Spotlight Computed Tomography with Natural Pixels

Van Emden Henson* Mark A. Limber† Stephen F. McCormick‡
Bruce T. Robinson§

Abstract

The problem of reconstructing an image from its Radon transform projections is outlined and a minimum norm, optimal grid formulation is described. We develop the formalism necessary to apply fast adaptive composite grids (FAC) in this setting to obtain high resolution reconstructions in a selected region of the image. This avoids utilizing a fine grid on the entire image space, which leads to intractably large problems. Theoretical details and numerical results will be published in a future paper, [3].

1 The Radon Transform and Image Reconstruction

Image reconstruction from projections is the underdetermined linear inverse problem that attempts to recover an unknown function (the image) from a collection of lower dimensional profiles of the unknown [5]. In the situation we consider, these profiles are obtained by integrating the unknown function $u(x, y)$, defined on $\Omega \subset \mathbf{R}^2$, along certain rays, orthogonal to a set of "projections". Mathematically, we are given values of the function $R : H_1 = L_2(\Omega) \to H_2$, defined for $\rho \in \mathbf{R}$ and $\theta \in [0, \pi]$ by

$$Ru(\rho, \theta) = \int_{\mathbf{R}^2} u(x, y)\delta(x \cos\theta + y \sin\theta - \rho)dxdy \tag{1}$$

where δ is the Dirac delta function and H_2 is an appropriate Hilbert space [11]. We say that the collection of all $R(\rho, \theta)$ for fixed θ is a *projection* of the function u. In case one is given Ru for all angles θ and for all values of ρ, there is an inverse formula for equation (1) given by Radon in 1917 [12] (for a translation, see [2]).

In applications, one gathers approximations to Ru for a finite number of angles θ and a finite number of *bins* specified by ρ. The underdetermined problem is: *given* $Ru(\rho_k, \theta_k) = f_k$ *for a finite number of* k*, find* u*.* This is called the discrete Radon inverse problem. This problem is ill posed both in the sense that a small change in the data can cause a large change in the image u and in the sense that there if there is any function which satisfies the data, there is an infinity of such functions.

2 Optimal Grid Methods

We make the assumption, which is well justified in emission tomography, for example, that the data collected does not represent the line integrals of some function u, but rather strip

*Department of Mathematics, Naval Postgraduate School, email: vhenson@math.nps.navy.mil
†CECM, Department of Mathematics, Simon Fraser University, email: malimber@cecm.sfu.ca
‡Program in Applied Mathematics, University of Colorado, email: stevem@newton.colorado.edu
§Department of Mathematics, Naval Postgraduate School, email: robinson@math.nps.navy.mil

integrals of the function u. We then have a "fat ray" operator $A : L_2(\Omega) \to \mathbf{R}^2$ defined by

$$(Au)_j = \int u(x,y)\psi_j(x,y)dxdy \tag{2}$$

where ψ_j is the characteristic function of the j^{th} strip which is essentially a *fattening* of the line given by (ρ_j, θ_j) in equation (1). The functions ψ_j have been termed *natural pixels* [1].

Given projection data from an unknown function u sampled at at M angles, each with K_m bins, or strips, let $N = \sum K_m$ be the total number of data points. Assuming that the data $\mathbf{f}$ is consistent, we select the image u that satisfies the data constraint and minimizes the L_2-norm of the image, considered as a function in $L_2(\Omega)$. The optimal image, $\hat{u}$, is given by $\hat{u} = A^*\mathbf{w}$ where $\mathbf{w}$ solves $AA^*\mathbf{w} = \mathbf{f}$ and $\mathbf{w} \in \mathbf{R}^N$. It is easy to see [4] that

$$[A^*\mathbf{w}](x,y) = \sum_{j=1}^{N} \mathbf{w}_j \psi_j(x,y) \tag{3}$$

and $B = AA^*$ is an $N \times N$ matrix with $(i,j)^{\text{th}}$ entry $b_{i,j} = \langle \psi_i, \psi_j \rangle$, the inner product in L_2 of ψ_i with ψ_j. . The minimum norm image reconstruction problem becomes the problem of solving the matrix equation $B\mathbf{w} = \mathbf{f}$.

Note that the optimal image $A^*\mathbf{w}$ is a function in $L_2(\Omega)$, but from equation (3), $A^*\mathbf{w}$ is in fact a piecewise constant function, on what has been called the *optimal grid* [6].

In a typical emission tomography applications, one obtains data from 64 angles and 64 bins per angle, resulting in 4096 pieces of data, which means the matrix B is 4096×4096. Transmission tomography may often utilize 500-1500 angles with 700-1500 bins per angle [7], so systems of 2.5 million equations in as many unknowns may be generated. We have used a Gauss-Seidel relaxation with multigrid acceleration to solve $B\mathbf{w} = \mathbf{f}$ on the smaller problems (64×64), the details of which can be found in [4].

3 Spotlight CT Image Reconstruction

In case one desires high resolution in a certain region of interest (ROI), for example, where a tumor is suspected, it is computationally more efficient to discretize the ROI with a fine grid and the remaining image space with a coarser grid. We call this *spotlighting* the ROI. The FAC methodology to do this is described in the following section.

3.1 Fast Adaptive Composite Grid (FAC)

The spotlight CT problem is essentially a composite grid problem, in which an operator equation $Lu = f$ must be solved on a composite grid $\Omega^{\underline{h}}$ comprised of a global coarse grid Ω^{2h} and one or more local refinement grid Ω^h (the refinement grid may itself be a composite grid, which permits recursive refinement). Fast adaptive composite grid methods (FAC) were developed [10, 9] in order to utilize multigrid technology to treat such problems efficiently.

FAC succeeds because it handles the composite grid as a nested sequence of regular grids that can be treated independently using any handy regular-grid method. The key ingredients lie in having appropriate representations of the operator and intergrid transfer operators. Thus grid functions u must be representable on the composite, global, and refinement grids $(u^{\underline{h}}, u^{2h}, u^h)$, and operators must exist to transfer grid functions between the grids ($I_{\underline{h}}^h : \Omega^{\underline{h}} \to \Omega^h$, $I_{\underline{h}}^{2h} : \Omega^{\underline{h}} \to \Omega^{2h}$, $I_h^{\underline{h}} : \Omega^h \to \Omega^{\underline{h}}$, and $I_{2h}^{\underline{h}} : \Omega^{2h} \to \Omega^{\underline{h}}$). Finally, it is critical that the operator L be representable on these grids $(L^{\underline{h}}, L^{2h}, L^h)$. In general,

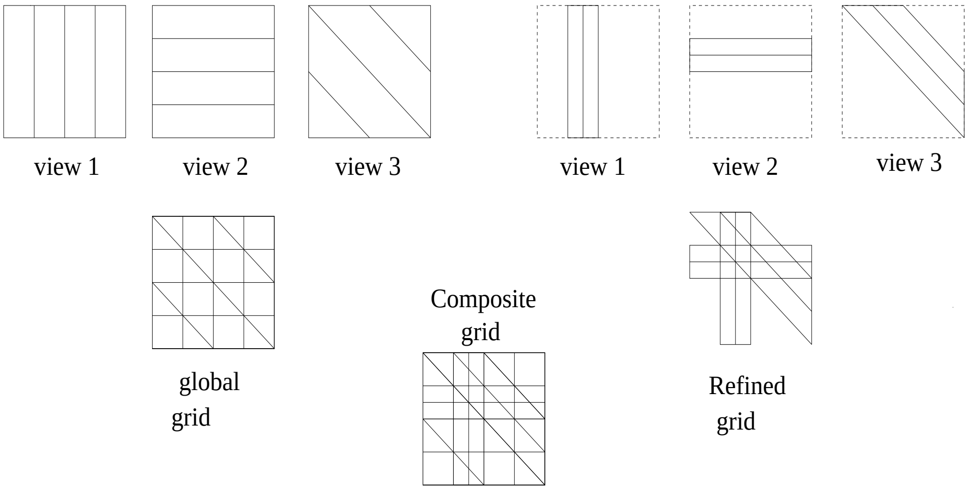

FIG. 1. The global grid Ω^h (bottom left) is generated by strips from three views (top left), while the refinement patch (bottom right) is generated by finer strips from the three views (top right). The union of all strips generates the composite grid (bottom center).

the details of these operators and representations can be developed in a straightforward fashion once the grids are defined; the details, however, are many and technical. Detailed treatments can be found in [8, 9, 10]. Once the operators and grids are defined, FAC proceeds in the following two-step sequence:

Step 1: *Set* $f^{2h} \leftarrow I_{\underline{h}}^{2h}(f^{\underline{h}} - L^{\underline{h}}u^{\underline{h}}), \quad u^{2h} \leftarrow (L^{2h})^{-1}f^{2h}, \quad u^{\underline{h}} \leftarrow u^{\underline{h}} + I_{2h}^{\underline{h}}u^{2h}$

Step 2: *Set* $f^{h} \leftarrow I_{\underline{h}}^{h}(f^{\underline{h}} - L^{\underline{h}}u^{\underline{h}}), \quad u^{h} \leftarrow (L^{h})^{-1}f^{h}, \quad u^{\underline{h}} \leftarrow u^{\underline{h}} + I_{h}^{\underline{h}}u^{h}.$

Despite this formal definition, FAC need not utilize exact solvers on the global coarse grid or the refinement patch. Historically, FAC has been used predominantly with iterative methods [9].

3.2 The spotlight grid

We utilize FAC methodology to devise a discretization for the spotlight CT problem. We begin with a global grid Ω^{2h} generated by the natural pixels ψ_j^{2h}. We next add a refinement grid Ω^h by forming strips ψ_j^h. For each view we choose pairs of strips whose union conforms exactly to one of the global strips. As an extremely simple example, consider the discretization given by three views, each consisting of four strip pixels (Figure 1, left). The resulting global (coarse grid) operator is the 12×12 matrix $B_{2h,2h}$, the matrix whose $(j,k)^{th}$ entry is $b_{jk}^{2h} = \left\langle \psi_j^{2h}, \psi_k^{2h} \right\rangle$. The refinement grid Ω^h is formed by partitioning one strip from each view into two strips. Hence ψ_2^{2h} generates ψ_1^h and ψ_2^h. Similarly, strips 3 and 4 on the refinement grid are a partition of ψ_7^{2h} and ψ_{11}^{2h} generates refinement strips 5 and 6 (Figure 1, right). The refinement strips ψ_k^h generate a refinement patch operator $B_{h,h}$, formed in the usual manner by $b_{jk}^h = \left\langle \psi_j^h, \psi_k^h \right\rangle$, giving a 6×6 matrix in the simple example. The composite grid $\Omega^{\underline{h}}$ formed in this manner is shown at the bottom of Figure 1.

The basic assumption of the discretization, that the unknown image is a linear combination of the strip functions, is unchanged. The total number of strips is $N =$

$N_{2h} + N_h$, the sum of the number of strips on the global grid and the refinement patch. Hence the image $u = A^{\underline{h}*}\mathbf{w}$ is given by (3) where $\psi_j = \psi_j^{2h}$ if $1 \le j \le N_{2h}$ and $\psi_j = \psi_{j-N_{2h}}^h$ if $N_{2h} < j \le N$. We can define the composite grid version of $A^{\underline{h}}$ by $(Au)_j = \langle \psi_j, u \rangle$ using the same ordering of the ψ_j's. The discretized composite grid operator $B_{\underline{h},\underline{h}}$ may then be computed in the standard way, by $B_{\underline{h},\underline{h}} = A^{\underline{h}}A^{\underline{h}*}$ and has, as its $(j,k)^{th}$ entry, the element $b_{jk}^{\underline{h}} = \langle \psi_j, \psi_k \rangle$. This method accounts for the interaction of the refinement grid with the global coarse grid, including the inner products between the fine strips and coarse strips. In fact, this leads to a natural partitioning of the composite grid operator as

$$B_{\underline{h},\underline{h}} = \begin{pmatrix} B_{2h,2h} & B_{2h,h} \\ B_{h,2h} & B_{h,h} \end{pmatrix}.$$

We observe that by the ordering of the ψ_j's and the definition of $A^{\underline{h}*}$ we are led to a natural definition of the composite grid unknown, namely $u^{\underline{h}} = (u^{2h} \;\; u^h)^T$. Naturally, we must have a compatible composite grid data vector $f^{\underline{h}} = (f^{2h} \;\; f^h)^T$. Conceivably, the coarse grid data and the refinement data could be acquired in separate recordings, but it is more likely that a single data set be generated, from which the coarse and refinement data are derived. The composite grid problem $B_{\underline{h},\underline{h}}u^{\underline{h}} = f^{\underline{h}}$ then becomes

$$\begin{pmatrix} B_{2h,2h} & B_{2h,h} \\ B_{h,2h} & B_{h,h} \end{pmatrix} \begin{pmatrix} u^{2h} \\ u^h \end{pmatrix} = \begin{pmatrix} f^{2h} \\ f^h \end{pmatrix}. \tag{4}$$

3.3 FAC Implementation

It can be shown [3, 9, 13] that FAC in this setting is exactly equivalent to applying two steps of *block* Gauss-Seidel iteration to the system (4). That is, FAC takes the two step form:

Step 1: *Set* $\quad u^{2h} \leftarrow B_{2h,2h}^{-1}(f^{2h} - B_{2h,h}u^h),$

Step 2: *Set* $\quad u^h \leftarrow B_{h,h}^{-1}(f^h - B_{h,2h}u^{2h})$

These steps are formal, of course, since we know that $B_{2h,2h}$ is singular. Nor do we *need* exact solvers. In principle we may apply any method to these subproblems: ART, filtered backprojection, Fourier methods. We expect, however, that they would commonly be treated with iterative methods, such as Gauss-Seidel or multigrid [4]. Noting that each of the steps are solving a "residual" equation on one of the grids, this process may be viewed as one of multilevel correction.

The composite grid operator $B_{\underline{h},\underline{h}}$ possesses a host of useful and interesting properties [13], related to a family of useful properties generated by the discretization method and inherent in the global and local operators $B_{2h,2h}$ and $B_{h,h}$. Space limitations do not permit elaboration here, nor is there room for a performance analysis. A rigorous treatment of the method, including a performance assesment, will appear in forthcoming works [3, 13].

4 Conclusions

We close with a few observations regarding the utility of the method. The primary scenario in which we envision using this method occurs when fine resolution is required over a small portion of an image while coarser resolution will suffice elsewhere. Our proposal is to collect data simultaneously for both the refined and global coarse grids. If necessary, the data can be aquired uniformly at the refined grid level; it is easy to construct the global coarse grid data from it using the intergrid transfers.

As a simple illustrative example, suppose that the global coarse problem is defined with N strips (of width $1/N$) per view, and the refinement region is over 10% of the strips on each view. Suppose that recursive refinement is carried out until the refinement region is uniformly covered with strips of width $1/(2^p N)$. The total number of strips per view generating $B_{\underline{h},\underline{h}}$ will be $N(1+0.1\sum_{j=1}^{p} 2^j)$, or $N(1+0.1(2^{p+1}-2))$. A global grid at this level of refinement would have $2^p N$ strips per view. For example, if we want $8\times$ refinement over 10% of 100 strips on each of 10 views, $B_{\underline{h},\underline{h}}$ is a 2400×2400 matrix, or 9% the size of the 8000×8000 matrix corresponding to the globally refined 800 strips per view.

References

[1] M. H. Bounocore, W. R. Brody, and A. Macovski, *A natural pixel decomposition for two dimensional image reconstruction*, IEEE Transactions on biomedical engineering, BME-28 (1981), pp. 69–78.

[2] S. R. Deans, *The Radon Transform and some of its Applications*, John Wiley and Sons, New York, N.Y., 1983.

[3] V. E. Henson, M. A. Limber, S. F. McCormick, and B. T. Robinson, *Multilevel projection methods for spotlight computed tomography*, in preparation.

[4] ———, *Multilevel optimal grid methods for image reconstruction from projections*, in Proceedings of the Colorado Conference on Iterative Methods, to appear.

[5] A. C. Kak and M. Slaney, *Principles of Computerized Tomographic Imaging*, IEEE Press, New York, N.Y., 1988.

[6] M. A. Limber, T. A. Manteuffel, S. F. McCormick, and D. S. Sholl, *Optimal resolution in maximum entropy image reconstruction from projections with multigrid acceleration*, in Proceedings of the Sixth Annual Copper Mountain Conference on Multigrid Methods, 1993.

[7] A. K. Louis, *Medical imaging: the state of the art and future development*, Inverse Problems, 8 (1992), pp. 709–738.

[8] J. Mandel and S. F. McCormick, *Iterative solution of elliptic equations with refinement: the model multilevel case*, in Domain decomposition methods, T. F. Chan, ed., Philadelphia, PA, 1987, Society for Industrial and Applied Mathematics, pp. 81–92.

[9] S. F. McCormick, *Multilevel adaptive methods for partial differential equations*, vol. 6 of Frontiers in applied mathematics, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1989.

[10] S. F. McCormick and J. Thomas, *The fast adaptive composite grid method (FAC) for elliptic boundary value problems*, Math. Comp., 46 (1986), pp. 439–456.

[11] F. Natterer, *A Sobolev space analysis of picture reconstruction*, SIAM Journal of Applied Mathematics, 39 (1980), pp. 402–410.

[12] J. Radon, *Über die bestimmung von funktionen durch ihre integralwerte längs gewisser mannigfaltigkeiten*, Berichte Sächsische Akadamie der Wissenschaften, Leipzig,Math.-Phys. Kl., 69 (1917), pp. 262–267.

[13] B. T. Robinson, *Multilevel Methods of Image Reconstruction*, PhD thesis, Naval Postgraduate School, 1994 (anticipated).