\epsilon_{xx} + \epsilon_{yy} + \epsilon_{zz} = 2\cdot(1-\delta)\cdot\frac{\partial^2}{\partial z^2}\varphi_2$$

(1.3)

Stresses:

$$\sigma_{xx} = 2\mu\cdot\frac{\partial^2}{\partial x^2}\varphi_1 - 2\mu\cdot\frac{\partial^2}{\partial x^2}\varphi_2 - 4\mu\delta\cdot z\cdot\frac{\partial^3}{\partial x^2\partial z}\varphi_2 + 2\mu\cdot(3\delta-1)\cdot\frac{\partial^2}{\partial z^2}\varphi_2 + 2\mu\frac{\partial^2}{\partial x\partial y}\varphi_3$$

$$\sigma_{xy} = 2\mu\cdot\frac{\partial^2}{\partial x\partial y}\varphi_1 - 2\mu\cdot\frac{\partial^2}{\partial x\partial y}\varphi_2 - 4\mu\delta\cdot z\cdot\frac{\partial^3}{\partial x\partial y\partial z}\varphi_2 + \mu\cdot\left(\frac{\partial^2}{\partial y^2}\varphi_3 - \frac{\partial^2}{\partial x^2}\varphi_3\right)$$

$$\sigma_{xz} = 2\mu\cdot\frac{\partial^2}{\partial x\partial z}\varphi_1 - 2\mu\delta\cdot\frac{\partial^2}{\partial x\partial z}\varphi_2 - 4\mu\delta\cdot z\cdot\frac{\partial^3}{\partial z^2\partial x}\varphi_2 + \mu\cdot\frac{\partial^2}{\partial y\partial z}\varphi_3$$

$$\sigma_{yy} = 2\mu\cdot\frac{\partial^2}{\partial y^2}\varphi_1 + 2\mu\cdot\frac{\partial^2}{\partial y^2}\varphi_2 - 4\mu\delta\cdot z\cdot\frac{\partial^3}{\partial y^2\partial z}\varphi_2 + 2\mu\cdot(3\delta-1)\cdot\frac{\partial^2}{\partial z^2}\varphi_2$$
$$- 2\mu\cdot\frac{\partial^2}{\partial x\partial y}\varphi_3$$
$$\sigma_{yz} = 2\mu\cdot\frac{\partial^2}{\partial y\partial z}\varphi_1 - 2\mu\delta\cdot\frac{\partial^2}{\partial y\partial z}\varphi_2 - 4\mu\delta\cdot z\cdot\frac{\partial^3}{\partial z^2\partial y}\varphi_2 - \mu\cdot\frac{\partial^2}{\partial x\partial z}\varphi_3$$
$$\sigma_{zz} = 2\mu\cdot\frac{\partial^2}{\partial z^2}\varphi_1 + 2\mu\delta\cdot\frac{\partial^2}{\partial z^2}\varphi_2 - 4\mu\delta\cdot z\cdot\frac{\partial^3}{\partial z^3}\varphi_2 \tag{1.4}$$

Note: $2\lambda\cdot(1-\delta) = 2\mu\cdot(3\delta-1)$

Appendix 2: Sample potentials for some point sources

Point Force:

The displacement field due to a point force at the origin can be written as:

$$u_i = \frac{1}{4\pi\mu\cdot(1+\delta)}\cdot\left[p_i\frac{1}{r} + \delta\cdot p_k\cdot x_k\cdot x_i\cdot\frac{1}{r^3} \right] \tag{2.1}$$

where: $\delta \equiv \frac{\lambda+\mu}{\lambda+3\mu} \equiv \frac{1}{\kappa} \equiv \frac{1}{3-4\nu}$

$r^2 = x^2 + y^2 + z^2$

p_i are the magnitudes of the point forces in the i^{th}-direction

For the point force it can then be checked that the Hansen potentials are:

$$\varphi_1 = \frac{\beta}{2}\cdot\left[p_1\cdot\frac{x}{r\pm z} + p_2\cdot\frac{y}{r\pm z} \pm p_3\cdot\ln(r\pm z) \right]$$

$$\varphi_2 = \frac{\beta}{2}\cdot\left[-p_1\cdot\frac{x}{r\pm z} - p_2\cdot\frac{y}{r\pm z} \pm p_3\cdot\ln(r\pm z) \right] \tag{2.2}$$

$$\varphi_3 = \beta\cdot\left[p_1\cdot(1+\delta)\cdot\frac{y}{r\pm z} + p_2\cdot(1+\delta)\cdot\frac{x}{r\pm z} \right]$$

where: $\beta \equiv 1/[4\pi\mu\cdot(1+\delta)]$

Note that if the upper (lower) "sign" is chosen in one expression, the upper (lower) "signs" must be chosen throughout for all the potentials. Also note that taking r+z (r−z) in the expressions makes the potentials (but not necessarily the displacements) singular when $x=y=0$ and $z<0$ ($z>0$).

<u>Line forces at $x = 0$ parallel to the $z = 0$ plane</u>

The displacement field due to a line force can be written (for plain strain) as:

$$u_i = \frac{\alpha}{4\pi\mu\delta} \cdot \left[-p_i \cdot \ln\xi + \delta \cdot p_k \cdot x_k \cdot x_i \cdot \frac{1}{\xi^2} \right] \qquad \text{for } i,k = 1,3$$

and (2.3)

$$u_2 = 0$$

where: $\alpha \equiv \dfrac{\lambda+\mu}{\lambda+2\mu}$ $\qquad \delta \equiv \dfrac{\lambda+\mu}{\lambda+3\mu}$

$$\xi^2 = x^2 + z^2$$

p_1 and p_3 are the magnitude of the line forces

For the line force it can be checked that the Hansen potentials are:

$$\varphi_1 = \frac{\alpha}{8\pi\mu\delta} \cdot \Big[\; p_1 \cdot \left[z \cdot \arctan(\tfrac{z}{x}) - x \cdot \ln\xi + (1+\delta) \cdot x \right] - p_3 \cdot \left[z \cdot \ln\xi - z + x \cdot \arctan(\tfrac{z}{x}) \right] \Big]$$

$$\varphi_2 = \frac{\alpha}{8\pi\mu\delta} \cdot \Big[-p_1 \cdot \left[z \cdot \arctan(\tfrac{z}{x}) - x \cdot \ln\xi + (1+\delta) \cdot x \right] - p_3 \cdot \left[z \cdot \ln\xi - z + x \cdot \arctan(\tfrac{z}{x}) \right] \Big]$$

$$\varphi_3 = 0$$

(2.4)

Dislocations parallel to the z=0 plane:

The displacement due to a dislocation along the y-axis (plane strain) can be written as:

$$u_i = \left[\frac{d_i}{2\pi}\cdot\arctan\left(\frac{z}{x}\right) - \epsilon_{ik}\cdot\frac{d_k}{2\pi}\cdot\ln\xi \right]$$

$$+ \frac{\alpha}{4\pi\mu\delta}\cdot\left[-(2\mu\epsilon_{ik}d_k)\cdot\ln\xi + (2\mu\epsilon_{nk}d_k)\cdot\delta\cdot x_n\cdot x_i\cdot\frac{1}{\xi^2} \right]$$

for i,k,n = 1,3

and: (2.5)

$$u_2 = 0$$

where: $$\epsilon_{ik} = \begin{cases} +1 & \text{for } i = 1,\ k = 3 \\ -1 & \text{for } i = 3,\ k = 1 \\ 0 & \text{otherwise} \end{cases}$$

d_1 and d_3 are the slip magnitude of the dislocations

We note that the terms in the second brackets expressing the displacements are of the form of line force expressions with equivalent magnitudes of $2\mu\epsilon_{ik}d_k$ and thus their Hansen potentials are already known. The Hansen potentials for the terms in the first bracket can be shown to be:

$$\varphi_1^{\text{first bracket}} = \frac{1}{2\pi}\cdot\left[d_1\cdot\left[z\cdot\ln\xi - z + x\cdot\arctan\left(\frac{z}{x}\right) \right] + d_3\cdot\left[z\cdot\arctan\left(\frac{z}{x}\right) - x\cdot\ln\xi + x \right] \right]$$

$$\varphi_2^{\text{first bracket}} = 0 \qquad \varphi_3^{\text{first bracket}} = 0$$

(2.6)

Appendix 3: Derivation of the image method algorithm

The eigenfunction expansion method for elasticity problems in layered media was first formulated by Ben-Menahem and Singh 1968. We have used this method to derive the algorithm discussed in this paper. The notation (as far as possible) is the same as in the 1968 reference paper, although some new temporary terms have been defined in order to simplify the algebra for this specific implementation.

Any elastic displacement field satisfying the equilibrium equations:

$$\nabla^2 \underline{u} + (1+\lambda/\mu)\cdot\nabla\nabla\cdot\underline{u} = \underline{0} \tag{3.1}$$

can be written as the sum of $\underline{N}$, $\underline{F}$ and $\underline{M}$:

$$\begin{aligned} \underline{N} &= \nabla\varphi_1 \\ \underline{F} &= 2\cdot\hat{e}_z\cdot\frac{\partial}{\partial z}\varphi_2 - \nabla\varphi_2 - 2\cdot\delta\cdot z\cdot\nabla\cdot\frac{\partial}{\partial z}\varphi_2 \\ \underline{M} &= \nabla\times[\hat{e}_z\cdot\varphi_3] \end{aligned} \tag{3.2}$$

where: $\delta \equiv (\lambda+\mu)/(\lambda+3\mu)$

$$\nabla^2\varphi_1 = \nabla^2\varphi_2 = \nabla^2\varphi_3 = 0$$

Using the method of the separation of variables in cylindrical coordinates on the potentials φ_1, φ_2 and φ_3 in the form:

$$\varphi = R(r)\cdot F(\theta)\cdot Z(z)$$

We get: (3.3)

$$\varphi = \exp(\pm kz)\cdot J_m(kr)\cdot\exp(\pm im\theta)$$

where:

J_m is Bessel's function of the first kind m^{th} order

Reexpressing φ_1, φ_2 and φ_3 in the above form and carrying out the ∇ and $\nabla\times$ and $\frac{\partial}{\partial z}$ operations, we get:

$$\underline{u} = \sum_{m=1}^{\infty}\int_0^{\infty}\left[\, A_m^+\cdot N_m^+ + A_m^-\cdot N_m^- + B_m^+\cdot F_m^+ + B_m^-\cdot F_m^- + C_m^+\cdot M_m^+ + C_m^-\cdot M_m^- \,\right]\cdot dk \tag{3.4}$$

where: $A_m^{\pm}$, $B_m^{\pm}$ and $C_m^{\pm}$ are constants dependent on 'm' only.

$$\underline{N}_m^{\pm} = \exp(\pm kz)\cdot\left[\, \pm\underline{P}_m + \underline{B}_m \,\right]$$

$$\underline{F}_m^{\pm} = \exp(\pm kz)\cdot\left[\, (\pm 1-2\delta kz)\cdot\underline{P}_m - (1\pm 2\delta kz)\cdot\underline{B}_m \,\right]$$

$$\underline{M}_m^{\pm} = \exp(\pm kz)\cdot\underline{C}_m$$

and:

$$\underline{P}_m = \hat{e}_z\cdot J_m(kr)\cdot\exp(im\theta)$$

$$\underline{B}_m = (\hat{e}_r\cdot\frac{\partial}{\partial kr} + \hat{e}_\theta\cdot\frac{1}{kr}\cdot\frac{\partial}{\partial\theta})J_m(kr)\cdot\exp(im\theta)$$

$$\underline{C}_m = (\hat{e}_r\cdot\frac{1}{kr}\cdot\frac{\partial}{\partial\theta}\cdot - \hat{e}_\theta\cdot\frac{\partial}{\partial kr})J_m(kr)\cdot\exp(im\theta) \tag{3.5}$$

In the above expressions for $\underline{P}_m$, $\underline{B}_m$ and $\underline{C}_m$ there is the implicit understanding that we can consider either the real or imaginary components of the expressions seperately.

From the above expressions for the displacements, we can find the expressions for the tractions at a plane z=constant, and we rewrite the above as:

$$\underline{u} = \sum_{m=1}^{\infty} \int_0^{\infty} (\underline{u}_m^R + \underline{u}_m^L) \cdot dk \tag{3.6}$$

$$\underline{T}^z = \sum_{m=1}^{\infty} \int_0^{\infty} (\underline{T}_m^R + \underline{T}_m^L) \cdot dk \tag{3.7}$$

where:

$$\begin{aligned}
\underline{u}_m^R &= x_m \cdot \underline{P}_m + y_m \cdot \underline{B}_m \\
\underline{T}_m^R &= 2k \cdot X_m \cdot \underline{P}_m + 2k \cdot Y_m \cdot \underline{B}_m \\
\underline{u}_m^L &= z_m \cdot \underline{C}_m \\
\underline{T}_m^L &= k \cdot Z_m \cdot \underline{C}_m
\end{aligned} \tag{3.8}$$

and:

$$\begin{aligned}
x_m = \; & A_m^+ \cdot \exp(kz) - A_m^- \cdot \exp(-kz) \\
& + B_m^+ \cdot (1-2\delta kz) \cdot \exp(kz) + B_m^- \cdot (-1-2\delta kz) \cdot \exp(-kz) \\
\\
y_m = \; & A_m^+ \cdot \exp(kz) + A_m^- \cdot \exp(-kz) \\
& + B_m^+ \cdot (-1-2\delta kz) \cdot \exp(kz) + B_m^- \cdot (-1+2\delta kz) \cdot \exp(-kz) \\
\\
z_m = \; & C_m^+ \cdot \exp(kz) + C_m^- \cdot \exp(-kz) \\
\\
X_m = \; & A_m^+ \cdot \mu \cdot \exp(kz) + A_m^- \cdot \mu \cdot \exp(-kz) \\
& + B_m^+ \cdot \mu\delta \cdot (1-2kz) \cdot \exp(kz) + B_m^- \cdot \mu\delta \cdot (1+2kz) \cdot \exp(-kz) \\
\\
Y_m = \; & A_m^+ \cdot \mu \cdot \exp(kz) - A_m^- \cdot \mu \cdot \exp(-kz) \\
& + B_m^+ \cdot \mu\delta \cdot (-1-2kz) \cdot \exp(kz) + B_m^- \cdot \mu\delta \cdot (1-2kz) \cdot \exp(-kz) \\
\\
Z_m = \; & C_m^+ \cdot \mu \cdot \exp(kz) - C_m^- \cdot \mu \cdot \exp(-kz)
\end{aligned} \tag{3.9}$$

We notice that that $\underline{u}_m^R$ components are uncoupled from the $\underline{u}_m^L$ in the sense that the $A_m^{\pm}$, $B_m^{\pm}$ coefficients do not affect the $\underline{u}_m^L$ components and the $C_m^{\pm}$ coefficients do not affect the $\underline{u}_m^R$ components. We therefore treat the $\underline{u}_m^R$ and the $\underline{u}_m^L$ components seperately when analyzing a specific problem in terms of the Hansen potentials.

Now we consider the specific geometry shown in figure 3.1. The region consists of two elastic materials seperated by a planar interface. The material elasticity parameters used to characterize the regions are taken to be μ_1, δ_1 and μ_2, δ_2. A point source exists at the position z=-h. We are required to find the displacement fields for region 1 (z<0) and for region 2 (z>0) under the influence of the point source, such that the displacements and the tractions are continuous across the interface plane (z=0).

In what follows, we are manipulating $\underline{u}_m^R$ and $\underline{u}_m^L$ in equation 3.6 for a fixed 'm', but the 'm' subscript will be dropped. First, we express the displacement and traction (on a z-plane) for (z+h) > 0 (which includes z=0) of a point source of arbitrary nature (using the eigenfunction expansion method and expressing the m'th component in matrix form) in the following way:

$$\begin{bmatrix} \underline{u}(\underline{P}) \\ \hline \underline{u}(\underline{B}) \\ \hline \underline{T}(\underline{P}) \\ \hline \underline{T}(\underline{B}) \end{bmatrix}_0 = \left[\begin{array}{c|c} -1 & -1 - 2\delta_1 k\cdot(z+h) \\ \hline +1 & -1 + 2\delta_1 k\cdot(z+h) \\ \hline +2k\mu_1 & +2k\mu_1\delta_1\cdot[1+2k\cdot(z+h)] \\ \hline -2k\mu_1 & +2k\mu_1\delta_1\cdot[1-2k\cdot(z+h)] \end{array}\right]\cdot\begin{bmatrix} A_0^- \\ \hline B_0^- \end{bmatrix}\cdot\exp(-k|z+h|)$$

$$\left[\begin{array}{c} \underline{u}(\underline{C}) \\ \hline \underline{T}(\underline{C}) \end{array}\right]_0 = \left[\begin{array}{c} +1 \\ \hline -k\mu_1 \end{array}\right] \left[\, C_0^- \right] \cdot \exp(-k|z+h|) \tag{3.10}$$

and we define:

$$\left[\begin{array}{c} S_1 \\ \hline S_2 \\ \hline S_3 \\ \hline S_4 \end{array}\right] \equiv -1 \cdot \left[\begin{array}{c} \underline{u}(\underline{P}) \\ \hline \underline{u}(\underline{B}) \\ \hline \underline{T}(\underline{P}) \\ \hline \underline{T}(\underline{B}) \end{array}\right]_0 \Bigg|_{z=0} \qquad \left[\begin{array}{c} S_5 \\ \hline S_6 \end{array}\right] = -1 \cdot \left[\begin{array}{c} \underline{u}(\underline{C}) \\ \hline \underline{T}(\underline{C}) \end{array}\right]_0 \Bigg|_{z=0} \tag{3.11}$$

Now the elastic fields in region 1 are expressible as:

$$\left[\begin{array}{c} \underline{u}(\underline{P}) \\ \hline \underline{u}(\underline{B}) \\ \hline \underline{T}(\underline{P}) \\ \hline \underline{T}(\underline{B}) \end{array}\right]_1 = \left[\begin{array}{c|c} +1 & +1 - 2\delta_1 k \cdot z \\ \hline +1 & -1 - 2\delta_1 k \cdot z \\ \hline +2k\mu_1 & +2k\mu_1\delta_1 \cdot (1-2kz) \\ \hline +2k\mu_1 & +2k\mu_1\delta_1 \cdot (-1-2kz) \end{array}\right] \cdot \left[\begin{array}{c} A_1^+ \\ \hline B_1^+ \end{array}\right] \cdot \exp(kz)$$

$$+ \left[\begin{array}{c} \text{terms due to the} \\ \text{point source} \\ \text{as given above} \end{array}\right]$$

$$\left[\begin{array}{c} \underline{u}(\underline{C}) \\ \hline \underline{T}(\underline{C}) \end{array}\right]_1 = \left[\begin{array}{c} +1 \\ \hline +k\mu_1 \end{array}\right] \left[\, C_1^+ \right] \cdot \exp(kz) + \left[\begin{array}{c} \text{terms due to the} \\ \text{point source} \\ \text{as given above} \end{array}\right] \tag{3.12}$$

And the elastic fields in region 2 are expressible as:

$$\begin{bmatrix} \underline{u}(\underline{P}) \\ \underline{u}(\underline{B}) \\ \underline{T}(\underline{P}) \\ \underline{T}(\underline{B}) \end{bmatrix}_2 = \begin{bmatrix} -1 & -1 - 2\delta_2 kz \\ +1 & -1 + 2\delta_2 kz \\ +2k\mu_2 & +2k\mu_2\delta_2\cdot(1+2kz) \\ -2k\mu_2 & +2k\mu_2\delta_2\cdot(1-2kz) \end{bmatrix} \cdot \begin{bmatrix} A_2^- \\ B_2^- \end{bmatrix} \cdot \exp(-kz)$$

$$\begin{bmatrix} \underline{u}(\underline{C}) \\ \underline{T}(\underline{C}) \end{bmatrix}_2 = \begin{bmatrix} +1 \\ -k\mu_2 \end{bmatrix} \begin{bmatrix} C_2^- \end{bmatrix} \cdot \exp(-kz)$$

(3.13)

Applying the condition that $\underline{u}_m$ and $\underline{T}_m$ are to be continuous (for each m) along the interface plane z=0, we get:

$$\begin{bmatrix} +1 & +1 & +1 & +1 \\ +1 & -1 & -1 & +1 \\ +2k\mu_1 & +2k\mu_1\delta_1 & -2k\mu_2 & -2k\mu_2\delta_2 \\ +2k\mu_1 & -2k\mu_1\delta_1 & +2k\mu_2 & -2k\mu_2\delta_2 \end{bmatrix} \cdot \begin{bmatrix} A_1^+ \\ B_1^+ \\ A_2^- \\ B_2^- \end{bmatrix} = \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix}$$

$$\begin{bmatrix} +1 & -1 \\ +k\mu_1 & +k\mu_2 \end{bmatrix} \cdot \begin{bmatrix} C_1^+ \\ C_2^- \end{bmatrix} = \begin{bmatrix} S_5 \\ S_6 \end{bmatrix}$$

(3.14)

Now we solve for A_1^+, B_1^+, C_1^+ and A_2^-, B_2^-, C_2^- by inverting the 4x4 and 2x2 system of equations. We obtain:

$$\begin{bmatrix} A_1^+ \\ B_1^+ \\ A_2^- \\ B_2^- \end{bmatrix} = \frac{1}{\Delta} \cdot \begin{bmatrix} +\Delta/2-\gamma-\delta_1 & +\Delta/2-\gamma-\delta_1 & +\frac{\gamma+\delta_1}{2k\mu_1} & +\frac{\gamma+\delta_1}{2k\mu_1} \\ +\Delta/2-\gamma\delta_1\delta_2-\delta_1 & -\Delta/2+\gamma\delta_1\delta_2+\delta_1 & +\frac{\gamma\delta_2+1}{2k\mu_1} & -\frac{\gamma\delta_2+1}{2k\mu_1} \\ +\gamma\delta_1\delta_2+\delta_1 & -\gamma\delta_1\delta_2-\delta_1 & -\frac{\gamma\delta_2+1}{2k\mu_1} & +\frac{\gamma\delta_2+1}{2k\mu_1} \\ +\gamma+\delta_1 & +\gamma+\delta_1 & -\frac{\gamma+\delta_1}{2k\mu_1} & -\frac{\gamma+\delta_1}{2k\mu_1} \end{bmatrix} \cdot \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{bmatrix}$$

$$\begin{bmatrix} C_1^+ \\ C_2^- \end{bmatrix} = \frac{1}{\gamma+1} \cdot \begin{bmatrix} +\gamma & +\frac{1}{k\mu_1} \\ -1 & +\frac{1}{k\mu_1} \end{bmatrix} \cdot \begin{bmatrix} S_5 \\ S_6 \end{bmatrix} \tag{3.15}$$

where: $\gamma \equiv \mu_2/\mu_1$

$\Delta \equiv 2\cdot(\gamma+\delta_1)\cdot(\gamma\delta_2+1)$

Expressing the S's in terms of A_0^-, B_0^- and C_0^-, and simplifying the expressions we get:

$$\begin{bmatrix} A_1^+ \\ B_1^+ \\ A_2^- \\ B_2^- \end{bmatrix} = \begin{bmatrix} 0 & 1 - b \\ 1 - a & 2\delta_1\cdot(1-a)\cdot kh \\ a & 2\delta_1 a\cdot kh \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} A_0^- \\ B_0^- \end{bmatrix} \cdot \exp(-kh)$$

$$\begin{bmatrix} C_1^+ \\ C_2^- \end{bmatrix} = \begin{bmatrix} \frac{1-\gamma}{1+\gamma} \\ \frac{2}{1+\gamma} \end{bmatrix} \cdot \begin{bmatrix} C_0^- \end{bmatrix} \cdot \exp(-kh) \tag{3.16}$$

where: $a \equiv (\delta_1+1)/(\gamma+\delta_1)$

(3.17)

$b \equiv (\delta_1+1)/(\gamma\cdot\delta_2+1)$

Therefore we find that the displacement field (for a given m) in region 1 and region 2 can be written as:

$$\left[\begin{array}{c} \underline{u}(\underline{P}) \\ \hline \underline{u}(\underline{B}) \end{array}\right]_1 = \left[\begin{array}{c|c} +1 & +1-2\delta_1 kz \\ \hline +1 & -1-2\delta_1 kz \end{array}\right] \cdot \left[\begin{array}{c|c} 0 & 1-b \\ \hline 1-a & 2\delta_1 \cdot (1-a) \cdot kh \end{array}\right] \cdot \left[\begin{array}{c} A_0^- \\ \hline B_0^- \end{array}\right] \cdot \exp[k(z-h)]$$

$$+ \left[\text{ source terms }\right]$$

$$\left[\underline{u}(\underline{C})\right]_1 = \left[+1\right] \cdot \left[\frac{1-\gamma}{1+\gamma}\right] \cdot \left[C_0^-\right] \cdot \exp[k(z-h)] \quad + \left[\text{ source terms }\right] \tag{3.18}$$

$$\left[\begin{array}{c} \underline{u}(\underline{P}) \\ \hline \underline{u}(\underline{B}) \end{array}\right]_2 = \left[\begin{array}{c|c} -1 & -1-2\delta_1 kz \\ \hline +1 & -1+2\delta_1 kz \end{array}\right] \cdot \left[\begin{array}{c|c} +a & 2\delta_1 a \cdot kh \\ \hline 0 & +b \end{array}\right] \cdot \left[\begin{array}{c} A_0^- \\ \hline B_0^- \end{array}\right] \cdot \exp[-k(z+h)]$$

$$\left[\underline{u}(\underline{C})\right]_2 = \left[+1\right] \cdot \left[\frac{2}{1+\gamma}\right] \cdot \left[C_0^-\right] \cdot \exp[-k(z+h)] \tag{3.19}$$

The $\underline{u}(\underline{C})$ terms are in a form from which we can deduce the algorithm, however, the $\underline{u}(\underline{P})$ and $\underline{u}(\underline{B})$ terms have to be further manipulated. We now try to express the $\underline{u}(\underline{P})$ and $\underline{u}(\underline{B})$ terms in the following manner:

$$\left[\begin{array}{c} \underline{u}(\underline{P}) \\ \hline \underline{u}(\underline{B}) \end{array}\right]_1 = \left[\begin{array}{c|c} +1 & +1-2\delta_1 k \cdot (z-h) \\ \hline +1 & -1-2\delta_1 k \cdot (z-h) \end{array}\right] \cdot \left[\begin{array}{c} A_a^+ \\ \hline B_a^+ \end{array}\right] \cdot \exp[k(z-h)]$$

$$+ \frac{\partial}{\partial z} \left[\begin{array}{c|c} +1 & +1-2\delta_1 k \cdot (z-h) \\ \hline +1 & -1-2\delta_1 k \cdot (z-h) \end{array}\right] \cdot \left[\begin{array}{c} A_b^+ \\ \hline B_b^+ \end{array}\right] \cdot \exp[k(z-h)]$$

$$+\frac{\partial^2}{\partial z^2}\left[\begin{array}{c} +1 \\ \hline +1 \end{array}\right]\cdot\left[\begin{array}{c} A_c^+ \end{array}\right]\cdot\exp[k(z-h)] \qquad + \left[\begin{array}{c} \text{source terms} \end{array}\right] \tag{3.20}$$

$$\left[\begin{array}{c} \underline{u}(\underline{P}) \\ \hline \underline{u}(\underline{B}) \end{array}\right]_2 = \left[\begin{array}{c|c} -1 & -1-2\delta_1 k\cdot(z+h) \\ \hline +1 & -1+2\delta_1 k\cdot(z+h) \end{array}\right]\cdot\left[\begin{array}{c} A_a^- \\ \hline B_a^- \end{array}\right]\cdot\exp[-k(z+h)]$$

$$\frac{\partial}{\partial z}\left[\begin{array}{c|c} -1 & -1-2\delta_1 k\cdot(z+h) \\ \hline +1 & -1+2\delta_1 k\cdot(z+h) \end{array}\right]\cdot\left[\begin{array}{c} A_b^- \\ \hline B_b^- \end{array}\right]\cdot\exp[-k(z+h)]$$

$$\frac{\partial^2}{\partial z^2}\left[\begin{array}{c} -1 \\ \hline +1 \end{array}\right]\cdot\left[\begin{array}{c} A_c^- \end{array}\right]\cdot\exp[-k(z+h)] \tag{3.21}$$

Noting that:

$$\frac{\partial^n}{\partial z^n}\exp[k(z-h)] = k^n\cdot\exp[k(z-h)]$$

and

$$\frac{\partial^n}{\partial z^n}\exp[-k(z+h)] = (-k)^n\cdot\exp[-k(z+h)] \tag{3.22}$$

We obtain:

$$A_a^+ = (1-b)\cdot B_0^- \qquad\qquad B_a^+ = (1-a)\cdot A_0^-$$

$$A_b^+ = -2\delta_1\cdot(1-a)\cdot h\cdot A_0^- + 4\delta_1^2\cdot(1-a)\cdot h\cdot B_0^- \tag{3.23}$$

$$B_b^+ = 2\delta_1\cdot(1-a)\cdot h\cdot B_0^- \qquad\qquad A_c^+ = -4\delta_1^2\cdot(1-a)\cdot h^2\cdot B_0^-$$

and:

$$A_a^- = a \cdot A_0^- \qquad B_a^- = b \cdot B_0^-$$

$$A_b^- = 2 \cdot (\delta_2 b - \delta_1 a) \cdot h \cdot B_0^- \qquad B_b^- = A_c^- = 0 \tag{3.24}$$

Since the above relations are true for each component of a potential, then they must be true for the whole potential and we get:

if:

$$\underline{u}^0 = \underline{N}(-h, \varphi_1^0) + \underline{E}(\delta_1, -h, \varphi_2^0) + \underline{M}(-h, \varphi_3^0)$$

Then:

$$\begin{aligned}
\underline{u}^1 = \underline{u}^0 &+ \underline{N}(h, (1-b) \cdot \bar{\varphi}_2^0) + \frac{\partial}{\partial z}\underline{N}(h, -2\delta_1 \cdot (1-a) \cdot h \cdot \bar{\varphi}_1^0) \\
&+ \frac{\partial}{\partial z}\underline{N}(h, +4\delta_1^2 \cdot (1-a) \cdot h \cdot \bar{\varphi}_2^0) \\
&+ \frac{\partial^2}{\partial z^2}\underline{N}(h, -4\delta_1^2 \cdot (1-a) \cdot h^2 \cdot \bar{\varphi}_2^0) \\
&+ \underline{E}(\delta_1, h, (1-a)\bar{\varphi}_1^0) \\
&+ \frac{\partial}{\partial z}\underline{E}(\delta_1, h, 2\delta_1 \cdot (1-a) \cdot h \cdot \bar{\varphi}_2^0) \\
&+ \underline{M}(h, \frac{1-\gamma}{1+\gamma} \cdot \bar{\varphi}_3^0)
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
\underline{u}^2 = \underline{N}(-h, a\varphi_1^0) &+ \frac{\partial}{\partial z}\underline{N}(-h, 2 \cdot (\delta_2 b - \delta_1 a) \cdot h \cdot \varphi_2^0) \\
&+ \frac{\partial}{\partial z}\underline{E}(\delta_2, -h, b\varphi_2^0) \\
&+ \underline{M}(-h, \frac{2}{1+\gamma} \cdot \varphi_3^0)
\end{aligned} \tag{3.26}$$

Noting that:

$$\frac{\partial^n}{\partial z^n}\underline{N}(h,cst\cdot\boldsymbol{\varphi}) = \underline{N}(h,cst\cdot\frac{\partial^n}{\partial z^n}\boldsymbol{\varphi})$$

and (3.27)

$$\frac{\partial}{\partial z}F(\boldsymbol{\delta},h,cst\cdot\boldsymbol{\varphi}) = N(h,-2\boldsymbol{\delta}_1\cdot cst\cdot\frac{\partial}{\partial z}\boldsymbol{\varphi}) + F(\boldsymbol{\delta},h,cst\cdot\frac{\partial}{\partial z}\boldsymbol{\varphi})$$

where: cst is a constant

We obtain the algorithm given in the main body of this paper (two minor differences are: i) The statement of the algorithm in the paper considers region 1 to be at $z>0$ and hence $z=+h$ instead of $z=-h$ to be the location of the source point and ii) A formalism in terms of matrix operators is implemented in the main text).

Appendix 4: Analytic check of the image method algorithm

Referring to figure 1 and using the notation defined in the main text, the image algorithm states that if:

$$\underline{u}^0 = \underline{N}(h,\boldsymbol{\varphi}_1^0) + \underline{E}(\boldsymbol{\delta}_1,h,\boldsymbol{\varphi}_2^0) + \underline{M}(h,\boldsymbol{\varphi}_3^0)$$

Then:

$$\underline{u}^1 = \underline{u}^0 + \underline{u}^1(-h,\boldsymbol{\delta}_1,\overline{\underline{\boldsymbol{\varphi}}}_R^1,\overline{\underline{\boldsymbol{\varphi}}}_L^1)$$

$$\underline{u}^2 = u^2(h,\boldsymbol{\delta}_2,\underline{\boldsymbol{\varphi}}_R^2,\underline{\boldsymbol{\varphi}}_L^2)$$

where:

$$\begin{aligned}
\overline{\underline{\boldsymbol{\varphi}}}_R^1 &= \underline{\underline{R}}_R(h,a,b,\boldsymbol{\delta}_1)\cdot\overline{\underline{\boldsymbol{\varphi}}}_R^0 \\
\overline{\underline{\boldsymbol{\varphi}}}_L^1 &= \underline{\underline{R}}_L(\gamma)\cdot\overline{\underline{\boldsymbol{\varphi}}}_L^0 \\
\underline{\boldsymbol{\varphi}}_R^2 &= \underline{\underline{T}}_R(h,a,b,\boldsymbol{\delta}_2,\boldsymbol{\delta}_1)\cdot\underline{\boldsymbol{\varphi}}_R^0 \\
\underline{\boldsymbol{\varphi}}_L^2 &= \underline{\underline{T}}_L(\gamma)\cdot\underline{\boldsymbol{\varphi}}_L^0
\end{aligned} \tag{4.1}$$

We have already expressed the Cartesian components of the displacement and stress fields of any given Hansen potentials (Appendix 1). In this section, we will check whether the Cartesian components of the displacement and stress fields (in terms of Hansen's potentials) that are generated by the image algorithm satisfy the conditions of displacement and traction continuity along the interface plane.

In order to simplify the checking of the algorithm, we will consider the following 3 cases seperately:

i) $\boldsymbol{\varphi}_1 = \boldsymbol{\varphi}^0$; $\quad \boldsymbol{\varphi}_2 = \boldsymbol{\varphi}_3 = 0$

ii) $\boldsymbol{\varphi}_2 = \boldsymbol{\varphi}^0$; $\quad \boldsymbol{\varphi}_1 = \boldsymbol{\varphi}_3 = 0$

iii) $\boldsymbol{\varphi}_3 = \boldsymbol{\varphi}^0$; $\quad \boldsymbol{\varphi}_1 = \boldsymbol{\varphi}_2 = 0$

We will also note the following:

$$\frac{\partial}{\partial z}\bar{\varphi} = -\left[\overline{\frac{\partial}{\partial(-z)}\varphi}\right] \qquad \Rightarrow \frac{\partial}{\partial z}\varphi\bigg|_{z=0} = -\frac{\partial}{\partial z}\bar{\varphi}\bigg|_{z=0}$$

$$\frac{\partial}{\partial x}\varphi\bigg|_{z=0} = \frac{\partial}{\partial x}\bar{\varphi}\bigg|_{z=0} \qquad \text{and} \qquad \frac{\partial}{\partial y}\varphi\bigg|_{z=0} = \frac{\partial}{\partial y}\bar{\varphi}\bigg|_{z=0} \tag{4.2}$$

Also define:

$$\frac{\partial^n}{\partial(x,y,z)^n}\varphi \equiv \frac{\partial^n}{\partial(x,y,z)^n}\varphi^0\bigg|_{z=0}$$

$$\frac{\partial^n}{\partial(x,y,z)^n}\bar{\varphi} \equiv \frac{\partial^n}{\partial(x,y,z)^n}\bar{\varphi}^0\bigg|_{z=0} \tag{4.3}$$

<u>Case I: $\varphi_1 = \varphi^0;\ \varphi_2 = \varphi_3 = 0$</u>

$$\begin{aligned}\underline{u}^1\bigg|_{z=0} = {} & \hat{e}_x\cdot\left[\frac{\partial}{\partial x}\varphi + 2\delta_1\cdot(1-a)\cdot h\cdot\frac{\partial^2}{\partial x\partial z}\bar{\varphi} + (1-a)\cdot\left(-\frac{\partial}{\partial x}\bar{\varphi} - 2\delta\cdot h\cdot\frac{\partial^2}{\partial x\partial z}\bar{\varphi}\right)\right] \\ & + \hat{e}_y\cdot\left[\frac{\partial}{\partial y}\varphi + 2\delta_1\cdot(1-a)\cdot h\cdot\frac{\partial^2}{\partial y\partial z}\bar{\varphi} + (1-a)\cdot\left(-\frac{\partial}{\partial y}\bar{\varphi} - 2\delta\cdot h\cdot\frac{\partial^2}{\partial y\partial z}\bar{\varphi}\right)\right] \\ & + \hat{e}_z\cdot\left[\frac{\partial}{\partial z}\varphi + 2\delta_1\cdot(1-a)\cdot h\cdot\frac{\partial^2}{\partial z^2}\bar{\varphi} + (1-a)\cdot\left(\frac{\partial}{\partial z}\bar{\varphi} - 2\delta\cdot h\cdot\frac{\partial^2}{\partial z^2}\bar{\varphi}\right)\right]\end{aligned} \tag{4.4}$$

$$\underline{u}^2\bigg|_{z=0} = \hat{e}_x\cdot\left[\ a\cdot\frac{\partial}{\partial x}\varphi\ \right] + \hat{e}_y\cdot\left[\ a\cdot\frac{\partial}{\partial y}\varphi\ \right] + \hat{e}_z\cdot\left[\ a\cdot\frac{\partial}{\partial z}\varphi\ \right] \tag{4.5}$$

We notice that the displacement field is continuous across z=0 (the interface plane). Next we consider the $\underline{T}^z$ traction continuity (equilibrium).

$$
\begin{aligned}
\underline{T}^{z1}\Big|_{z=0} = \hat{e}_x \cdot \Big[\, & 2\mu_1 \cdot \frac{\partial^2}{\partial x \partial z}\varphi + 2\delta_1 \cdot (1-a) \cdot 2\mu_1 \cdot h \cdot \frac{\partial^3}{\partial x \partial z^2}\bar{\varphi} \\
& + (1-a) \cdot \Big(-2\mu_1\delta_1 \cdot \frac{\partial^2}{\partial x \partial z}\bar{\varphi} - 4\mu_1\delta_1 \cdot h \cdot \frac{\partial^3}{\partial x \partial z^2}\bar{\varphi}\Big) \Big] \\
+ \hat{e}_y \cdot \Big[\, & 2\mu_1 \cdot \frac{\partial^2}{\partial y \partial z}\varphi + 2\delta_1 \cdot (1-a) \cdot 2\mu_1 \cdot h \cdot \frac{\partial^3}{\partial y \partial z^2}\bar{\varphi} \\
& + (1-a) \cdot \Big(-2\mu_1\delta_1 \cdot \frac{\partial^2}{\partial y \partial z}\bar{\varphi} - 4\mu_1\delta_1 \cdot h \cdot \frac{\partial^3}{\partial y \partial z^2}\bar{\varphi}\Big) \Big] \\
+ \hat{e}_z \cdot \Big[\, & 2\mu_1 \cdot \frac{\partial^2}{\partial z^2}\varphi + 2\delta_1 \cdot (1-a) \cdot 2\mu_1 \cdot h \cdot \frac{\partial^3}{\partial z^3}\bar{\varphi} \\
& + (1-a) \cdot \Big(2\mu_1\delta_1 \cdot \frac{\partial^2}{\partial z^2}\bar{\varphi} - 4\mu_1\delta_1 \cdot h \cdot \frac{\partial^3}{\partial z^3}\bar{\varphi}\Big) \Big]
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
\underline{T}^{z2}\Big|_{z=0} = {} & \hat{e}_x \cdot \Big[\, 2\mu_2 a \cdot \frac{\partial^2}{\partial x \partial z}\varphi \,\Big] \\
& + \hat{e}_y \cdot \Big[\, 2\mu_2 a \cdot \frac{\partial^2}{\partial y \partial z}\varphi \,\Big] \\
& + \hat{e}_z \cdot \Big[\, 2\mu_2 a \cdot \frac{\partial^2}{\partial z^2}\varphi \,\Big]
\end{aligned}
\tag{4.7}
$$

Noting that:

$$2\mu_2 \cdot [1 + (1-a) \cdot \delta_1] = 2\mu_2 \cdot a \tag{4.8}$$

since

$$2\mu_1 \cdot \left[\frac{\gamma + \delta_1 + \delta_1\gamma - \delta_1}{\gamma + \delta_1}\right] = 2\mu_2 \cdot \frac{\delta_1 + 1}{\gamma + \delta_1}$$

We find that the tractions across the interface plane (z=0) are continuous.

<u>Case II: $\varphi_2 = \varphi^0;\ \varphi_1 = \varphi_3 = 0$</u>

$$\underline{u}^1\Big|_{z=0} = \hat{e}_x \cdot \Big[-\frac{\partial}{\partial x}\varphi + 2\delta_1 h \cdot \frac{\partial^2}{\partial x \partial z}\varphi + (1-b)\cdot\frac{\partial}{\partial x}\bar{\varphi} - 4\delta_1^2\cdot(1-a)\cdot h^2 \frac{\partial^3}{\partial x \partial z^2}\bar{\varphi}$$
$$- 2\delta_1\cdot(1-a)\cdot h\cdot(-\frac{\partial^2}{\partial x\partial z}\bar{\varphi} - 2\delta_1\cdot h\cdot\frac{\partial^3}{\partial x\partial z^2}\bar{\varphi}) \Big]$$
$$+ \hat{e}_y \cdot \Big[-\frac{\partial}{\partial y}\varphi + 2\delta_1 h \cdot \frac{\partial^2}{\partial y \partial z}\varphi + (1-b)\cdot\frac{\partial}{\partial y}\bar{\varphi} - 4\delta_1^2\cdot(1-a)\cdot h^2 \frac{\partial^3}{\partial y \partial z^2}\bar{\varphi}$$
$$- 2\delta_1\cdot(1-a)\cdot h\cdot(-\frac{\partial^2}{\partial y\partial z}\bar{\varphi} - 2\delta_1\cdot h\cdot\frac{\partial^3}{\partial y\partial z^2}\bar{\varphi}) \Big]$$
$$+ \hat{e}_z \cdot \Big[\frac{\partial}{\partial z}\varphi + 2\delta_1 h \cdot \frac{\partial^2}{\partial z^2}\varphi + (1-b)\cdot\frac{\partial}{\partial z}\bar{\varphi} - 4\delta_1^2\cdot(1-a)\cdot h^2 \frac{\partial^3}{\partial z^3}\bar{\varphi}$$
$$- 2\delta_1\cdot(1-a)\cdot h\cdot(\frac{\partial^2}{\partial z^2}\bar{\varphi} - 2\delta_1\cdot h\cdot\frac{\partial^3}{\partial z^3}\bar{\varphi}) \Big] \tag{4.9}$$

$$\underline{u}^2\Big|_{z=0} = \hat{e}_x \cdot \Big[-2\cdot h\cdot(\delta_2 b - \delta_1 a)\cdot\frac{\partial^2}{\partial x\partial z}\varphi + b\cdot(-\frac{\partial}{\partial x}\varphi + 2\delta_2\cdot h\cdot\frac{\partial^2}{\partial x\partial z}\varphi) \Big]$$
$$+ \hat{e}_y \cdot \Big[-2\cdot h\cdot(\delta_2 b - \delta_1 a)\cdot\frac{\partial^2}{\partial y\partial z}\varphi + b\cdot(-\frac{\partial}{\partial y}\varphi + 2\delta_2\cdot h\cdot\frac{\partial^2}{\partial y\partial z}\varphi) \Big]$$
$$+ \hat{e}_z \cdot \Big[-2\cdot h\cdot(\delta_2 b - \delta_1 a)\cdot\frac{\partial^2}{\partial z^2}\varphi + b\cdot(\frac{\partial}{\partial z}\varphi + 2\delta_2\cdot h\cdot\frac{\partial^2}{\partial z^2}\varphi) \Big] \tag{4.10}$$

We notice that the displacement field is continuous across z=0 (the interface plane). Next we consider the $\underline{T}^z$ traction continuity (equilibrium).

$$\underline{T}^{z1}\Big|_{z=0} = \hat{e}_x \cdot \Big[-2\mu_1\delta_1\cdot\frac{\partial^2}{\partial x\partial z}\varphi + 4\mu_1\cdot\delta_1\cdot h\cdot\frac{\partial^3}{\partial x\partial z^2}\varphi + 2\mu_1\cdot(1-b)\cdot\frac{\partial^2}{\partial x\partial z}\bar{\varphi}$$

$$
\begin{aligned}
&- 4\delta_1^2\cdot(1-a)\cdot 2\mu_1\cdot h^2\cdot\frac{\partial^4}{\partial x\partial z^3}\bar{\varphi} \\
&- 2\delta_1\cdot(1-a)\cdot h\cdot\left(-2\mu_1\delta_1\cdot\frac{\partial^3}{\partial x\partial z^2}\bar{\varphi} - 4\mu_1\delta_1\cdot h\cdot\frac{\partial^4}{\partial x\partial z^3}\bar{\varphi}\right) \Big] \\
+ \hat{e}_y\cdot\Big[&-2\mu_1\delta_1\cdot\frac{\partial^2}{\partial y\partial z}\varphi + 4\mu_1\cdot\delta_1\cdot h\cdot\frac{\partial^3}{\partial y\partial z^2}\varphi + 2\mu_1\cdot(1-b)\cdot\frac{\partial^2}{\partial y\partial z}\bar{\varphi} \\
&- 4\delta_1^2\cdot(1-a)\cdot 2\mu_1\cdot h^2\cdot\frac{\partial^4}{\partial y\partial z^3}\bar{\varphi} \\
&- 2\delta_1\cdot(1-a)\cdot h\cdot\left(-2\mu_1\delta_1\cdot\frac{\partial^3}{\partial y\partial z^2}\bar{\varphi} - 4\mu_1\delta_1\cdot h\cdot\frac{\partial^4}{\partial y\partial z^3}\bar{\varphi}\right) \Big] \\
+ \hat{e}_z\cdot\Big[&\ 2\mu_1\delta_1\cdot\frac{\partial^2}{\partial z^2}\varphi + 4\mu_1\cdot\delta_1\cdot h\cdot\frac{\partial^3}{\partial z^3}\varphi + 2\mu_1\cdot(1-b)\cdot\frac{\partial^2}{\partial z^2}\bar{\varphi} \\
&- 4\delta_1^2\cdot(1-a)\cdot 2\mu_1\cdot h^2\cdot\frac{\partial^4}{\partial z^4}\bar{\varphi} \\
&- 2\delta_1\cdot(1-a)\cdot h\cdot\left(2\mu_1\delta_1\cdot\frac{\partial^3}{\partial z^3}\bar{\varphi} - 4\mu_1\delta_1\cdot h\cdot\frac{\partial^4}{\partial z^4}\bar{\varphi}\right) \Big]
\end{aligned}
\tag{4.11}
$$

$$
\begin{aligned}
\underline{T}^{z2}\Big|_{z=0} = \hat{e}_x\cdot\Big[&-2h\cdot(\delta_2 b-\delta_1 a)\cdot 2\mu_2\cdot\frac{\partial^3}{\partial x\partial z^2}\varphi \\
&+ b\cdot\left(-2\mu_2\delta_2\cdot\frac{\partial^2}{\partial x\partial z}\varphi + 4\mu_2\delta_2\cdot h\cdot\frac{\partial^3}{\partial x\partial z^2}\varphi\right) \Big] \\
+ \hat{e}_y\cdot\Big[&-2h\cdot(\delta_2 b-\delta_1 a)\cdot 2\mu_2\cdot\frac{\partial^3}{\partial y\partial z^2}\varphi \\
&+ b\cdot\left(-2\mu_2\delta_2\cdot\frac{\partial^2}{\partial y\partial z}\varphi + 4\mu_2\delta_2\cdot h\cdot\frac{\partial^3}{\partial y\partial z^2}\varphi\right) \Big] \\
+ \hat{e}_z\cdot\Big[&-2h\cdot(\delta_2 b-\delta_1 a)\cdot 2\mu_2\cdot\frac{\partial^3}{\partial z^3}\varphi \\
&+ b\cdot\left(2\mu_2\delta_2\cdot\frac{\partial^2}{\partial z^2}\varphi + 4\mu_2\delta_2\cdot h\cdot\frac{\partial^3}{\partial z^3}\varphi\right) \Big]
\end{aligned}
\tag{4.12}
$$

Noting that:

$$2\mu_1 \cdot [\delta_1 + (1-b)] = 2\mu_2 \cdot b \cdot \delta_2$$

since

$$2\mu_1 \cdot \left[\frac{\gamma\delta_2\delta_1 + \gamma\delta_2}{1 + \gamma\delta_2} \right] = 2\mu_2 \cdot \frac{\delta_1\delta_2 + \delta_2}{1 + \gamma\delta_2}$$

and

$$4\mu_1[\delta_1 + \delta_1^2 \cdot (1-a)] = 4\mu_2\delta_1 \cdot a \tag{4.13}$$

We find that the tractions across the interface plane (z=0) are continuous.

Case III: $\varphi_3 = \varphi^0; \ \varphi_1 = \varphi_2 = 0$

$$\underline{u}^1 \Big|_{z=0} = \hat{e}_x \cdot \left[\frac{\partial}{\partial y}\varphi + \frac{1-\gamma}{1+\gamma} \cdot \frac{\partial}{\partial y}\bar{\varphi} \right] + \hat{e}_y \cdot \left[-\frac{\partial}{\partial x}\varphi - \frac{1-\gamma}{1+\gamma} \cdot \frac{\partial}{\partial x}\bar{\varphi} \right] + \hat{e}_z \cdot [\ 0\ ] \tag{4.14}$$

$$\underline{u}^2 \Big|_{z=0} = \hat{e}_x \cdot \left[\frac{2}{1+\gamma} \cdot \frac{\partial}{\partial y}\varphi \right] + \hat{e}_y \cdot \left[-\frac{2}{1+\gamma} \cdot \frac{\partial}{\partial x}\varphi \right] + \hat{e}_z \cdot [\ 0\ ] \tag{4.15}$$

We notice that the displacement field is continuous across z=0 (the interface plane). Next we consider the $\underline{T}^z$ traction continuity (equilibrium).

$$\underline{T}^{z1} \Big|_{z=0} = \hat{e}_x \cdot \left[\mu_1 \cdot \frac{\partial^2}{\partial y \partial z}\varphi + \frac{1-\gamma}{1+\gamma} \cdot \mu_1 \cdot \frac{\partial^2}{\partial y \partial z}\bar{\varphi} \right]$$

$$+ \hat{e}_y \cdot \left[-\mu_1 \cdot \frac{\partial^2}{\partial x \partial z}\varphi - \frac{1-\gamma}{1+\gamma} \cdot \mu_1 \cdot \frac{\partial^2}{\partial x \partial z}\bar{\varphi} \right] \qquad + \hat{e}_z \cdot \left[0 \right] \tag{4.16}$$

$$\begin{aligned} \underline{T}^{z2}\Big|_{z=0} &= \hat{e}_x \cdot \left[\mu_2 \cdot \frac{2}{1+\gamma} \cdot \frac{\partial^2}{\partial y \partial z}\varphi \right] \\ &+ \hat{e}_y \cdot \left[-\mu_2 \cdot \frac{2}{1+\gamma} \cdot \frac{\partial^2}{\partial x \partial z}\varphi \right] \qquad + \hat{e}_z \cdot \left[0 \right] \end{aligned} \tag{4.17}$$

Noting that:

$$\mu_1 \cdot \left[1 - \frac{1-\gamma}{1+\gamma} \right] = \mu_2 \cdot \frac{2}{1+\gamma} \tag{4.18}$$

We find that the tractions across the interface plane (z=0) are continuous.

Therefore, all three cases check correctly, and the algorithm is correct.

Appendix 5: Derivation of some sample Green's functions through the use of the image algorithm

In this section, we consider displacement fields in Cartesian components for some sample point source problems on or in a halfspace with a free surface. The solutions that will be rederived are readily available (and established) in the literature and hence serve as an empirical check (see appendix 4 for an analytic check) of the algorithm. In addition, these specific examples help clarify details of the application of the algorithm.

By considering a halfspace problem with a free surface, we obtain the following simplifications:

$$\gamma = 0 \qquad 1-a = -1/\delta_1 \qquad 1-b = -\delta_1$$

call:

$$\delta \equiv \delta_1$$

then:

$$\underline{\underline{R}}_R(h,a,b,\delta_1) = \left[\begin{array}{c|c} -2h\cdot\frac{\partial}{\partial z} & -\delta + 4\delta h^2\cdot\frac{\partial^2}{\partial z^2} \\ \hline -1/\delta & +2h\cdot\frac{\partial}{\partial z} \end{array}\right] \qquad R_L(\gamma) = \begin{bmatrix} 1 \end{bmatrix} \tag{5.1}$$

The following example problems will be considered:

I. Screw dislocation (Antiplane problem).

II. Line force (x=0, z=0) acting on a free surface.

i) The line force is in the x direction.

ii) The line force is in the z direction (Flamant's solution).

III. Point force acting on a free surface.

i) The point force is in the x direction (Cerruti's solution).

ii) The point force is in the z direction (Boussinesq solution).

IV. Point force acting interior to the halfspace with a free surface (Mindlin's solution).

i) The point force is in the x direction.

ii) The point force is in the z direction.

I. Screw dislocation (antiplane problem)

For the antiplane problem, all that is required is to obtain the image potential with respect to the interface plane since the $\underline{\underline{R}}_L$ matrix is the identity operator. Also, we notice that getting the image of a given $\underline{M}$ type (see main text) displacement field is equal to the $\underline{M}$ displacement field of the image potential describing that field (i.e. $\overline{\underline{M}(h,\underline{\mathbf{P}})} = \underline{M}(-h,\overline{\underline{\mathbf{P}}})$), and hence we can directly operate on a given displacement field when using the algorithm for a purely antiplane problem. This corresponds to the scalar field image method for the antiplane case.

As an example we consider the field due to a screw dislocation in the plane perpendicular to the x-z plane at location z=h and x=0. The field due to the dislocation in infinite space is:

$$u_y = \arctan[(z-h)/x] \tag{5.2}$$

The image field will be $\overline{u}_y$ which implies that the combined fields give:

$$u_y = \arctan[(z-h)/x] - \arctan[(z+h)/x] \tag{5.3}$$

Of course this is but a simple application of the scalar image method.

II. Line force ($x=0$, $z=0$) acting on a free surface.

For this problem the potentials are given in appendix 2. We note that $\varphi_3 = 0$ (i.e. there is no antiplane mode in a plane strain problem as is trivially known).

We note that for the case when the point source is at the interface (i.e. h=0), we get a significant simplification in the $\underline{\underline{R}}_R$ matrix operator in the following way:

$$\underline{\underline{R}}_R(0,a,b,\delta_1) = \left[\begin{array}{c|c} 0 & -\delta \\ \hline 1/\delta & 0 \end{array}\right] \tag{5.4}$$

This means that for the displacement fields all we need to calculate are the following (e.g. see equations (2)):

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right] \cdot \left[\bar{\varphi}_2^0\right] \qquad \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial z}, \frac{\partial^2}{\partial x \partial z}, \frac{\partial^2}{\partial z^2}\right] \cdot \left[\bar{\varphi}_1^0\right]$$

Now we perform the above differentiations for the image potentials due to a line force. These potentials are linear combinations of the following functions:

$$\begin{aligned} \bar{\varphi}_A &= (z \cdot \arctan(z/x) - x \cdot \ln\xi + (1+\delta) \cdot x)/2 = \varphi_A \\ \bar{\varphi}_C &= (z \cdot \ln\xi - z + x \cdot \arctan(z/x))/2 = -\varphi_C \end{aligned} \tag{5.5}$$

where: $\xi^2 = x^2 + z^2$

We get:

$$\frac{\partial}{\partial x}\bar{\varphi}_A = -\ln\xi/2 + \delta/2 \qquad \frac{\partial}{\partial x}\bar{\varphi}_C = [\arctan(z/x)]/2$$

$$\frac{\partial}{\partial z}\bar{\varphi}_A = [\arctan(z/x)]/2 \qquad \frac{\partial}{\partial z}\bar{\varphi}_C = \ln\xi/2$$

$$\frac{\partial^2}{\partial x \partial z}\bar{\varphi}_A = -(z/\xi^2)/2 \qquad \frac{\partial^2}{\partial x \partial z}\bar{\varphi}_C = (x/\xi^2)/2$$

$$\frac{\partial^2}{\partial z^2}\bar{\varphi}_A = (x/\xi^2)/2 \qquad \frac{\partial^2}{\partial z^2}\bar{\varphi}_C = (z/\xi^2)/2$$

(5.6)

i) Case when the force is acting in the x direction:

We get:

$$\underline{u} = \underline{u}^0 + [\alpha/(4\pi\mu\delta)]\cdot\Big[\hat{e}_x\cdot[-\delta\cdot\ln\xi/2 - \ln\xi/(2\delta) - z^2/\xi^2 + \text{constants}]\;\; \hat{e}_z\cdot[\delta\cdot\arctan(z/x)/2 - \arctan(z/x)/(2\delta) + xz/\xi^2]\Big]$$

(5.7)

Noting the following identities:

$$z^2/\xi^2 = 1 - x^2/\xi^2 \qquad \delta/2-1/(2\delta) = -\mu\cdot(1+\delta)/(\lambda+\mu)$$

$$\delta/2+1/(2\delta)+1 = (1+\delta)^2/(2\delta) \qquad \alpha = 2\cdot\delta/(\delta+1)$$

(5.8)

We find that the above solution coincides with that given in Love 1927 (article 151), except for a rigid body motion.

i) Case when the force is acting in the z direction (Flamant's solution):

We get:

$$\underline{u} = \underline{u}^0 - [\alpha/(4\pi\mu\delta)]\cdot\Big[\hat{e}_x\cdot[\delta\cdot\arctan(z/x)/2 - \arctan(z/x)/(2\delta) - xz/\xi^2] + \hat{e}_z\cdot[\delta\cdot\ln\xi/2 + \ln\xi/(2\delta) - z^2/\xi^2] \Big] \tag{5.9}$$

Noting the same identities mentioned above (5.8), we find that the above solution coincides with that given in Love 1927 (article 151), except for a rigid body motion.

III. Point force acting on a free surface.

For this problem the potentials for the point source in infinite space are given in appendix 2. However, we have a choice of where to locate the singularities of the potentials. Since we do not want the image potentials to introduce any new sources inside the halfspace, we choose the infinite space potentials to have all their singularities in that halfspace.

Again, if we are only interested in the case when the point force acts on the free surface (h=0), we get equation (5.4). This means that all we need to calculate are the following:

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right]\cdot\left[\bar{\varphi}^0_2\right] \qquad \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial^2}{\partial x\partial z}, \frac{\partial^2}{\partial y\partial z}, \frac{\partial^2}{\partial z^2}\right]\cdot\left[\bar{\varphi}^0_1\right]$$

Now we perform the above differentiations for the image potentials due to a point force. These potentials are linear combinations of the following functions:

$$\bar{\varphi}_A = x/[2\cdot(r+z)]$$
$$\bar{\varphi}_C = -[\ln(r+z)]/2 \qquad (5.10)$$

In addition, we have an antiplane potential for this case which is a linear multiple of the following function:

$$\bar{\varphi}_B = y/[2\cdot(r+z)] \qquad (5.11)$$

and we will also have to calculate:

$$\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]\cdot\left[\bar{\varphi}_B\right]$$

We get:

$$\frac{\partial}{\partial x}\bar{\varphi}_A = \left[\ 1/(r+z) - x^2/[r\cdot(r+z)^2]\ \right]/2$$

$$\frac{\partial}{\partial y}\bar{\varphi}_A = \left[\ -(xy)/[r\cdot(r+z)^2]\ \right]/2$$

$$\frac{\partial}{\partial z}\bar{\varphi}_A = \left[\ -x/[r\cdot(r+z)]\ \right]/2$$

$$\frac{\partial^2}{\partial x\partial z}\bar{\varphi}_A = \left[\ -1/[r\cdot(r+z)] + x^2/[r^3\cdot(r+z)] + x^2/[r^2\cdot(r+z)^2]\ \right]/2$$

$$\frac{\partial^2}{\partial y\partial z}\bar{\varphi}_A = \left[\ (xy)/[r^3\cdot(r+z)] + (xy)/[r^2\cdot(r+z)^2]\ \right]/2$$

$$\frac{\partial^2}{\partial z^2}\bar{\varphi}_A = \left[\ x/r^3\ \right]/2$$

(5.12)

$$\frac{\partial}{\partial x}\bar{\varphi}_C = \left[\ -x/[r\cdot(r+z)]\ \right]/2 \qquad \frac{\partial^2}{\partial x\partial z}\bar{\varphi}_C = \left[\ x/r^3\ \right]/2$$

$$\frac{\partial}{\partial y}\bar{\varphi}_C = \left[-y/[r\cdot(r+z)] \right]/2 \qquad \frac{\partial^2}{\partial y \partial z}\bar{\varphi}_C = \left[y/r^3 \right]/2 \tag{5.13}$$

$$\frac{\partial}{\partial z}\bar{\varphi}_C = \left[-1/r \right]/2 \qquad \frac{\partial^2}{\partial z^2}\bar{\varphi}_C = \left[z/r^3 \right]/2$$

$$\begin{aligned} \frac{\partial}{\partial x}\bar{\varphi}_B &= \left[-(xy)/[r\cdot(r+z)^2] \right]/2 \\ \frac{\partial}{\partial y}\bar{\varphi}_B &= \left[1/(r+z) - y^2/[r\cdot(r+z)^2] \right]/2 \end{aligned} \tag{5.14}$$

i) Case when the force is acting in the x direction:

We get:

$$\begin{aligned} \underline{u} = \underline{u}^0 & \\ + [1/4\pi\mu(1+\delta)]\cdot \Big[& \\ \hat{e}_x\cdot \Big[& \delta\cdot/[2\cdot(r+z)] - (\delta\cdot x^2)/[2r\cdot(r+z)^2] \\ & + 1/[2\delta\cdot(r+z)] - x^2/[2\delta r\cdot(r+z)^2] \\ & - z/[r\cdot(r+z)] + (zx^2)/[r^3\cdot(r+z)] \\ & + (zx^2)/[r^2\cdot(r+z)^2)] \\ & + (1+\delta)/(r+z) - (1+\delta)\cdot y^2/[r\cdot(r+z)^2] \Big] \\ + \hat{e}_y\cdot \Big[& -(\delta\cdot xy)/[2r\cdot(r+z)^2] \\ & - (xy)/[2\delta r\cdot(r+z)^2] \\ & + (xyz)/[r^3\cdot(r+z)] \\ & + (xyz)/[r^2\cdot(r+z)^2)] \\ & + (1+\delta)\cdot(xy)/[r\cdot(r+z)^2] \Big] \end{aligned}$$

$$+ \hat{e}_z \cdot \Big[-(\delta\cdot x)/[2r\cdot(r+z)] + x/[2\delta r\cdot(r+z)] + (xz)/r^3 \Big] \Big] \tag{5.15}$$

Noting the following identities:

$$(x^2z)/[r^3\cdot(r+z)] + (x^2z)/[r^2\cdot(r+z)^2] = x^2/r^3 - x^2/[r\cdot(r+z)^2]$$
$$z/[r\cdot(r+z)] = 1/r - 1/(r+z)$$
$$y^2/[r\cdot(r+z)^2] = 2/(r+z) - 1/r - x^2/[r\cdot(r+z)^2]$$
$$(xyz)/[r^3\cdot(r+z)] + (xyz)/[r^2\cdot(r+z)^2)] = (xy)/r^3 - (xy)/[r\cdot(r+z)^2]$$
$$-\delta/2+1/(2\delta) = \mu\cdot(1+\delta)/(\lambda+\mu) \tag{5.16}$$

We find that the above result coincides with the published results (e.g. Love 1927, page 243).

ii) Case when the force is acting in the z direction:

We get:

$$\underline{u} = \underline{u}^0$$
$$- [1/4\pi\mu(1+\delta)]\cdot\Big[$$
$$\hat{e}_x\cdot\Big[-(\delta\cdot x)/[2r\cdot(r+z)] + x/[2\delta r\cdot(r+z)] -(xz)/r^3 \Big]$$
$$+ \hat{e}_y\cdot\Big[-(\delta\cdot y)/[2r\cdot(r+z)] + y/[2\delta r\cdot(r+z)] - (yz)/r^3 \Big]$$

$$+ \hat{e}_z \cdot \left[-\delta/(2r) - 1/(2\delta r) - z^2/r^3 \right] \Bigg] \tag{5.17}$$

Noting the identities in (5.16) concerning material parameters, we find that the above result coincides with the published results (e.g. Love 1927, page 191).

IV. Point force acting interior to a halfspace with a free surface.

The infinite space source potentials for this case are the same as for case III. However, since $h \neq 0$ in general, the matrix operator involved in the algorithm (5.1) requires us to further calculate in addition to the terms shown in case III, the following functions:

$$\frac{\partial}{\partial z} \cdot \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot \left[\bar{\varphi}_1^0\right] \qquad \frac{\partial^2}{\partial z^2} \cdot \left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \cdot \left[\bar{\varphi}_2^0\right]$$

$$\frac{\partial}{\partial z}\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial^2}{\partial x \partial z}, \frac{\partial^2}{\partial y \partial z}, \frac{\partial^2}{\partial z^2}\right] \cdot \left[\bar{\varphi}_2^0\right]$$

There are some repetition in the suggested functions to be calculated since the partial differentiation operations are commutative when the function is sufficiently smooth. Now we perform the additional required differentiations:

$$\frac{\partial^3}{\partial z^2 \partial x} \bar{\varphi}_A = \left[1/r^3 - 3 \cdot x^2/r^5 \right]/2$$

$$\frac{\partial^3}{\partial z^2 \partial y} \bar{\varphi}_A = \left[-3 \cdot (xy)/r^5 \right]/2 \tag{5.18}$$

$$\frac{\partial^3}{\partial z^3}\bar{\varphi}_A = \left[-3\cdot(xz)/r^5 \right]/2$$

$$\frac{\partial^3}{\partial z^2 \partial x}\bar{\varphi}_C = \left[-3\cdot(xz)/r^5 \right]/2$$

$$\frac{\partial^3}{\partial z^2 \partial y}\bar{\varphi}_C = \left[-3\cdot(yz)/r^5 \right]/2 \qquad (5.19)$$

$$\frac{\partial^3}{\partial z^3}\bar{\varphi}_C = \left[1/r^3 - 3\cdot z^2/r^5 \right]/2$$

i) Case when the force is acting in the x direction:

Defining:

$$r_2^2 = x^2 + y^2 + (z+h)^2$$

We get after simplification and the use of identities similar to those given in (5.16), but with z replaced by (z+h) and "r" replaced by "r_2" wherever they occur:

$$\begin{aligned}
\underline{u} = \underline{u}^0 \\
&+ [1/4\pi\mu(1+\delta)]\cdot\Big[\\
&\hat{e}_x\cdot\Big[[\delta/2 + 1/(2\delta) + 1 - 2\cdot(1+\delta) + (1+\delta)]/(r_2+z+h) \\
&\quad + [-\delta/2 - 1/(2\delta) - 1 + (1+\delta)]\cdot x^2/[r_2\cdot(r_2+z+h)^2] \\
&\quad + [-1 + (1+\delta)]/r_2 \\
&\quad + x^2/r_2^3 + 2\delta hz\cdot[1/r_2^3 - 3\cdot x^2/r_2^5] \Big] \\
&+ \hat{e}_y\cdot\Big[[-\delta/2 - 1/(2\delta) - 1 - (1+\delta)]\cdot(xy)/[r_2\cdot(r_2+z+h)^2] \\
&\quad + (xy)/r_2^3 + 2\delta hz\cdot[1/r_2^3 - 3\cdot x^2/r_2^5] \Big]
\end{aligned}$$

$$+ \hat{e}_z \cdot \Big[[-\delta/2 + 1/(2\delta)] \cdot x/[r_2 \cdot (r_2+z+h)] + (z-h) \cdot x/r_2^3 - 6\delta hz \cdot x \cdot (z+h)/r_2^5 \Big] \Bigg] \tag{5.20}$$

Noting that:

$$\begin{aligned} \delta/[4\pi\mu \cdot (1+\delta)] &= 1/[16\pi\mu \cdot (1-\nu)] \\ 1/\delta &= 3-4\nu \\ -\delta/2+1/(2\delta) &= 4\delta \cdot (1-\nu) \cdot (1-2\nu) \end{aligned} \tag{5.21}$$

We find that the above result (5.20) coincides with the solution first obtained by Mindlin (1936) and shown in Mura (1982).

ii) Case when the force is acting in the z direction:

After simplifications we get:

$$\begin{aligned} \underline{u} = \underline{u}^0 & \\ + [1/4\pi\mu(1+\delta)] \cdot \Bigg[& \\ \hat{e}_x \cdot \Big[& + [\delta/2 - 1/(2\delta)] \cdot x/[r_2 \cdot (r_2+z+h)] + (z-h) \cdot x/r_2^3 + 6\delta hz \cdot x \cdot (z+h)/r_2^5 \Big] \\ + \hat{e}_y \cdot \Big[& [\delta/2 - 1/(2\delta)] \cdot y/[r_2 \cdot (r_2+z+h)] + (z-h) \cdot y/r_2^3 + 6\delta hz \cdot y \cdot (z+h)/r_2^5 \Big] \\ + \hat{e}_z \cdot \Big[& [\delta/2 + 1/(2\delta)]/r_2 + [(z+h)^2 - 2\delta hz]/r_2^3 + 6\delta hz \cdot (z+h)^2/r_2^5 \Big] \Bigg] \end{aligned} \tag{5.22}$$

Noting the relations given in (5.21) and:

$$\delta/2+1/(2\delta) = \delta\cdot[8\cdot(1-\nu)^2 - (3-4\nu)] \tag{5.23}$$

We find that the above result (5.22) coincides with the solution first obtained by Mindlin (1936) and shown in Mura (1982).

Figure Captions

figure 1:

2 bonded elastic halfspaces with a point source at z=h.

figure 2:

An elastic plate of thickness H perfectly bonded to two elastic halfspaces, with a point source in region 1 (the elastic plate).

figure 3:

Location of the point source and image point sources for the elastic field in region 1 (left series of points) and region 2 (right series of points) when the point source is located in region 1 (the elastic plate).

figure 4:

Location of the point source and image point sources for the elastic field in region 1 (left series of points) and region 2 (right series of points) when the point source is located in region 2 (the lower halfspace).

figure 3.1:

2 bonded elastic halfspaces with a point source at z=-h.

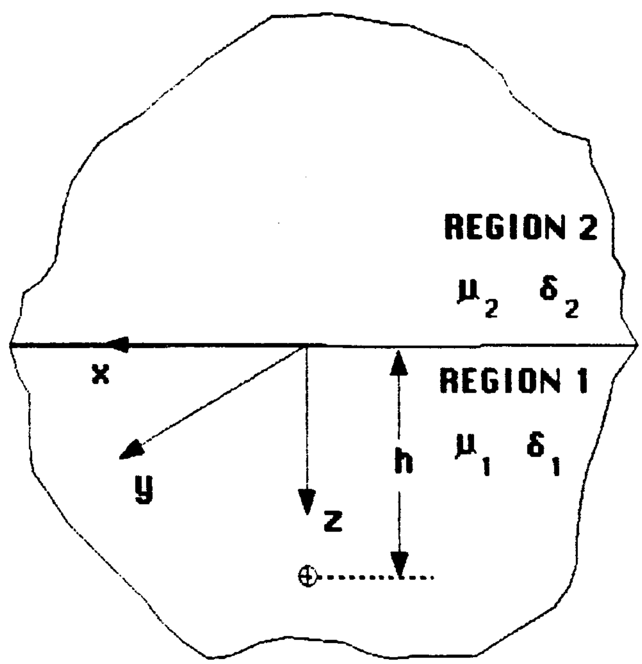

Figure 1

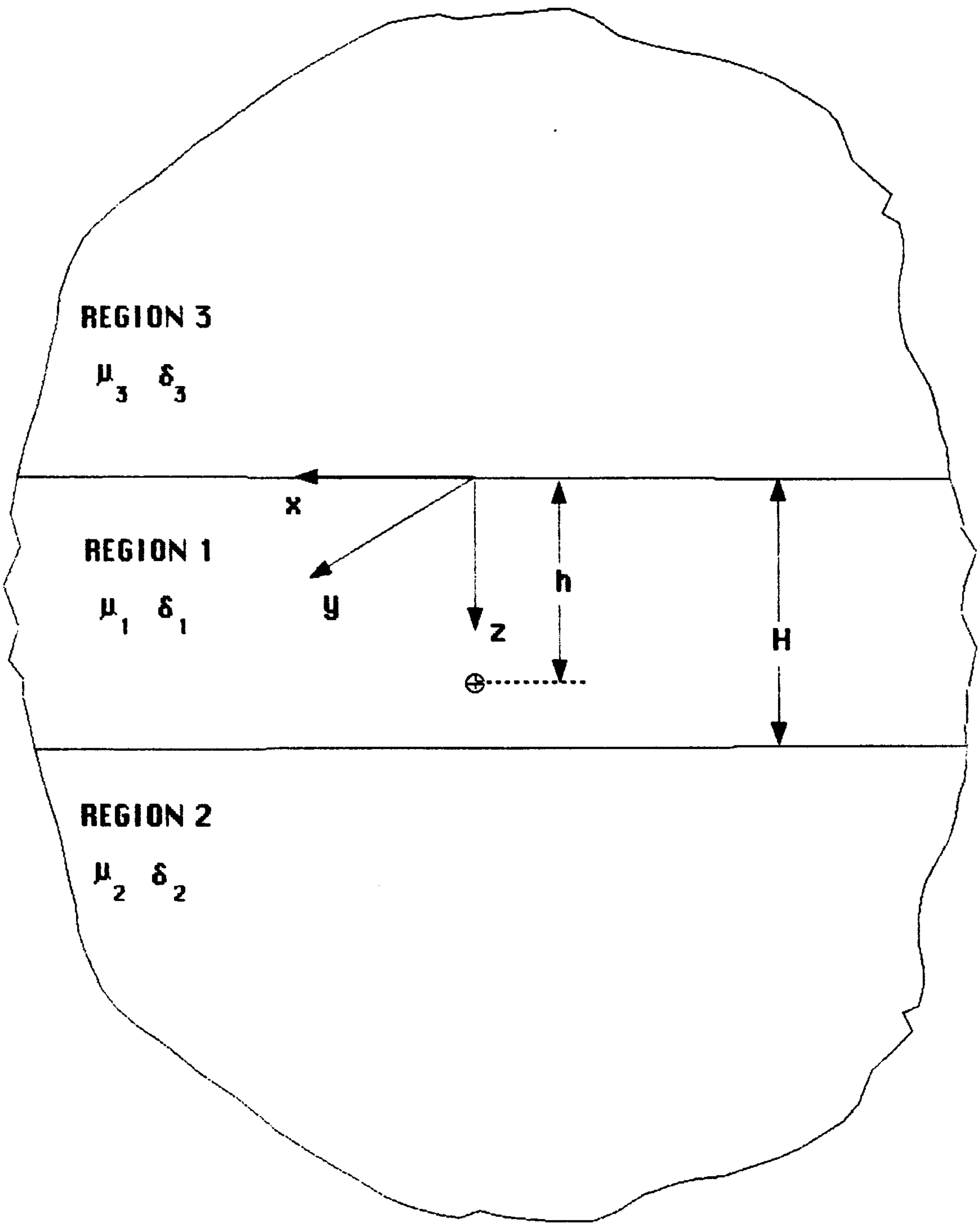
REGION 3
μ_3 δ_3
x
REGION 1
μ_1 δ_1
y
z
h
H
REGION 2
μ_2 δ_2

Figure 2

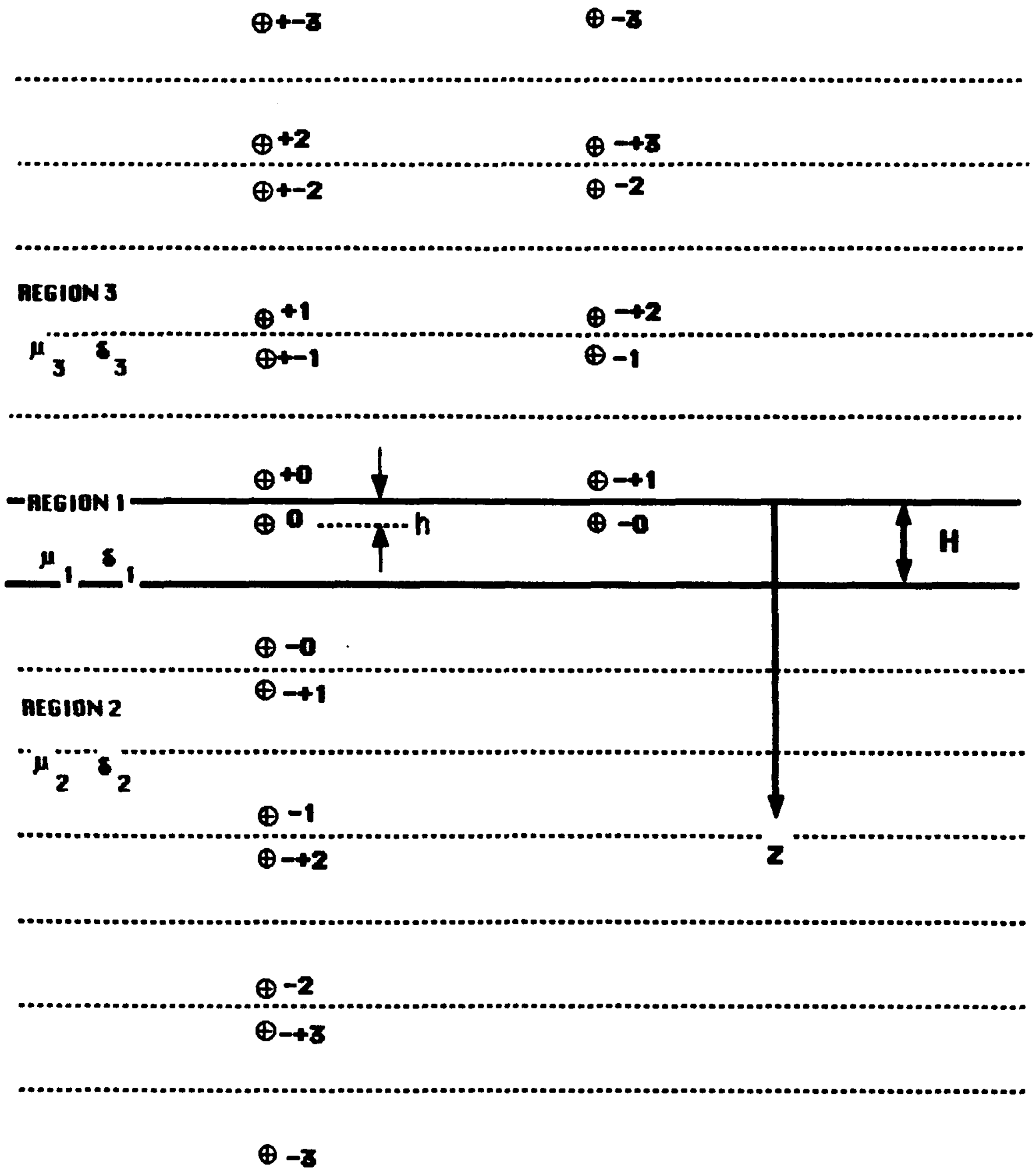
⊕+-3
⊕-3
⊕+2
⊕-+3
⊕+-2
⊕-2
REGION 3
⊕+1
⊕-+2
μ3 ε3
⊕+-1
⊕-1
⊕+0
⊕-+1
REGION 1
⊕0
h
⊕-0
H
μ1 ε1
⊕-0
⊕-+1
REGION 2
μ2 ε2
⊕-1
⊕-+2
z
⊕-2
⊕-+3
⊕-3

Figure 3

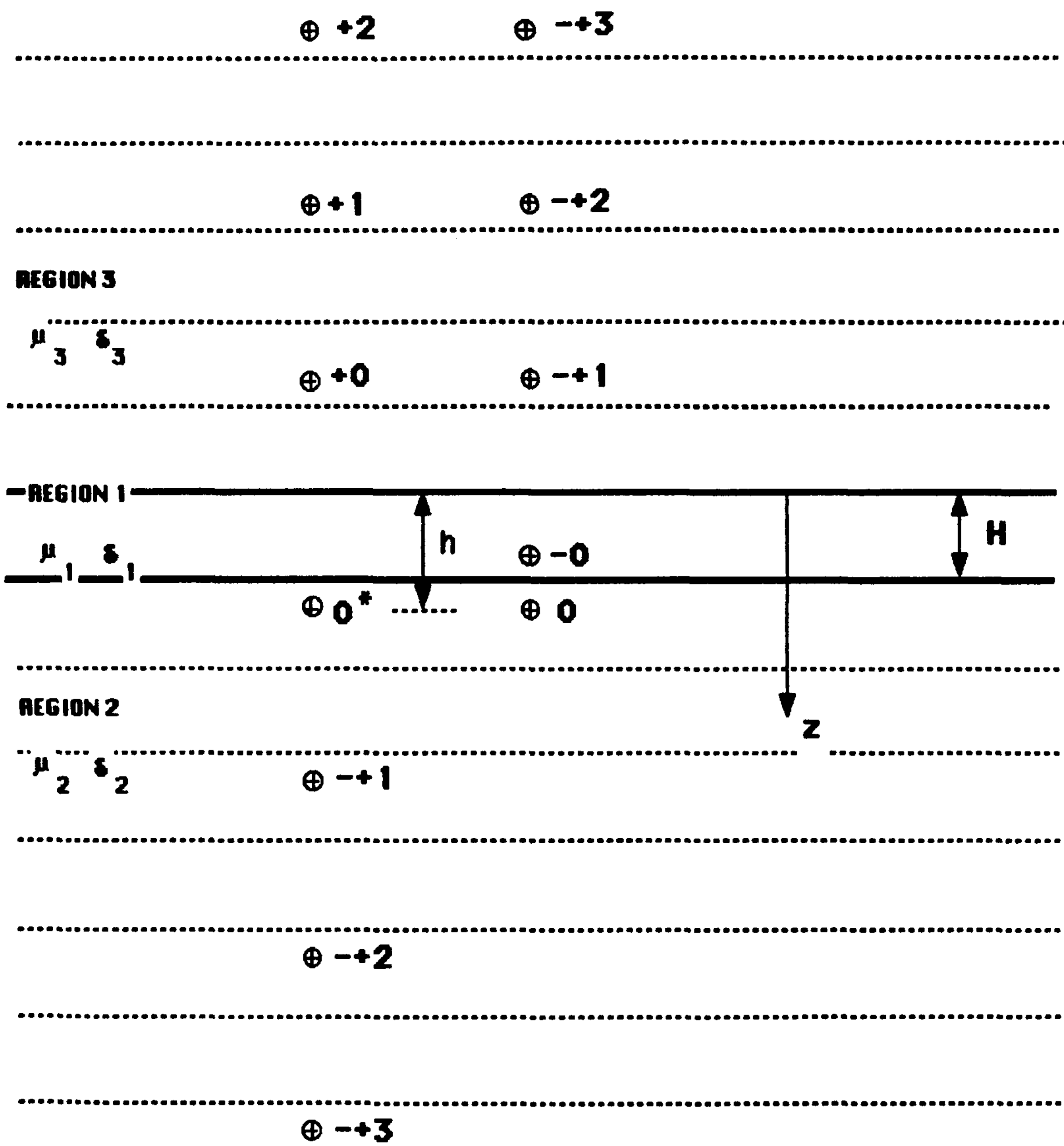

Figure 4

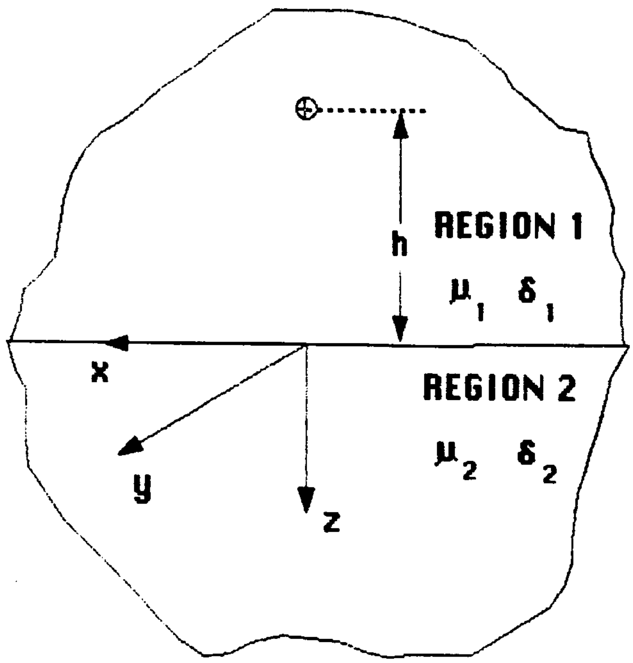

Figure 3.1