X-642-67-455

NASA TM X-55935

BESSEL FUNCTION SUBROUTINE FOR COMPUTING FUNCTIONS OF THE FIRST KIND $J_n(X)$ OR $J_{n+1/2}(X)$, FUNCTIONS OF THE SECOND KIND $Y_n(X)$ OR $J_{-n-1/2}(X)$, SPHERICAL BESSEL FUNCTIONS $j_n(X)$ OR $y_n(X)$, AND MODIFIED FUNCTIONS $I_n(X)$ OR $K_n(X)$

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SEPTEMBER 1967

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TITLE: Double Precision Bessel Function Subroutine

PROGRAM NAME: BSLFNX

LANGUAGE: IBM 360 FORTRAN G

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SUMMARY: This subroutine will compute any one of the following Bessel Function Arrays, as specified by the user, for each entry into the routine:

Bessel Functions of the 1st Kind

a. $J_0(x), J_1(x), \ldots J_n(x)$ integer orders

b. $J_{\frac{1}{2}}(x), J_{3/2}(x), \ldots J_{n+\frac{1}{2}}(x)$ half orders

Bessel Functions of the 2nd Kind

c. $Y_0(x), Y_1(x), \ldots Y_n(x)$ integer orders

d. $J_{-\frac{1}{2}}(x), J_{-3/2}(x), \ldots J_{-(n+\frac{1}{2})}(x)$ Negative half orders

Spherical Bessel Functions

e. $j_0(x), j_1(x), \ldots j_n(x)$ 1st Kind

f. $y_0(x), y_1(x), \ldots y_n(x)$ 2nd Kind

Modified Bessel Functions

g. $I_0(x), I_1(x), \ldots I_n(x)$ integer orders

h. $K_0(x), K_1(x), \ldots K_n(x)$ Integer orders

The argument, x, is real and must be in the range

$$0 < x \leq 6000$$

for all functions except the Modified Functions $I_n(x)$ and $K_n(x)$. For these functions, the bounds on x are

$$0 < x \leq 128.$$

All functions are accurate to at least seven significant digits.

METHOD

I. Regular Functions

The method used to compute the Regular Functions $J_n(x)$, $J_{n+\frac{1}{2}}(x)$, $j_n(x)$, and $I_n(x)$ is the "Backward Recurrence Method" described in Reference 1. The method essentially follows the description below. For more details see Reference 1.

Upon using the recurrence formula for solutions to Bessel's Differential Equation, in a backward fashion, or

$$F_{n-1}(x) = \frac{2n}{x} F_n(x) - F_{n+1}(x) \qquad (1)$$

with

$$F_{m+1}(x) = 0$$

and

$$F_m(x) = a$$

where, a, is any constant, one obtains an array of functions at some, n<m, which are all a constant multiple, β, of the regular function to any desired degree of accuracy.

That is, say for $J_n(x)$,

$$F_K(x) \cong \beta J_K(x), \qquad K=0,1,2,\ldots,n(<m)$$

Then, all that remains is to normalize all the functions after determining β.

β is conveniently determined in the computer using one of the following relationships, depending upon the function involved.

1. $\underline{J_n(x)}$

$$1 = J_0(x) + 2 \sum_{n=1}^{\infty} J_{2n}(x)$$

or for $F_n(x) = \beta J_n(x)$

$$\beta = F_0(x) + 2 \sum_{n=1}^{\infty} F_{2n}(x) \qquad (2)$$

2. $J_{n+\frac{1}{2}}(x)$ and $j_n(x)$

$$\beta = \frac{F_{\frac{1}{2}}(x)}{J_{\frac{1}{2}}(x)} \qquad \text{or} \qquad \beta = \frac{F_0(x)}{j_0(x)}$$

(see Formulas section for expressions for $J_{\frac{1}{2}}(x)$ and $j_0(x)$)

3. $I_n(x)$

$$e^x = I_0 + 2 \sum_{n=1}^{\infty} I_n(x)$$

or for $F_n(x) = \beta I_n(x)$

$$\beta = e^{-x} \left[F_0 + 2 \sum_{n=1}^{\infty} F_n(x) \right] \qquad (3)$$

In actual use, the summations in (2), and (3) need only go to, m, since the $F_n(x)$'s are a decreasing function as, n, increases. Determination of, m, is described under USAGE. Of course, when using the method for computing the modified function, $I_n(x)$, the recurrence formula (1) must be replaced with

$$F_{n-1}(x) = \frac{2n}{x} F_n(x) + F_{n+1}(x).$$

II. Irregular Functions

The method used for the regular functions is the Forward Recurrence Method using equation (1) but with, n, increasing, or

$$F_{n+1}(x) = \frac{2n}{x} F_n(x) - F_{n-1}(x). \qquad (4)$$

In using equation (4) for the irregular functions, no accuracy is lost in the forward recurrence for either $n<x$, or $n>x$ since the irregular function is an increasing function as, n, increases. Therefore, as, n, increases one actually gains more significant digits of accuracy.

In using equation (4), one must change the sign before the $F_{n-1}(x)$ term when recurring for $K_n(x)$, or

$$K_{n+1}(x) = \frac{2n}{x} K_n(x) + K_{n-1}(x).$$

Therefore all one needs are two starting values for the irregular functions and use the recurrence formula with, n, increasing.

Several methods are available for determining starting values. The methods used in BSLFNX gave accuracies of at least seven significant digits for all function starting values.

The expressions used for $Y_n(x)$ and $K_n(x)$ were polynomials described in Reference 2. Direct analytical expressions were used for $J_{-(n+\frac{1}{2})}(x)$ and $y_n(x)$, and are listed under the Formula section.

USAGE

The calling sequence is

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CALL BSLFNX (FNX, ARG, N, NTYPE)
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where ARG, N, and NTYPE must be defined prior to entry into BSLFNX. FNX must be singly dimensioned with the size dependent upon the number of functions desired and the value of, M, defined below. Both FNX and ARG must be typed Double Precision.

FNX   the array of computed functions,

$F_K(x)$, F_{K+1}, . . . F_{K+n+1}; $K=0$, or $\frac{1}{2}$

ARG   argument, in $F_n(ARG)$

N = n — the highest order of the array of Bessel Functions computed. For integer orders, one obtains $F_0(x)$, $F_1(x)$, ... $F_n(x)$, and for plus or minus half orders one obtains

$$F_{\pm\frac{1}{2}}(x),\ F_{\pm 3/2}(x),\ \ldots\ F_{\pm(n+\frac{1}{2})}(x).$$

NTYPE — option for the particular function desired. It can assume the following values:

NTYPE	Function Computed
1	$J_n(x)$
2	$J_{n+\frac{1}{2}}(x)$
3	$j_n(x)$
4	$Y_n(x)$
5	$J_{-(n+\frac{1}{2})}(x)$
6	$y_n(x)$
7	$I_n(x)$
8	$K_n(x)$

m Values

The values of, m, obtained from the following emperical formulas yield the accuracies specified for all regular functions, and will compute a maximum accurate array size such that the difference in magnitude between $F_0(x)$ to $F_n(x)$ is of the order of 10^{20}. Larger arrays could be obtained by adding to the constant term any amount within practical limits of the computer.

$$m = \begin{cases} 5x+20, & 0 < x < 10 \\ 1.48x+55, & 10 \le x < 150 \\ 1.05x+115, & 150 < x \end{cases}$$

Accuracy

For details on the accuracies obtained for the regular functions, see Reference 1. At least seven significant digits were obtained for all orders and the following arguments, checked on the IBM 360/65 computer.

$$x = 1, 2, 5, 10, 50, 100$$

Reference 1 pertains to accuracies obtained on the IBM 7094 computer where the word length is only 36 bits. Better accuracies should be expected with the IBM 360 Double Precision word length.

Subroutines Required

No special subroutines are required for BSLFNX except the standard library routines supplied with the IBM OS 360.

Program Restrictions

The subroutine stops the program if the order, n, asked for exceeds the value of, (m-5), (where, m, was calculated from the argument) for all regular functions. The stop is STOP 1.

There is no limit on, n, for the irregular functions except to be within the practical limits of the computer.

Function Formulas

1. $J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \quad \sin(x)$

2. $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \quad \cos(x)$

3. $J_{-3/2}(x) = \sqrt{\frac{2}{\pi x}} \quad -\sin(x) - \frac{\cos(x)}{x}$

4. $j_n(x) = \sqrt{\frac{\pi}{2x}} \quad J_{n+\frac{1}{2}}(x)$

5. $y_n(x) = \sqrt{\frac{\pi}{2x}} \quad Y_{n+\frac{1}{2}}(x) = (-1)^{n+1} \frac{\pi}{2x} \quad J_{-(n+\frac{1}{2})}(x)$

6. $1 = J_0(x) + \sum_{n=1}^{\infty} J_{2n}(x)$

7. $$e^x = I_0(x) + \sum_{n=1}^{\infty} I_n(x)$$

Wronskians

8. $$I_n(x)\, K_{n+1}(x) + I_{n+1}(x)\, K_n(x) = \frac{1}{x}$$

9. $$J_{v+1}(x)\, J_{-v}(x) + J_v(x)\, J_{-(v+1)}(x) = -\frac{2\sin(v\pi)}{\pi z}$$

(v can be either, n, or n+½)

10. $$J_{n+1}(x)\, Y_n(x) - J_n(x)\, Y_{n+1}(x) = \frac{2}{\pi z}$$

References

1. Michels, T. E., "The Backward Recurrence Method for Computing the Regular Bessel Functions," NASA, Technical Note D-2141, Department of Commerce, Washington, D. C., May, 1964.

2. Abramowity, M. and Stegun, I. A., "Handbook of Mathematical Functions, U. S. Government Printing Office, Washington, D. C., June 1964.

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      SUBROUTINE BSLFNX(FNX,ARG,N,NTYPE)                                BSLFNX00
C***********************************************************************BSLFNX01
C   SUBROUTINE BSLFNX COMPUTES AN ARRAY OF BESSEL FUNCTIONS CORRESPONDINGBSLFNX02
C  TO THE VALUE OF NTYPE, DEFINED BELOW, FOR REAL ARGUMENT(ARG), AND    BSLFNX03
C  ORDER (K), FOR K=(0+NU),(1+NU),...,(N+NU), WHERE NU = 0 OR 1/2.      BSLFNX04
C                                                                       BSLFNX05
C  CALLING SEQUENCE                                                     BSLFNX06
C     CALL BSLFNX(FNX,ARG,N,NTYPE)                                      BSLFNX07
C  WHERE                                                                BSLFNX08
C       FNX  =  ARRAY OF BESSEL FUNCTIONS COMPUTED CORRES. TO NTYPE.    BSLFNX09
C               FNX IS DEFINE DOUBLE PRECISION AND MUST BE DIMENSIONED  BSLFNX0A
C               AT LEAST THE VALUE OF 'M' DEFINED BELOW:                BSLFNX0B
C                  M=5*ARG + 20      ,  0 [illegible] ARG LT 10                 BSLFNX0C
C                  M=1.48*ARG + 55   .  [illegible]0 GE ARG LT 150              BSLFNX0D
C                  M=1.05*ARG + 115 ,  ARG GE 150                       BSLFNX0E
C       ARG  =  ARGUMENT OF BESSEL FUNCTION, DEFINE DOUBLE PRECISION    BSLFNX0F
C        N   =  HIGHEST ORDER BESSEL FUNCTION TO COMPUTE FOR ARRAY      BSLFNX10
C               OF INTEGER ORDERS 0,1,...,N                             BSLFNX11
C               OR HALF ORDERS 1/2, 3/2,..., (N+1/2) DEPENDING UPON     BSLFNX12
C               THE TYPE REQUESTED BY THE USER                          BSLFNX13
C      NTYPE =  TYPE OF BESSEL FUNCTION REQUESTED BY THE USER ACCORDING BSLFNX14
C               TO THE VALUES BELOW                                     BSLFNX15
C    NTYPE          TYPE OF BESSEL FUNCTION COMPUTED                    BSLFNX16
C    -----          --------------------------------                    BSLFNX17
C      1       J SUB (N) OF (ARG)                                       BSLFNX18
C              B. FUNCT. OF THE FIRST KIND-INTEGER ORDER                BSLFNX19
C      2       J SUB (N+1/2) OF (ARG)                                   BSLFNX1A
C              B. FUNCT. OF  THE FIRST KIND-HALF ORDER                  BSLFNX1B
C      3       SMALL J SUB (N) OF (ARG)                                 BSLFNX1C
C              SPHERICAL B. FUNCT. OF THE FIRST KIND                    BSLFNX1D
C      4       Y SUB (N) OF (ARG)                                       BSLFNX1E
C              B. FUNCT. OF THE SECOND KIND-INTEGER ORDER               BSLFNX1F
C      5       J SUB -(N+1/2) OF (ARG)                                  BSLFNX20
C              B. FUNCT. OF THE SECOND KIND-HALF ORDER                  BSLFNX21
C      6       SMALL Y SUB (N) OF (ARG)                                 BSLFNX22
C              SPHERICAL B. FUNCT. OF THE SECOND KIND                   BSLFNX23
C      7       I SUB (N) OF (ARG)                                       BSLFNX24
C              MODIFIED B. FUNCT.-INTEGER ORDER                         BSLFNX25
C      8       K SUB (N) OF (ARG)                                       BSLFNX26
C              MODIFIED B. FUNCT.-INTEGER ORDER                         BSLFNX27
C***********************************************************************BSLFNX28
      IMPLICIT REAL*8 (A-H,O-Z)                                         BSLFNX29
      DIMENSION FNX(1)                                                  BSLFNX2A
      DATA G1,G2,G3,G4,G5,G6/-.07832358D0,.02189568D0,-.01062446D0,     BSLFNX2B
     G.00587872D0,-.00251540D0,.53208D-3/                               BSLFNX2C
      DATA F1,F2,F3,F4,F5,F6/.42278420D0,.23069176D0,.03488590D0,       BSLFNX2D
     F.00262698D0,.10750D-3,.740D-5/                                    BSLFNX2E
      DATA H1,H2,H3,H4,H5,H6/-7.7D-7,-.0055274D0,-9.512D-5,.00137237D0  BSLFNX2F
     H-7.2805D-4,1.4476D-4/                                             BSLFNX30
      DATA E1,E2,E3,E4,E5,E6/-.04166397D0,-3.954D-5,.00262573D0,        BSLFNX31
     E-5.4125D-4,-2.9333D-4,1.3558D-4/                                  BSLFNX32
      DATA W1,W2,W3,W4,W5,W6/1.56D-6,.01659667D0,1.7105D-4,-.00249511D0,BSLFNX33
     W1.13653D-3,-2.0033D-4/                                            BSLFNX34
      DATA Y1,Y2,Y3,Y4,Y5,Y6/.12499612D0,5.65D-5,-6.37879D-3,7.4348D-4, BSLFNX35
     Y7.9824D-4,-2.9166D-4/                                             BSLFNX36
      DATA C1,C2,C3,C4,C5,C6 /-2.2499997D0,1.2656208D0,-0.3163866D0,    BSLFNX37
     C4.44479D-2,-3.9444D-3,2.1D-4/                                     BSLFNX38
      DATA D1,D2,D3,D4,D5,D6/0.60559366D0,-.74350384D0,.25300117D0,     BSLFNX39
     D-4.261214D-2,4.27916D-3,-2.4846D-4/                               BSLFNX3A
      DATA O1,O2,O3,O4,O5,O6/-.56249985D0,.21093573D0,-3.954289D-2,     BSLFNX3B
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     04.43319D-3,-3.1761D-4,1.109D-5/                                    BSLFNX3C
      DATA P1,P2,P3,P4,P5,P6/.221209lD0,2.1682709D0,-1.3164827D0,        BSLFNX3D
     P.3123951D0,-4.00976D-2,2.7873D-3/                                  BSLFNX3E
      DATA ONE,TWO,THREE,PIO2/1.D0,2.D0,3.D0,1.5707963267948966D0/       BSLFNX3F
      FACT(A,B,C,D,E,F,G) = G*(A+G*(B+G*(C+G*(D+G*(E+G*F)))))            BSLFNX40
      NN=N+1                                                             BSLFNX41
      X=ARG                                                              BSLFNX42
      NU=1                                                               BSLFNX43
      SIGN=-ONE                                                          BSLFNX44
      M1=3                                                               BSLFNX45
      M2=2                                                               BSLFNX46
      GO TO(10,10,10,155,160,160,10,10),NTYPE                            BSLFNX47
   10 IF(X.GE.10.D0 ) GO TO 20                                           BSLFNX48
      M=5.*X+15.                                                         BSLFNX49
      GO TO 40                                                           BSLFNX4A
   20 IF(X.GE.150.D0) GO TO 30                                           BSLFNX4B
      M=1.48*X + 48.                                                     BSLFNX4C
      GO TO 40                                                           BSLFNX4D
   30 M=1.05*X+112.                                                      BSLFNX4E
   40 IF((M-5).GE.N) GO TO (60,70,80,240,240,240,50,50),NTYPE            BSLFNX4F
      IF(NTYPE.EQ.8) GO TO 50                                            BSLFNX50
      M=M-5                                                              BSLFNX51
      WRITE (6,45) NTYPE,N,M,X                                           BSLFNX52
   45 FORMAT('0 ORDER ASKED FOR IN BSLFNX TOO LARGE.'/' FUNCTION TYPE ISBSLFNX53
     F'I2/' ORDER ASKED FOR IS'I4,' BUT CANNOT BE GREATER THAN 'I4,      BSLFNX54
     F'---REDUCE ORDER AND RUN AGAIN'/' ARGUMENT IS'F12.8)               BSLFNX55
      STOP 1                                                             BSLFNX56
   50 M1=2                                                               BSLFNX57
      M2=1                                                               BSLFNX58
      SIGN=ONE                                                           BSLFNX59
   60 NU=0                                                               BSLFNX5A
      GO TO 90                                                           BSLFNX5B
   70 CONTINUE                                                           BSLFNX5C
      Z=DSIN(X)/DSQRT(PIO2*X)                                            BSLFNX5D
      GO TO 90                                                           BSLFNX5E
   80 Z=DSIN(X)/X                                                        BSLFNX5F
   90 FNX(M+2)=0.D0                                                      BSLFNX60
      FNX(M+1)=1.D-60                                                    BSLFNX61
      DO 100 I=1,M                                                       BSLFNX62
      K=M-I+1                                                            BSLFNX63
  100 FNX(K)=DFLOAT(2*K+NU)*FNX(K+1)/X+SIGN*FNX(K+2)                     BSLFNX64
      GO TO TO(110,130,130,240,240,240,110,110),NTYPE                    BSLFNX65
  110 Z=0.D0                                                             BSLFNX66
      MM=M-2                                                             BSLFNX67
      DO 120 I=M1,MM,M2                                                  BSLFNX68
  120 Z=Z+FNX(I)                                                         BSLFNX69
      Z=FNX(1)+TWO*Z                                                     BSLFNX6A
      IF(NTYPE.GT.6) Z=Z/DEXP(X)                                         BSLFNX6B
      GO TO 140                                                          BSLFNX6C
  130 Z=FNX(1)/Z                                                         BSLFNX6D
  140 DO 150 I=1,NN                                                      BSLFNX6E
      IF(I.EQ.3.AND.NTYPE.EQ.8) GO TO 152                                BSLFNX6F
  150 FNX(I)=FNX(I)/Z                                                    BSLFNX70
      GO TO 500                                                          BSLFNX71
  152 IF(X.GE.TWO) GO TO 153                                             BSLFNX72
      C=(X/TWO)**2                                                       BSLFNX73
      FKN=-DLOG(X/TWO)*FNX(1)-.57721566D0+FACT(F1,F2,F3,F4,F5,F6,C)      BSLFNX74
      GO TO 154                                                          BSLFNX75
  153 C=TWO/X                                                            BSLFNX76
      FKN=(1.25331414D0+FACT(G1,G2,G3,G4,G5,G6,C))/(DEXP(X)*DSQRT(X))    BSLFNX77
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  154 FNX(2)=(ONE-FNX(2)*FKN*X)/(FNX(1)*X)                             BSLFNX78
      FNX(1)=FKN                                                       BSLFNX79
      GO TO 170                                                        BSLFNX7A
  155 NU=0                                                             BSLFNX7B
      GO TO 190                                                        BSLFNX7C
  160 DIV=DSQRT(PIO2*X)                                                BSLFNX7D
      FNX(1)=DCOS(X)/DIV                                               BSLFNX7E
      FNX(2)=-(DSIN(X)+DCOS(X)/X)/DIV                                  BSLFNX7F
      IF(NTYPE.EQ.5) GO TO 170                                         BSLFNX80
      FNX(1)=-FNX(1)*DIV/X                                             BSLFNX81
      FNX(2)=FNX(2)*DIV/X                                              BSLFNX82
  170 DO 180 I=2,NN                                                    BSLFNX83
      K=I-1                                                            BSLFNX84
  180 FNX(I+1)=DFLOAT(2*K+NU)*FNX(I)/X+SIGN*FNX(I-1)                   BSLFNX85
      GO TO 500                                                        BSLFNX86
  190 IF(X.LE.THREE) GO TO 220                                         BSLFNX87
      C=THREE/X                                                        BSLFNX88
      A=.79788456D0                                                    BSLFNX89
      B=.78539816D0                                                    BSLFNX8A
      F=A + FACT(H1,H2,H3,H4,H5,H6,C)                                  BSLFNX8B
      T=X-B+FACT(E1,E2,E3,E4,E5,E6,C)                                  BSLFNX8C
      FNX(1)=F*DSIN(T)/DSQRT(X)                                        BSLFNX8D
      B=2.35619449D0                                                   BSLFNX8E
      T=X-B+FACT(Y1,Y2,Y3,Y4,Y5,Y6,C)                                  BSLFNX8F
      F=A + FACT(W1,W2,W3,W4,W5,W6,C)                                  BSLFNX90
      FNX(2)=F*DSIN(T)/DSQRT(X)                                        BSLFNX91
      GO TO 170                                                        BSLFNX92
  220 C=(X/THREE)**2                                                   BSLFNX93
      B=DLOG(X/TWO)/PIO2                                               BSLFNX94
      TOM=.36746691D0                                                  BSLFNX95
      FNX(1)=B*(ONE+FACT(C1,C2,C3,C4,C5,C6,C))+TOM+ FACT(D1,D2,D3,D4,D5,BSLFNX96
     1D6,C)                                                            BSLFNX97
      TOM=-.6366198D0                                                  BSLFNX98
      FNX(2)=(B*(0.5D0+FACT(O1,O2,O3,O4,O5,O6,C))+  TOM+ FACT(P1,P2,P3, BSLFNX99
     1 P4,P5,P6,C))/X                                                  BSLFNX9A
      GO TO 170                                                        BSLFNX9B
  240 WRITE(6,250) NTYPE                                               BSLFNX9C
  250 FORMAT(1H2,57HMACHINE TROUBLE---COULD NOT GET TO STATEMENT WITH NTBSLFNX9D
     1YPE =I3)                                                         BSLFNX9E
      STOP 2                                                           BSLFNX9F
  500 RETURN                                                           BSLFNXA0
      END                                                              BSLFNXA1
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