Chapter 2

The Biot model in poroelasticity

2.1 Mathematical model

Mechanical processes in porous media (for example, soil) involve two basic elements: fluid flow and rock deformation. In reservoir simulations, the change in fluid pressure, which is caused by fluid injection or production, perturbs the stress state of the rock. The change in the stress state triggers the rock deformation which, in turn, affects the fluid flow processes. To describe the interaction between the flow and deformation, the concept of *effective stress* (the total stress of rock plus the fluid pressure) has been introduced by K.Terzaghi in 1923. This concept was further generalized by Maurice Biot in a number of papers [9, 10, 11, 12]. The Biot theory of *poroelasticity* has been widely used in civic, mining and petroleum engineering, acoustic wave propagation in saturated media, and biology for several decades.

The mathematics models for fluid flow in elastic porous media are derived from physical principles based on certain physical assumptions [9, 22, 40, 53, 79]. Hereafter, we make the following assumptions: (a) temperature of the reservoir is constant, (b) there is no mass exchange between the rock phase and the fluid phase, (c) the rock is elastic material, and (d) the fluid is incompressible. Two fundamental physical principles used below are: mass conservation and momentum balance for both fluid and rock phases.

The momentum balance for the solid phase is given by the well known equilibrium conditions for the stress tensor field $\boldsymbol{\sigma}$:

$$\mathrm{div}\boldsymbol{\sigma} = 0. \tag{2.1}$$

According to the poroelasticity theory [9, 10, 22], the effective stress $\hat{\boldsymbol{\sigma}}$ is defined as

$$\hat{\boldsymbol{\sigma}} = \boldsymbol{\sigma} + c_{up} p \boldsymbol{I}, \tag{2.2}$$

where c_{up} is a positive coefficient less than 1 and p is a fluid pressure. Generally speaking, c_{up} could be a function of the fluid pressure and rock deformation. However, for the simplified model, it is reasonable to assume that c_{up} is the physical constant [9].

Let $\boldsymbol{u}$ be the displacement and $\boldsymbol{\epsilon}$ be the strain. The elastic strain-displacement relationship is defined as

$$\boldsymbol{\epsilon} = D(\boldsymbol{u}) = \frac{1}{2}\left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T\right).$$

The variables $\hat{\boldsymbol{\sigma}}$ and $\boldsymbol{\epsilon}$ are not independent, but instead, they are coupled by a linear constitutive equation

$$\hat{\boldsymbol{\sigma}} = \frac{E}{1+\nu}\left(\boldsymbol{\epsilon} + \frac{\nu}{1-2\nu}\mathrm{trace}\,(\boldsymbol{\epsilon})\boldsymbol{I}\right), \tag{2.3}$$

where E is the modulus of elasticity and ν is the Poisson ratio. Substituting (2.2)–(2.3) into the equilibrium condition (2.1), we get the governing equation for the rock deformation:

$$-\mathrm{div}[\mu(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)] - \nabla(\lambda\, \mathrm{div}\boldsymbol{u}) + \nabla(c_{up} p) = 0, \tag{2.4}$$

where μ and λ are the Lamé constants:

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad \text{and} \quad \mu = \frac{E}{2(1+\nu)}.$$

Note that μ is sometimes called the shear modulus of the rock.

To derive the second governing equation, we shall follow [9]. It has been shown there that the fundamental relation describing the increment θ of fluid volume per unit volume of soil may be written as

$$\theta = c_{pu}\mathrm{div}\boldsymbol{u} + c_{pp}p, \tag{2.5}$$

where c_{pu} and c_{pp} are some physical parameters. The first one measures the ratio of the fluid volume squeezed out to the volume change of the soil if the latter is compressed while allowing the fluid to escape. The second one is a measure of the amount of fluid which can be forced into the soil under pressure while the volume of the soil is kept constant [9].

The momentum balance for the fluid is interpreted as the Darcy law for the fluid velocity:

$$\boldsymbol{f} = -\frac{\rho\boldsymbol{k}}{\mu_f}\nabla p \equiv -\boldsymbol{K}\nabla p, \tag{2.6}$$

where μ_f stands for fluid viscosity, $\boldsymbol{k}$ for the soil permeability, and ρ for the fluid density.

Since the fluid is incompressible, the mass conservation law states that the rate of fluid content in an element of soil must be equal to the volume of fluid entering per second through the surface of the element plus a flow rate S_f from an external source or sink:

$$\frac{\partial\theta}{\partial t} = -\mathrm{div}\boldsymbol{f} + S_f. \tag{2.7}$$

Combining formulae (2.5), (2.6) and (2.7), we obtain the governing equations for the fluid flow:

$$\frac{\partial(c_{pu}\mathrm{div}\boldsymbol{u})}{\partial t} + \frac{\partial(c_{pp}p)}{\partial t} + \mathrm{div}\boldsymbol{f} = S_f, \qquad \boldsymbol{f} = -\boldsymbol{K}\nabla p. \tag{2.8}$$

The coupled system consists of the Lamé equation (2.4) and the diffusion equation in the mixed form (2.8). We assume that the physical parameters c_{pu} and c_{pp} are absolute constants. We shall use the first order implicit Euler scheme for time discretization. Let Δt be the time step and $\boldsymbol{u}^n$, $\boldsymbol{f}^n$ and p^n be the values of physical variables at time $t^n = n\Delta t$. Replacing time derivatives by conventional finite differences, we get a system of three PDEs:

$$\begin{array}{rcrcrcl} -\mathrm{div}(2\mu D(\boldsymbol{u}^n)) - \nabla(\lambda\,\mathrm{div}\,\boldsymbol{u}^n) & & & + & c_{up}\nabla p^n & = & 0, \\ & & \boldsymbol{K}^{-1}\boldsymbol{f}^n & + & \nabla p^n & = & 0, \\ -c_{pu}\mathrm{div}\,\boldsymbol{u}^n & - & \mathrm{div}\,\boldsymbol{f}^n & - & c_{pp}p^n & = & -S_f, \end{array} \tag{2.9}$$

where $c_{pu} \colon = c_{pu}/\Delta t$, $c_{pp} \colon = c_{pp}/\Delta t$, and

$$S_f \colon = S_f + \frac{1}{\Delta t}\left(c_{pu}\,\mathrm{div}\boldsymbol{u}^{n-1} + c_{pp}\,p^{n-1}\right).$$

Hereafter, we shall study numerical methods for solving the coupled system (2.9). In order to simplify the notations, we shall omit superscript n in the sequel. Summarizing the above assumptions, we recall that μ, λ, c_{up}, c_{pu}, c_{pp} are positive constants, and $\boldsymbol{K}$ is the positive definite symmetric tensor with bounded entries. Under these assumptions, system (2.9) can be easily symmetrized by rescaling the displacement variable as follows:

$$\boldsymbol{u} \colon = \frac{c_{pu}}{c_{up}}\boldsymbol{u}.$$

Without loss of generality, we assume hereafter that $c_{pu} = c_{up} = 1$, i.e., the coupled problem is symmetric.

Let Γ be a part of $\partial\Omega$ with non-zero measure $|\Gamma| > 0$. Then, system (2.9) is closed by imposing the following boundary conditions:

$$\begin{array}{rclrclcl} \hat{\boldsymbol{\sigma}}\cdot\boldsymbol{n} & = & 0, & p & = & 0 & \text{on} & \Gamma, \\ \boldsymbol{u} & = & 0, & \boldsymbol{f}\cdot\boldsymbol{n} & = & 0 & \text{on} & \partial\Omega\setminus\bar{\Gamma}, \end{array} \tag{2.10}$$

where $\boldsymbol{n}$ is the outward normal vector.

2.2 Finite element discretization

Throughout this thesis, we shall use notations $\|\cdot\|_{1,D}$, $\|\cdot\|_{\mathrm{div},D}$ and $\|\cdot\|_D$ for the norms on the Hilbert spaces $H^1(D)$, $H(\mathrm{div};\,D)$ and $L_2(D)$, respectively, for some set

$D \subset \Omega$:

$$\|p\|_D = \left(\int\limits_D p^2 \, \mathrm{d}x \right)^{1/2}, \qquad \|p\|_{1,\,D} = \left(\|p\|_D^2 + \int\limits_D |\nabla p|^2 \, \mathrm{d}x \right)^{1/2}$$

and

$$\|\boldsymbol{f}\|_{\mathrm{div},\,D} = \left(\sum_{i=1}^{3} \|f_i\|_D^2 + \int\limits_D (\mathrm{div}\,\boldsymbol{f})^2 \, \mathrm{d}x \right)^{1/2}.$$

In addition, we denote by $(\cdot,\cdot)_D$ the inner product on D of either scalar or vector functions, and by $|\cdot|_{1,\,D}$ the semi-norm on $H^1(D)$. Let $\|\cdot\|_\Omega \equiv \|\cdot\|$, $\|\cdot\|_{1,\,\Omega} \equiv \|\cdot\|_1$, and $(\cdot,\cdot) \equiv (\cdot,\cdot)_\Omega$. In order to simplify notation, we define three additional Hilbert spaces:

$$\begin{aligned} V &= \{\boldsymbol{v} \in (H^1(\Omega))^3 : \ \boldsymbol{v} = 0 \ \text{ on } \ \partial\Omega \setminus \bar{\Gamma}\}, \\ X &= \{\boldsymbol{g} \in H(\mathrm{div};\, \Omega) : \ \boldsymbol{g} \cdot \boldsymbol{n} = 0 \ \text{ on } \ \partial\Omega \setminus \bar{\Gamma}\}, \end{aligned}$$

and $Q \equiv L_2(\Omega)$. Hereafter, we shall use letters $\boldsymbol{u}$ and $\boldsymbol{v}$ to denote vector functions from space V. The letters $\boldsymbol{f}$, $\boldsymbol{g}$ and $\boldsymbol{w}$ will be reserved for fluxes from space X. Finally, letters p and q will be used for scalar functions living in space Q. In addition, we shall use notations $\|\boldsymbol{v}\|_D$ and $\|\boldsymbol{v}\|_{1,D}$ for L_2 and H^1 norms, respectively, of vector function $\boldsymbol{v}$:

$$\|\boldsymbol{v}\|_D = \left(\sum_{i=1}^{3} \|v_i\|_D^2 \right)^{1/2} \quad \text{and} \quad \|\boldsymbol{v}\|_{1,D} = \left(\sum_{i=1}^{3} \|v_i\|_{1,D}^2 \right)^{1/2}.$$

The mixed finite element formulation of (2.9), (2.10) is as follows: Find $(\boldsymbol{u}, \boldsymbol{f}, p) \in V \times X \times Q$ such that

$$\begin{array}{rcrcrcl} \mathcal{A}(\boldsymbol{u},\,\boldsymbol{v}) & & & - & \mathcal{B}(p,\,\boldsymbol{v}) & = & 0, \\ & & \mathcal{M}(\boldsymbol{f},\,\boldsymbol{g}) & - & \mathcal{B}(p,\,\boldsymbol{g}) & = & 0, \\ -\mathcal{B}(q,\,\boldsymbol{u}) & - & \mathcal{B}(q,\,\boldsymbol{f}) & - & c_{pp}\,\Sigma(p,\,q) & = & -(S_f,\,q) \end{array} \tag{2.11}$$

holds for all $(\boldsymbol{v}, \boldsymbol{g}, q) \in V \times X \times Q$. Here

$$\mathcal{A}(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega (2\mu D(\boldsymbol{u}) : D(\boldsymbol{v}) + \lambda \operatorname{div} \boldsymbol{u} \operatorname{div} \boldsymbol{v})\, \mathrm{d}x,$$

$$\mathcal{B}(p,\, \boldsymbol{g}) = \int_\Omega p \operatorname{div} \boldsymbol{g}\, \mathrm{d}x, \qquad \mathcal{M}(\boldsymbol{f},\, \boldsymbol{g}) = \int_\Omega \boldsymbol{K}^{-1} \boldsymbol{f} \cdot \boldsymbol{g}\, \mathrm{d}x,$$

$$\Sigma(p,\, q) = \int_\Omega p\, q\, \mathrm{d}x.$$

Solvability of the coupled problem

According to the general theory of saddle point problems [18], solvability of (2.11) can be proved under different assumptions. The simplest ones require only coercivity of bilinear forms $\mathcal{A}(\cdot, \cdot)$, $\mathcal{M}(\cdot, \cdot)$ and $\Sigma(\cdot, \cdot)$. Unfortunately, the stability estimates depend on the value of c_{pp}^{-1}. When c_{pp} is small, this usually results in a severe loss of accuracy for a discrete solution. The more complicated set of assumptions includes the well known *inf-sup* condition. Let us define two new bilinear forms:

$$\mathbb{A}((\boldsymbol{u},\, \boldsymbol{f}),\ (\boldsymbol{v},\, \boldsymbol{g})) = \mathcal{A}(\boldsymbol{u}, \boldsymbol{v}) + \mathcal{M}(\boldsymbol{f},\, \boldsymbol{g}),$$

$$\mathbb{B}(q,\ (\boldsymbol{v},\, \boldsymbol{g})) = \mathcal{B}(q,\, \boldsymbol{v}) + \mathcal{B}(q,\, \boldsymbol{g}).$$

Furthermore, let

$$W = \{(\boldsymbol{v},\, \boldsymbol{g}) :\ \mathbb{B}(q,\ (\boldsymbol{v},\, \boldsymbol{g})) = 0 \quad \forall q \in Q\}.$$

We equip the space $V \times X$ with a norm

$$\|(\boldsymbol{v},\, \boldsymbol{g})\|_1 = \left(\|\boldsymbol{v}\|_1^2 + \|\boldsymbol{g}\|_{\mathrm{div},\Omega}^2 \right)^{1/2}.$$

The Korn inequality (3.6) implies that the bilinear form $\mathcal{A}(\cdot, \cdot)$ generates a norm on V equivalent to the H^1-norm. Let us show that $\mathbb{A}(\cdot,\ \cdot)$ is coercive on W. The definition of the subspace W implies that

$$\operatorname{div} \boldsymbol{v} + \operatorname{div} \boldsymbol{g} = 0 \qquad \forall (\boldsymbol{v},\, \boldsymbol{g}) \in W.$$

Using inequality (3.7) and spectral properties of $\boldsymbol{K}$, we get

$$\begin{aligned}\mathbb{A}((\boldsymbol{v},\,\boldsymbol{g}),\,(\boldsymbol{v},\,\boldsymbol{g})) &\geq c_1\|\boldsymbol{v}_1\|_1^2 + c_3\|\boldsymbol{K}^{-1/2}\boldsymbol{g}\|^2 \\ &\geq \frac{c_1}{2}\|\boldsymbol{v}\|_1^2 + \frac{c_1}{4}\|\mathrm{div}\,\boldsymbol{v}\|^2 + \nu_{\min}(\boldsymbol{K}^{-1})\,\|\boldsymbol{g}\|^2 \\ &\geq \alpha(\|\boldsymbol{v}\|_1^2 + \|\boldsymbol{g}\|_{\mathrm{div},\Omega}^2) = \alpha\|(\boldsymbol{v},\,\boldsymbol{g})\|_1^2 \qquad \forall(\boldsymbol{v},\,\boldsymbol{g})\in W,\end{aligned} \tag{2.12}$$

where $\nu_{\min}(\boldsymbol{K}^{-1})$ is the minimal eigenvalue of $\boldsymbol{K}^{-1}$, and $\alpha = \min(c_1/4,\,\nu_{\min}(\boldsymbol{K}^{-1}))$. The coercivity (2.12) is the first stability condition for Brezzi's theory [18]. The second one is the famous Ladyshenskaya-Babuška-Brezzi (LBB) condition:

$$\sup_{(\boldsymbol{v},\boldsymbol{g})\in V\times X} \frac{\mathbb{B}(q,\,(\boldsymbol{v},\,\boldsymbol{g}))}{\|(\boldsymbol{v},\,\boldsymbol{g})\|_1} \geq \beta\|q\| \qquad \forall q\in Q, \tag{2.13}$$

where we assume implicitly that the left side has to be evaluated only for $\|(\boldsymbol{v},\boldsymbol{g})\|_1 \neq 0$. Restricting the supremum to a subset of functions $(0,\,\boldsymbol{g})$, we obtain a lower estimate for β:

$$\sup_{\boldsymbol{g}\in X} \frac{\mathcal{B}(q,\,\boldsymbol{g})}{\|\boldsymbol{g}\|_{\mathrm{div},\Omega}} \geq \tilde{\beta}\|q\| \qquad \forall q\in Q. \tag{2.14}$$

The existence of $\tilde{\beta} > 0$ is proved in [13, 18]. Since $\beta \geq \tilde{\beta}$, we get the second stability condition for Brezzi's theory, namely, the LBB condition (2.13). Therefore, the following result follows immediately from Brezzi's theory (see [18, p. 47, Theorem 1.2]).

Theorem 2.1 *The saddle point problem: Find* $((\boldsymbol{u},\,\boldsymbol{f}),\,p)\in(V\times X)\times Q$ *such that*

$$\begin{aligned}\mathbb{A}((\boldsymbol{u},\,\boldsymbol{f}),\,(\boldsymbol{v},\,\boldsymbol{g})) \;-\; \mathbb{B}(p,\,(\boldsymbol{v},\,\boldsymbol{g})) &= 0 &&\forall(\boldsymbol{v},\boldsymbol{g})\in V\times X,\\ -\mathbb{B}(q,\,(\boldsymbol{u},\,\boldsymbol{f})) \;-\; c_{pp}\,\Sigma(p,\,q) &= -(S_f,\,q) &&\forall q\in Q\end{aligned}$$

has a unique solution. Moreover

$$\|(\boldsymbol{u},\boldsymbol{f})\|_1 + \|p\| \leq c\,\|S_f\|$$

where c *is bounded when* $c_{pp}\to 0$.

Discrete problem

Given finite dimensional subspaces V_h, X_h and Q_h of V, X and Q, respectively, we formulate a discrete problem: Find $(\boldsymbol{u}^h, \boldsymbol{f}^h, p^h) \in V^h \times X^h \times Q^h$ such that

$$\begin{array}{rcrcrcl} \mathcal{A}(\boldsymbol{u}^h,\, \boldsymbol{v}^h) & & & - & \mathcal{B}(p^h,\, \boldsymbol{v}^h) & = & 0, \\ & & \mathcal{M}(\boldsymbol{f}^h,\, \boldsymbol{g}^h) & - & \mathcal{B}(p^h,\, \boldsymbol{g}^h) & = & 0, \\ -\mathcal{B}(q^h,\, \boldsymbol{u}^h) & - & \mathcal{B}(q^h,\, \boldsymbol{f}^h) & - & c_{pp}\Sigma(p^h,\, q^h) & = & -(S_f,\, q^h) \end{array} \tag{2.15}$$

holds for all $(\boldsymbol{v}^h, \boldsymbol{g}^h, q^h) \in V_h \times X_h \times Q_h$. The discrete spaces can not be chosen independently of each other. They have to satisfy discrete versions of (2.12) and (2.13). For a conforming approximation, satisfying $\operatorname{div} V_h \subseteq Q_h$ and $\operatorname{div} X_h \subseteq Q_h$, we immediately get a coercivity condition identical to (2.12):

$$\mathbb{A}((\boldsymbol{v}^h,\, \boldsymbol{g}^h),\ (\boldsymbol{v}^h,\, \boldsymbol{g}^h)) \geq \alpha_h \|(\boldsymbol{v}^h,\, \boldsymbol{g}^h)\|_1^2 \qquad \forall (\boldsymbol{v}^h,\, \boldsymbol{g}^h) \in W_h,$$

where $\alpha_h = \alpha$ and

$$W_h = \{(\boldsymbol{v}^h,\, \boldsymbol{g}^h) :\ \mathbb{B}(q^h,\ (\boldsymbol{v}^h,\, \boldsymbol{g}^h)) = 0 \quad \forall q^h \in Q_h\}.$$

Since $\operatorname{div} W_h \subseteq Q_h$, the second stability condition is reduced to verification of a discrete analog of (2.14):

$$\sup_{\boldsymbol{g}^h \in X_h} \frac{\mathcal{B}(q^h,\, \boldsymbol{g}^h)}{\|\boldsymbol{g}^h\|_{\mathrm{div},\Omega}} \geq \tilde{\beta}_h \|q^h\| \qquad \forall q^h \in Q_h. \tag{2.16}$$

The existence of $\tilde{\beta}_h > 0$ is proved in [13, 18]. Then, the solvability of the discrete problem and accuracy estimates for the discrete solution follow immediately from Brezzi's theory [18]. In particular, we get that the discretization error is estimated from above by the best interpolation error:

$$\begin{aligned} \|(\boldsymbol{u}, \boldsymbol{f}) - (\boldsymbol{u}^h, \boldsymbol{f}^h)\|_1 + \|p - p^h\| \ \leq \ c \Bigg(& \inf_{(\boldsymbol{v}^h, \boldsymbol{g}^h) \in V^h \times X^h} \|(\boldsymbol{u}, \boldsymbol{f}) - (\boldsymbol{v}^h, \boldsymbol{g}^h)\|_1 \\ & + \inf_{q^h \times Q^h} \|p - q^h\| \Bigg) \end{aligned}$$

where c depends only on α, $\tilde{\beta}$ and $\tilde{\beta}_h$.

Let $\mathcal{T}_h = \{e_1, \ldots, e_m\}$ be a partition of the domain Ω into m non-overlapping elements. We shall also use terms mesh and computational mesh for the partition $\mathcal{T}_h$. Later, we shall study a few types of partitions. In this section we discuss a finite element discretization on tetrahedral partitions. In the next subsection we consider a finite difference method oriented on more general partitions. In all cases, we assume that the computational mesh is structured in some way. Note that meshes used in geoscience applications are usually structured. The structure details will be exploited in deriving efficient iterative solvers.

Let $\mathcal{T}_h$ be a tetrahedral partition and $P_k(e)$ be the space of polynomials of order k over tetrahedron $e \in \mathcal{T}_h$. We use P_1-continuous nodal approximation for the space of displacements:

$$V_h = \left\{ \boldsymbol{v}^h \in [C^0(\Omega)]^3 \cap V \colon \ \boldsymbol{v}^h|_e \in [P_1(e)]^3 \quad \forall e \in \mathcal{T}_h \right\}.$$

A few suitable choices for a stable pair $X_h \times Q_h$ were proposed and analyzed in [62]. Hereafter, we use the lowest order Raviart-Thomas (RT) finite element spaces. Let γ_{ij} denote a common interface (triangle for tetrahedral meshes) of mesh elements e_i and e_j. Then

$$\begin{aligned} X_h = \{\boldsymbol{g} \in X \colon \quad & \boldsymbol{g}|_e \in [P_0(e)]^3 + P_0(e)\boldsymbol{r} \quad \forall e \in \mathcal{T}_h, \\ & \boldsymbol{g}|_{e_i} \cdot \boldsymbol{n}_i + \boldsymbol{g}|_{e_j} \cdot \boldsymbol{n}_j = 0 \quad \forall \gamma_{ij}\}, \\ Q_h = \{q \in Q \colon \quad & q|_e \in P_0(e) \quad \forall e \in \mathcal{T}_h\}, \end{aligned} \tag{2.17}$$

where $\boldsymbol{n}_i$ is a unit vector normal to γ_{ij}, $\boldsymbol{n}_j = -\boldsymbol{n}_i$, and $\boldsymbol{r} = (x,\, y,\, z)^T \in \Re^3$. In other words, X_h is the space of piecewise linear vector functions with normal component continuous across element interfaces, and Q_h is the space of piecewise constant functions.

Now, the discrete problem (2.15) can be written in a matrix form

$$\mathcal{A}x = b. \tag{2.18}$$

Taking into account the block structure of (2.15), we may rewrite (2.18) in a 3×3 block form:

$$\begin{pmatrix} A_{uu} & 0 & A_{pu}^T \\ 0 & A_{ff} & A_{pf}^T \\ A_{pu} & A_{pf} & -c_{pp}A_{pp} \end{pmatrix} \begin{pmatrix} u \\ f \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ b_p \end{pmatrix} \tag{2.19}$$

with the obvious notations for the matrix, solution and right-hand side blocks. Later, we shall prove that A_{uu}, A_{ff} and A_{pp} are symmetric positive definite matrices. Moreover, matrix A_{pp} is diagonal.

2.3 Mimetic finite difference discretization

In many applications, the mathematical model is formulated initially as a system of first-order partial differential equations (PDEs), with each equation having a natural connection to physical aspects of the problem (e.g., mass conservation). Thus, the increasing demand for accurate and robust numerical simulations has generated considerable interest in discretizations of this first-order form. One such methodology, the mimetic finite difference technique based on support-operators (SO), has been successfully applied to several applications including diffusion [37, 38, 57, 68], electromagnetics [36], and gas dynamics [21].

Note that only the diffusion part of (2.9) is written as a system of first order PDEs. However, if the stress tensor is of greater interest than the displacement, the elastic part can also be written as a system of first order PDEs.

Recall that the third equation in (2.9) describes the mass conservation law for the fluid. Moreover, the symmetric nature of the problem follows from the fact that the negative divergence is self adjoint to the gradient. The mimetic approach of the SO discretization is based on preserving these properties on a discrete level through the use of integral identities (e.g., Green's theorem) and the definition of corresponding discrete inner products. In this manner, the SO discretization forms a weak specification of the coupled system that is connected to the mixed finite element formulation (2.15).

The SO methodology consists of a few steps. We begin with a definition of degrees of freedom for the physical variables. Then, each of the corresponding discrete spaces is equipped with a scalar product. This is the most difficult part of the SO methodology, because a mesh function is known at a finite number of points. Next, we choose a discrete approximation to the divergence operator which we call the *prime* operator. Finally, a discrete approximation of the gradient operator (the *derived* operator) will implicitly follow from a discrete Green formula.

In general, the choice of prime and derived operators depends on the grid location of physical variables, e.g., the gradient may be chosen as the prime operator, then the discrete divergence will be the derived operator. It is pertinent to note here that the partition $\mathcal{T}_h$ may include elements of different types: tetrahedra, pyramids, hexahedra, etc. For the sake of simplicity, we assume that all mesh interfaces are 2D polyhedrons.

The spaces of discrete functions

The displacement $\boldsymbol{u}$ is defined by its values at mesh points. Let V_d be the corresponding vector space. The dimension of V_d is triple the number of mesh points except points on $\overline{\partial\Omega \setminus \Gamma}$ where the displacement is zero. Therefore, V_d can be written as the tensor product space, $V_d = V_d^1 \times V_d^1 \times V_d^1$, where V_d^1 is a scalar space for each component of the displacement. The scalar product on V_d is given by

$$[\boldsymbol{v}^d,\, \boldsymbol{u}^d]_V = \sum_{i=1}^{m} [\boldsymbol{u}^d,\, \boldsymbol{v}^d]_{V,i} \qquad \forall\, \boldsymbol{u}^d,\, \boldsymbol{v}^d \in V_d, \tag{2.20}$$

where $[\boldsymbol{u}^d,\, \boldsymbol{v}^d]_{V,i}$ is an approximation of the scalar product of vector functions $\boldsymbol{u}^d$ and $\boldsymbol{v}^d$ on the i-th mesh element. Let e_i be a mesh element with $s_{i,v}$ vertices, and T_{i_j} be a neighborhood of the j-th vertex, $j = 1, \ldots, s_{i,v}$. We define the approximate scalar product as

$$[\boldsymbol{u}^d,\, \boldsymbol{v}^d]_{V,i} = \alpha_i \sum_{j=1}^{s_{i,v}} |T_{i_j}|\, \boldsymbol{u}^d_{i_j} \cdot \boldsymbol{v}^d_{i_j} \tag{2.21}$$

where $|T_{i_j}|$ is the volume of T_{i_j}, and α_i is a scaling factor. If both the neighborhoods T_{i_j} and element e_i are convex domains, the scaling factor is given by

$$\alpha_i = \frac{|e_i|}{\displaystyle\sum_{j=1}^{s_{i,v}} |T_{i_j}|}$$

where $|e_i|$ is the volume of e_i. The optimal choice of the neighborhoods is known for a few types of mesh elements. The basic idea here is to get a quadrature rule which is exact at least for linear functions. If e_i is a tetrahedron, then, $s_{i,v} = 4$, T_{i_j} coincides with e_i, and $\alpha_i = 1/3$. If e_i is a convex hexagon, then, $s_{i,v} = 8$, and T_{i_j} is a tetrahedron uniquely defined by three edges ending at the j-th vertex.

The flux $\boldsymbol{f}$ is defined by its orthogonal projections on directions perpendicular to mesh faces. Let X_d be the corresponding vector space. The dimension of X_d equals the number of mesh faces except faces on $\overline{\partial\Omega \setminus \Gamma}$ where the flux is zero. The scalar product on X_d is given by

$$[\boldsymbol{f}^d,\, \boldsymbol{g}^d]_X = \sum_{i=1}^{m} [\boldsymbol{f}^d,\, \boldsymbol{g}^d]_{X,i} \qquad \forall\, \boldsymbol{f}^d,\, \boldsymbol{g}^d \in X_d, \tag{2.22}$$

where $[\boldsymbol{f}^d,\, \boldsymbol{g}^d]_{X,i}$ is an approximation of the scalar product of vector functions $\boldsymbol{K}^{-1}\boldsymbol{f}^d$ and $\boldsymbol{g}^d$ on the i-th mesh element. Recall that values of these vector functions are not available explicitly. However, at each vertex of the mesh element e_i, a vector function can be recovered from three orthogonal projections. Let us denote the recovered values by $\boldsymbol{f}^d_{i_j}$ and $\boldsymbol{g}^d_{i_j}$, $j = 1, \ldots, s_v$. We define the approximate scalar product as

$$[\boldsymbol{f}^d,\, \boldsymbol{g}^d]_{X,i} = \alpha_i \sum_{j=1}^{s_{i,v}} |T_{i_j}|\, \boldsymbol{K}^{-1}_{e_i}\, \boldsymbol{f}^d_{i_j} \cdot \boldsymbol{g}^d_{i_j} \tag{2.23}$$

where $\boldsymbol{K}_{e_i}$ is a value of $\boldsymbol{K}$ at the mass center of element e_i, T_{i_j} is a neighborhood of the j-th vertex, $j = 1, \ldots, s_{i,v}$, and α_i is the scaling factor defined above.

The pressure p is defined by its values at mass centers of mesh elements. Let Q_d be the corresponding vector space. Obviously, the dimension of Q_d equals to the number of mesh cells, i.e., $\dim(Q_d) = m$. The scalar product on Q_d is given by

$$[p^d,\, q^d]_Q = \sum_{i=1}^{m} |e_i|\, p_i^d\, q_i^d, \qquad \forall p^d,\ q^d \in Q_d, \tag{2.24}$$

where $p^d = (p_1^d, \ldots, p_m^d)^T$ and q^d is defined similarly.

The prime and derived operators

In the case of face discretization of the vector function $\boldsymbol{f}^d$, it is natural to choose the divergence operator as the prime operator. The definition of a discrete divergence $\mathrm{DIV}_f : X_d \to Q_d$ follows naturally from the Gauss divergence theorem:

$$(\mathrm{div}\boldsymbol{f})(\boldsymbol{x}) = \lim_{|e|\to 0} \frac{1}{|e|} \oint_{\partial e} \boldsymbol{f} \cdot \boldsymbol{n}\, \mathrm{d}x \tag{2.25}$$

where $\boldsymbol{n}$ is the external normal vector, and e is a neighborhood of $\boldsymbol{x}$.

Let e_i be a mesh element with faces γ_{i_j}, $j = 1, \ldots, s_{i,f}$, and corresponding normal vectors $\boldsymbol{n}_{i_j}$ which are not necessary the external vectors. We assume that the orientation of normal vectors is fixed *a priori* for each mesh face. Furthermore, let $f_{i_j}^d$ be the orthogonal projection of $\boldsymbol{f}^d \in X_d$ on the normal vector $\boldsymbol{n}_{i_j}$, i.e., $f_{i_j}^d = \boldsymbol{f}^d \cdot \boldsymbol{n}_{i_j}$. Then, the simplest central point quadrature rule results in

$$(\mathrm{DIV}_f\, \boldsymbol{f}^d)|_{e_i} = \frac{1}{|e_i|} \sum_{j=1}^{s_{i,f}} f_{i_j}^d\, |\gamma_{i_j}|\, \Bbbk_{i_j} \tag{2.26}$$

where $|\gamma_{i_j}|$ is the area of γ_{i_j}, and $\Bbbk_{i_j} = 1$ (resp., $\Bbbk_{i_j} = -1$) if $\boldsymbol{n}_{i_j}$ is the external (resp., internal) normal vector (see Fig. 2.1 for details).

The derived operator is a discrete gradient $\mathrm{GRAD}_f : Q_d \to X_d$. It is formally defined as $\mathrm{GRAD}_f = -\mathrm{DIV}_f^*$, where the adjoint operator is taken with respect to scalar products (2.22) and (2.24), i.e.,

$$[\mathrm{DIV}_f \boldsymbol{f}^d,\, p^d]_Q = -[\boldsymbol{f}^d,\, \mathrm{DIV}_f^* p^d]_X \equiv [\boldsymbol{f}^d,\, \mathrm{GRAD}_f p^d]_X \qquad \forall \boldsymbol{f}^d \in X_d, \quad \forall p^d \in Q_d,$$

$$(\mathrm{DIV}_f \boldsymbol{f}^d)_{e_i} = \frac{1}{|e_i|}\left(f_{i_1}|\gamma_{i_1}| - f_{i_3}|\gamma_{i_3}| + f_{i_2}|\gamma_{i_2}| - f_{i_4}|\gamma_{i_4}|\right)$$

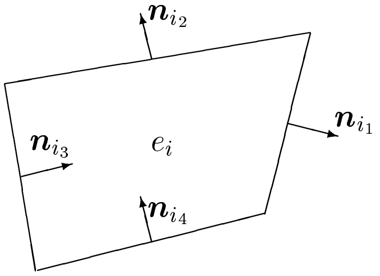

Figure 2.1: Definition of the discrete divergence for a quadrilateral.

which is nothing else but the discrete Green formula. In order to derive the explicit formula for GRAD_f, we consider an auxiliary scalar product $< \cdot, \cdot >$ and relate it to scalar products (2.22) and (2.24). Denote by $< \cdot, \cdot >$ the usual vector dot product. Then

$$[p^d,\, q^d]_Q =< \mathcal{D}_p p^d,\, q^d > \qquad \text{and} \qquad [\boldsymbol{f}^d,\, \boldsymbol{g}^d]_X =< \mathcal{M}_f\, \boldsymbol{f}^d,\, \boldsymbol{g}^d >$$

where $\mathcal{D}_p$ is a diagonal matrix, $\mathcal{D}_p = \mathrm{diag}\{|e_1|, \ldots, |e_m|\}$, and $\mathcal{M}_f$ is a sparse symmetric matrix (analog of a mass matrix in the FE theory) corresponding to the quadrature rule in (2.22). The non-zero entries of $\mathcal{M}_f$ connect face-based unknowns if and only if the corresponding faces have a common edge. Combining the last two formulae, we get

$$\begin{aligned} -[\boldsymbol{f}^d,\, \mathrm{DIV}_f^* p^d]_X &= - < \boldsymbol{f}^d,\, \mathcal{M}_f\, \mathrm{DIV}_f^* p^d > \;=\; [\mathrm{DIV}_f \boldsymbol{f}^d,\, p^d]_Q \\ &= \quad < \boldsymbol{f}^d,\, \mathrm{DIV}_f^T\, \mathcal{D}_p\, p^d > \qquad \forall \boldsymbol{f}^d \in X_d, \quad \forall p^d \in Q_d, \end{aligned}$$

where DIV_f^T is the adjoint of DIV_f with respect to the auxiliary scalar product. Therefore,

$$\mathrm{GRAD}_f = \mathcal{M}_f^{-1}\, \mathrm{DIV}_f^T\, \mathcal{D}_p. \tag{2.27}$$

Since $\mathcal{M}_f^{-1}$ is a dense matrix, the discrete gradient operator GRAD_f has a non-local stencil. One can ask here: Why do we use the weighted scalar product (2.23)? The flux $-\boldsymbol{K}\nabla p$ has physical meaning and is continuous across discontinuities in $\boldsymbol{K}$, but

∇p has neither. Since we are interested in the real flux, the weighted scalar product is a natural consequence of the Green formula:

$$-\int_\Omega p \,\mathrm{div}\, \boldsymbol{f} \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \nabla p \,\mathrm{d}x = \int_\Omega \boldsymbol{f} \boldsymbol{K}^{-1}(\boldsymbol{K}\nabla p)\,\mathrm{d}x \qquad \forall \boldsymbol{f} \in X, \quad \forall p \in H^1(\Omega,\, \Gamma),$$

where $H^1(\Omega,\, \Gamma)$ is a space of functions from $H^1(\Omega)$ vanishing on Γ. In the case of heterogeneous boundary conditions, a boundary term will appear in the Green formula and in all related scalar products.

In the case of nodal discretization of the vector function $\boldsymbol{u}^d$, it is also natural to use divergence as the prime operator. The discrete divergence $\mathrm{DIV}_u : V_d \to Q_d$ is derived from the Gauss theorem. Let $\boldsymbol{u}^d_{i_j}$, $j = 1, \ldots, s_{i,v}$, be values of $\boldsymbol{u}^d \in V_d$ at vertices of the mesh element e_i. Furthermore, let $\hat{\boldsymbol{u}}_{i_j}$, be a value of $\boldsymbol{u}^d$ at the mass center of face γ_{i_j}, $j = 1, \ldots, s_{i,f}$. It is computed as a linear combination of $\boldsymbol{u}^d_{i_j}$ associated with vertices of γ_{i_j} (e.g., for a triangular face, all weights in the linear combination equal to 1/3). Then, the central point quadrature rule results in

$$(\mathrm{DIV}_u\, \boldsymbol{u}^d)|_{e_i} = \frac{1}{|e_i|} \sum_{j=1}^{s_{i,f}} \hat{\boldsymbol{u}}_{i_j} \cdot \boldsymbol{n}_{i_j} |\gamma_{i_j}| \tag{2.28}$$

where $\boldsymbol{n}_{i_j}$ are *a priori* fixed normal vectors described above.

The derived operator is a discrete gradient $\mathrm{GRAD}_u : Q_d \to V_d$. It is derived similar to GRAD_f. Therefore, it is sufficient to relate the auxiliary scalar product to the scalar product (2.20):

$$[\boldsymbol{u}^d,\, \boldsymbol{v}^d]_V =< \mathcal{M}_u\, \boldsymbol{u}^d,\, \boldsymbol{v}^d >$$

where $\mathcal{M}_u$ is a positive definite diagonal matrix. The diagonal entries of $\mathcal{M}_u$ are nodal volumes given by coefficients in (2.21). Then, an analog of (2.27) for the new pair of discrete spaces (Q_d and V_d) is given by:

$$\mathrm{GRAD}_u = \mathcal{M}_u^{-1}\, \mathrm{DIV}_u^T\, \mathcal{D}_p. \tag{2.29}$$

Let us rewrite the first equation in (2.9) as follows:

$$-\mu\Delta\boldsymbol{u} - (\lambda + \mu)\,\nabla(\mathrm{div}\,\boldsymbol{u}) + \nabla p = 0.$$

It is obvious now that the discrete operators constructed above can used to approximate the last two terms. However, they can not be combined to derive a discrete analog of the vector Laplace operator acting from V_d to V_d. The solution of this problem is to construct the third pair of operators, $\mathrm{GRAD}_e : V_d^1 \to E_d$ and $\mathrm{DIV}_e : E_d \to V_d^1$, where E_d is a space of edge based vector functions. Each function in E_d is uniquely defined by its orthogonal projections on mesh edges. When $\mathcal{T}_h$ is a tetrahedral partition, E_d has the same degrees of freedom as the lowest order Nedelec finite element space.

The definition of a scalar product in space E_d is similar to that in space X_d. Let $\boldsymbol{a}^d, \boldsymbol{b}^d \in E_d$. Then

$$[\boldsymbol{a}^d,\, \boldsymbol{b}^d]_E = \sum_{i=1}^{m} [\boldsymbol{a}^d,\, \boldsymbol{b}^d]_{E,i} \tag{2.30}$$

where $[\boldsymbol{a}^d,\, \boldsymbol{b}^d]_{E,i}$ is an approximation of the scalar product of vector functions $\boldsymbol{a}^d$ and $\boldsymbol{b}^d$ on the i-th mesh element. Since the vector functions are defined by their projections on mesh edges, their actual values can be recovered only at mesh points. Let us denote the recovered values by $\boldsymbol{a}^d_{i_j}$ and $\boldsymbol{b}^d_{i_j}$, $j = 1, \ldots, s_{i,v}$. Then, $[\boldsymbol{a}^d,\, \boldsymbol{b}^d]_{E,i}$ is given by (2.23) with $\boldsymbol{a}^d$ (resp., $\boldsymbol{b}^d$) instead of $\boldsymbol{f}^d$ (resp., $\boldsymbol{g}^d$) and $\boldsymbol{I}$ instead of $\boldsymbol{K}_{e_i}$. The scalar product (2.30) is related to the auxiliary scalar product as follows:

$$[\boldsymbol{a}^d,\, \boldsymbol{b}^d]_E = < \mathcal{L}_u\, \boldsymbol{a}^d,\, \boldsymbol{b}^d >$$

where $\mathcal{L}_u$ is a sparse symmetric matrix (an analog of a mass matrix in the FE theory) corresponding to the quadrature rule in (2.30). The non-zero entries of $\mathcal{L}_u$ connect edge-based unknowns if and only if the corresponding edges have a common point.

For the pair of discrete spaces E_d and V_d^1, it is more natural to pick the gradient operator as the prime operator. A discrete GRAD_e operator is obtained from the definition of directional derivative $\partial/\partial l$,

$$\frac{\partial u_1^d}{\partial l} = \nabla u_1^d \cdot \boldsymbol{l},$$

which is the orthogonal projection of ∇u_1^d onto direction given by $\boldsymbol{l}$. In the discrete case for a scalar function $u_1^d \in V_d^1$ defined at mesh points, this relationship leads to a coordinate invariant definition of GRAD_e. Let $\boldsymbol{l}_i$ be a mesh edge with *a priori* fixed orientation, and u_{1,i_1}^d and u_{1,i_2}^d be values of function u_1^d at end points of the edge. The simplest central point quadrature rule gives

$$(\mathrm{GRAD}_e \boldsymbol{u}^d)|_{\boldsymbol{l}_i} = \frac{u_{1,i_2}^d - u_{1,i_1}^d}{|\boldsymbol{l}_i|} \tag{2.31}$$

where $|\boldsymbol{l}_i|$ is the length of $\boldsymbol{l}_i$.

The derived operator DIV_e is defined by the discrete Green formula, i.e., $\mathrm{DIV}_e = -\mathrm{GRAD}_e^*$ where the adjoint operator is computed with respect to scalar products (2.20) and (2.30). It is obvious that

$$\mathrm{DIV}_e = \mathcal{M}_u^{-1}\,\mathrm{GRAD}_e^T\,\mathcal{L}_u. \tag{2.32}$$

The finite difference approximation of the coupled problem (2.9) may be summarized as follows:

$$\begin{array}{rcrcl} -\mu\mathrm{DIV}_e\mathrm{GRAD}_e\,\boldsymbol{u}^d - (\mu+\lambda)\mathrm{GRAD}_u\mathrm{DIV}_u\,\boldsymbol{u}^d & + & \mathrm{GRAD}_u p^d & = & 0, \\ \boldsymbol{f}^d & + & \mathrm{GRAD}_f\, p^d & = & 0, \\ -\mathrm{DIV}_u\,\boldsymbol{u}^d - \mathrm{DIV}_f\,\boldsymbol{f}^d & - & c_{pp}p^d & = & S^d \end{array} \tag{2.33}$$

where $S^d = (S_1, \ldots, S_m)^T \in Q_d$ and S_i is the integral average of the right hand side $-S_f$ over mesh element e_i, $i = 1, \ldots, m$.

Since $\mathcal{M}_f^{-1}$ is a dense matrix, the discrete operator GRAD_f has a non-local stencil. Therefore, before solving (2.33), we have to multiply the second equation by $\mathcal{M}_f$.

There is a lack of theoretical results concerning accuracy of the solution to (2.33). We return to this problem in Chapter 5 where we shall prove the first accuracy estimated for the SO method.