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Balanced H_∞ and H_2 Controllers

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Abstract-- A balanced H_∞ controller is defined and analyzed in this paper. Gains of an H_∞ controller are obtained from the constrained solutions of two Riccati equations. If the solutions are equal and diagonal, the controller is H_∞ balanced. The transformation which generates the H_∞ balanced solution is derived. Also, properties of the balanced H_∞ controller, as well as its relationship to an H_2 balanced controller and to an open-loop balanced system, are presented.

A characteristic property of flexible structures is that they have almost independent components in Moore balanced coordinates. It is shown in this paper that the H_∞ balanced components are also almost independent and that the open-loop and the H_∞ balanced representations almost coincide, This property makes it possible to design reduced-order H_∞ controllers of comparable performance to full-order controllers.

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1. INTRODUCTION

BALANCED REPRESENTATION of open-loop systems is a tool for system order reduction, see Moore (198 1), Parnebo and Silverman (1982), Gawronski and Juang (1990), and many other authors. Its simplicity and efficiency has encouraged investigations of the balanced representations of model-based controllers, with a goal to obtain reduced-order controllers of comparable performance to full-order controllers. Balanced H_2 controllers were studied by Jonckheere and Silverman (1983), Opendacker and Silverman (1985), and Gawronski (1993), while balanced H_∞ controllers were investigated by Mustafa (1988). Note, however, that the balanced H_∞ controller was obtained by Mustafa for a special case of collocated control and exogenous inputs, and collocated measured and controlled outputs.

Considerable attention has recently been given to the design of H_∞ and H_2 controllers for flexible structures, see for example Balas and Doyle (1991), Carrier et *al.* (1991), Lim, Maghami, Joshi (1992), Lim and Balas (1992), Gawronski (1993). In this paper a generic H_∞ controller is analyzed. The transformation to an H_∞ balanced representation is derived, and the relationships between H_∞ and H_2 characteristic values, H_∞ characteristic values and Hankel singular values, and H_2 characteristic values and Hankel singular values were obtained. It is shown that in the case of flexible structures the open-loop balanced representation and the H_∞ balanced representation almost coincide, and that the components of a balanced controller are almost independent. Based on these facts, approximate closed-form formulas for H_∞ characteristic values, for their upper and lower bounds and for the closed-loop pole shift are derived. A controller reduction index is introduced to facilitate a stable reduction of a controller that preserves the performance of the full-order controller. Finally, the balanced H_2 controller is obtained as a special case of the balanced H_∞ controller.

2. BALANCED CONTROLLERS

Open-loop balanced system. Denote (A, B_k, C_k) and $k=1,2,$ as the state-space representations of stable, controllable, and observable open-loop systems, where A is NxN, B_k is $N\times p_k$, and C_k is $q_k\times N$. Their controllability and observability grammians W_{ck} and W_{ok} are positive-definite and satisfy the Lyapunov equations

$$A W_{ck} + W_{ck}A^T + B_kB_k^T = 0, \quad A^T W_{ok} + W_{ok}A + C_k^TC_k = 0 \tag{1}$$

$k=1,2.$ The system representation is balanced in the sense of Moore (cf. Moore (1981)) if its controllability and observability grammians are diagonal and equal

$$W_{ck} = W_{ok} = \Gamma_k^2, \quad \Gamma_k = diag(\gamma_{k1}, \ldots, \gamma_{kN}), \quad k=1,2 \tag{2}$$

and $\gamma_{kj} > 0$ is the jth Hankel singular value of the kth sYstem.

Central H_∞ controller. Consider a representation of a closed-loop H_∞ system, with the plant transfer function $G(s)$, and the controller transfer function $K(s)$, such that

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = G(s) \begin{bmatrix} w(s) \\ u(s) \end{bmatrix}, \quad u(s) = K(s)y(s) \tag{3}$$

u, w are control and exogenous inputs, and y, z measured and controlled outputs, respectively. In the related state-space equations

$$\dot{x} = Ax + B_1w + B_2u, \; z = C_1x + D_{12}u, \; y = C_2x + D_{21}w \tag{4}$$

(A, B_2, C_2) is stabilizable and are detectable, the conditions

$$D_{12}^T[C_1 \;\; D_{12}] = [0 \;\; I], \; D_{21}[B_1^T \;\; D_{21}^T] = [0 \;\; I] \tag{5a}$$

are satisfied, and the matrices

$$\begin{bmatrix} A-j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}, \quad \begin{bmatrix} A-j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix} \tag{5b}$$

have full column rank, see Glover and Doyle (1988), and Doyle *et al.* (1989). Let G_{zw} be the transfer function of the closed-loop system from w to z, then there exists an admissible controller such that $\|G_{zw}\|_\infty < \rho$, where ρ is the smallest number such that the following three conditions hold:

1. $S_{\infty c} \geq 0$ solves the following central Riccati equation (HCARE)

$$S_{\infty c}A + A^T S_{\infty c} + C_1^T C_1 - S_{\infty c}(B_2 B_2^T - \rho^{-2} B_1 B_1^T) S_{\infty c} = 0 \tag{6a}$$

2. $S_{\infty e} \geq 0$ solves the following central Riccati equation (HFARE)

$$S_{\infty e}A^T + A S_{\infty e} + B_1 B_1^T - S_{\infty e}(C_2^T C_2 - \rho^{-2} C_1^T C_1) S_{\infty e} = 0 \tag{6b}$$

3. $$\lambda_{max}(S_{\infty c} S_{\infty e}) < \rho^2 \tag{6c}$$

and $\lambda_{max}(X)$ is the largest eigenvalue of X,

4. The Hamiltonian matrices

$$\begin{bmatrix} A & \rho^{-2}B_1B_1^T - B_2B_2^T \\ -C_1^TC_1 & -A^T \end{bmatrix}, \quad \begin{bmatrix} A^T & \rho^{-2}C_1^TC_1 - C_2^TC_2 \\ -B_1B_1^T & -A \end{bmatrix} \tag{6d}$$

do not have eigenvalues on the $j\omega$ axis.

Balanced H_∞ controller. An H_∞ controller is balanced if the related HCARE and HFARE solutions are equal and diagonal,

Definition 1. The solutions of HCARE and HFARE are H_∞ balanced if

$$S_{\infty c}=S_{\infty e}=M_{\infty},\quad M_{\infty}=diag(\mu_{\infty 1},\mu_{\infty 2},\ldots,\mu_{\infty N}),\qquad \mu_{\infty 1}\geq\mu_{\infty 2}\geq\ldots\geq\mu_{\infty N}>0 \tag{7}$$

where $\mu_{\infty i}$ is the i-th H_{∞} characteristic (or singular) value.

Let

$$P_{\infty c}=S_{\infty c}^{1/2},\qquad P_{\infty e}=S_{\infty e}^{1/2} \tag{8a}$$

denote $N_{\infty}=P_{\infty c}P_{\infty e}$, and let

$$N_{\infty}=V_{\infty}M_{\infty}U_{\infty}^{T} \tag{8b}$$

be the singular value decomposition of, N. Consider the transformation T_{∞} of the state x such that $\tilde{x}=T_{\infty}x$, then:

Proposition 1. For transformation T_{∞}

$$T_{\infty}=P_{\infty e}U_{\infty}M_{\infty}^{-1/2}=P_{\infty c}^{-1}V_{\infty}M_{\infty}^{1/2} \tag{9}$$

the representation $(T_{\infty}^{-1}AT_{\infty}, T_{\infty}^{-1}B_1, T_{\infty}^{-1}B_2, C_1T_{\infty}, C_2T_{\infty})$ is H_{∞} balanced.

Proof The solutions of HCARE and HFARE in new coordinates are $\bar{S}_{\infty c}=T_{\infty}^{T}S_{\infty c}T_{\infty}$, $\bar{S}_{\infty e}=T_{\infty}^{-1}S_{\infty e}T_{\infty}^{-T}$. Introducing T_{∞} as in Eq.(9) gives the balanced HCARE and HFARE solutions. □

For the H_{∞} balanced solution the condition in Eq. (6c) simplifies to

$$\mu_{\infty 1}<\rho,\ \text{and}\ \mu_{\infty N}>0 \tag{lo}$$

Let $X_1>X_2$ $(X_1\geq X_2)$ denote that X_1-X_2 is positive definite (positive semidefinite). The relationship between H_{∞} characteristic values and open-loop (Hankel) singular values is established, First note the following lemma:'

Lemma 1, Derese and Noldus (1980). For asymptotically stable A, and $V > O$, consider two Riccati equations:

$$A^T S_i + S_i A - S_i W_i S_i + V = 0, \quad i = 1,2 \tag{11}$$

then if $W_2 \geq W_1 \geq 0$, one obtains $S_1 \geq S_2 \geq 0$.

Let Γ_1 be a matrix of Hankel singular values of the representation (A, B_1, C_1), cf. Eq.(2), and M_∞ be a matrix of H_∞ characteristic values defined in Eq.(7). Then:

Proposition 2. For asymptotically stable A, and for $B_2B_2^T - \rho^{-2}B_1B_1^T \geq 0$, $C_2^TC_2 - \rho^{-2}C_1^TC_1 \geq 0$, one obtains

$$M_\infty \leq \Gamma_1^2, \quad \text{or} \quad \mu_{\infty i} \leq \gamma_{1i}^2, \quad i = 1, \ldots, N \tag{12}$$

Proof. This proposition is a consequence of Lemma 1 applied to Eq.(6a), and the second ~.(1), as well as Eq.(6b), and the first Eq.(1), obtaining $W_{c1} \geq S_{\infty c}$ and $W_{o1} \geq S_{\infty c}$. From the latter inequalities it follows that $\lambda_i(W_{c1}) \geq \lambda_i(S_{\infty c})$ and $\lambda_i(W_{o1}) \geq \lambda_i(S_{\infty c})$ (see Horn and Johnson (1985), Corollary 7.7.4, p.471), thus $\lambda_i(W_{c1}W_{o1}) \geq \lambda_i(S_{\infty c}S_{\infty c})$, or $M_\infty \leq \Gamma^2{}_1$.□

H_2 controller. An H_2 system is a special case of the H_∞ system, cf. Boyd and Barratt (1991). It has similar representation as the H_∞ system in Eq.(4), and its matrices A, B_1, B_2, C_1, C_2, D_{12}, D_{21} defined in the following. It consists of state x, control input u, measured output y, exogenous input $w^T = [v_u^T \; v_y^T]$, and regulated variable $z = C_1x + D_{12}u$, where vu, v_y are process and measurement noise, respectively. The noises vu and v_y are uncorrelated, and have constant power spectral density matrices *Vu* and V_y, respectively. For positive semidefinite matrix *VU,* the matrix B_1 has the following form:

$$B_1 = [V_u^{1/2} \; q \tag{13}$$

The task is to determine the controller gain (k_c) and estimator gain (k_e) such that the performance index J

$$J = E\left\{\int_0^\infty (x^T Qx + u^T Ru)dt\right\} \tag{14}$$

is minimal, where R *is* a positive definite input weight matrix, and Q a positive semi definite state weight matrix. The matrix C_1 is defined through the weight Q

$$C_1 = \begin{bmatrix} 0 \\ Q^{1/2} \end{bmatrix} \tag{15}$$

and, without loss of generality, assume $R=I$ and $V_y = I$, obtaining

$$D_{12} = \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad D_{21} = [0 \quad I] \tag{16}$$

The minimum of J *is* achieved for the feedback where the gain matrices (k_c and k_e)

$$k_c = -B_2^T S_{2c}, \quad k_e = -S_{2e} C_2^T \tag{17}$$

where S_{2c} and S_{2e} are solutions of the controller Riccati equation (CARE) and the estimator Riccati equation (FARE), respectively

$$S_{2c}A + A^T S_{2c} + C_1^T C_1 - S_{2c} B_2 B_2^T S_{2c} = 0 \tag{18a}$$

$$S_{2e}A^T + AS_{2e} + B_1 B_1^T - S_{2e} C_2^T C_2 S_{2e} = 0 \tag{18b}$$

Note by comparing Eqs.(6) and (18) that for $\rho^{-1} = 0$ the H_∞ solution becomes the H_2 solution.

Balanced H_2 controller. An H_2 controller is balanced if the related CARE and FARE solutions are equal and diagonal.

Definition 2. The solutions of CARE and FARE are H_2 balanced if

$$S_{2c}=S_{2e}=M_2, \quad M_2=diag(\mu_{21},\mu_{22},\ldots,\mu_{2N}), \quad \mu_{21}\geq\mu_{22}\geq\ldots\geq\mu_{2N}>0 \tag{19}$$

where μ_{2i} is the i-th H_2 characteristic (or singular) value.

Denote

$$N_2=P_{2c}P_{2e}, \quad \text{where} \quad P_{2c}=S_{2c}^{1/2}, \quad P_{2e}=S_{2e}^{1/2} \tag{20a}$$

and let N_2 have the following singular value decomposition

$$N_2=V_2M_2U_2^T \tag{20b}$$

then

Proposition 3, Gawronski (1993). The transformation T_2

$$T_2=P_{2e}U_2M_2^{-1/2} = P_{2c}^{-1}V_2M_2^{1/2} \tag{21}$$

balances the H_2 system.

Next, the relationship between H_∞ and H_2 characteristic values is derived.

Lemma 2. Let $\beta=inf\{\rho: M_\infty(\rho)\geq 0\}$. Then on the segment $(\beta,+\infty)$ all H_∞ characteristic values, $\mu_{\infty i}$ $i=1,\ldots,n,$ are smooth nonincreasing functions of ρ, and the maximal characteristic value $\mu_{\infty 1}$ is a nonincreasing convex function of ρ.

Proof. It is a straightforward corollary of the Theorem 3.1 of Li and Chang (1993).

Proposition 4. For $\rho\to\infty$, one obtains $M_\infty\to M_2$.

Proposition 5.

$$M_2 \leq M_\infty, \quad \text{or} \quad \mu_{2i} \leq \mu_{\infty i}, \qquad i=1,\ldots,N \tag{22}$$

Proof. $\mu_{\infty i}$ are increasing functions of ρ, and $\mu_{\infty i} \to \mu_{2i}$ as $\rho \to \infty$, thus $\mu_{2i} \leq \mu_{\infty i}$.□

The connection between the H_2 characteristic values and Hankel singular values is presented in the following proposition.

Proposition 6. For the H_2 characteristic values the following holds:

$$M_2 \leq \Gamma_1^2, \quad \text{or} \quad \mu_{2i} \leq \gamma_{1i}^2, \qquad i=1,\ldots,N \tag{23}$$

Proof. Eq. *(23)* follows as a special case of Proposition *2* for $p \to \infty$. □

3. BALANCED CONTROLLERS FOR FLEXIBLE STRUCTURES

Flexible structure. In this paper a flexible structure is defined as a nondefective, controllable, and observable linear system with distinct complex conjugate pairs of poles (N poles, N is even), and with small and negative real parts of the poles. Nondefective systems can have multiple poles, but the related eigenvectors are independent. This definition is a narrow interpretation of a more general flexible structure concept, which includes heavily damped modes, defective matrix A, and an unobservable, or uncontrollable system. In this paper flexible structures are considered in the narrower sense only. In the Moore balanced coordinates they consist of n =N/2 components, see Gawronski and Juang (1990), Gawronski and Williams (1991), and each component consists of two states.

Approximate equality. In the following sections an approximate equality between two variables is used in the following sense. Two variables x and y are approximately equal ($x \cong y$) if $x=y+\varepsilon$, and $\|\varepsilon\|/\|y\| \ll 1$. For example, if $S_{\infty c}$ and $S_{\infty e}$ are diagonally dominant, M is a diagonal matrix, and if $S_{\infty c} \cong S_{\infty e} \cong M$, then the system is approximately H_∞ balanced (their diagonal terms $s_{\infty ci}$, $s_{\infty ei}$, satisfy $s_{\infty ci}+\varepsilon_{ci}=\mu_i$, $s_{\infty ei}+\varepsilon_{ei}=\mu_i$, with ε_{ci} and ε_{ei} small ($|\varepsilon_{ci}/s_{\infty ci}| \ll 1$, $|\varepsilon_{ei}/s_{\infty ei}| \ll 1$).

Modal representation. Let Φ be the modal matrix of a flexible structure, $\Phi=[\phi_1,\phi_2, \ldots,\phi_n]$, where ϕ_i is the i-th flexible mode. In the modal state-space representation (A_m,B_m, C_m), matrix A_m is block-diagonal, with 2x2 blocks on the main diagonal

$$A_m=diag(A_i), \qquad A_i=\begin{bmatrix} -\zeta_i\omega_i & -\omega_i \\ \omega_i & -\zeta_i\omega_i \end{bmatrix}, \quad i=1,\ldots, n \tag{24}$$

where ω_i is the i-th natural frequency of the structure, and ζ_i is the i-th modal damping. The matrices B_m, C_m are not unique - they depend on normalization of the modal matrix Φ. The modal and balanced coordinates are almost identical, and they required re-scaling only, cf. Jonckheere (1984), and Gregory (1984). In fact, the transformation R_m from the modal to the balanced representation

$$(A_b,B_b,C_b) \cong (A_m,R_m^{-1}B_m,C_mR_m) \tag{25a}$$

is diagonally dominant, and its diagonal entries depend on the scaling of the modal matrix Φ

$$\varepsilon_{mi}=r_{mi}/\|r_{mi}\| \cong e_i, \quad i=1,\ldots,n \tag{25b}$$

where r_{mi} is the i-th colon of R_m, and e_i is the unit vector (all but one zero components, the nonzero component equal to 1).

Open-loop balanced flexible structure. Denote (A,B_k,C_k), and $k=1,2$ as the state-space representations of a flexible structure. Its controllability and observability grammians W_{ck} and W_{ok} are positive-definite and satisfy the

Lyapunov equations, Eq.(1), for $k=1,2$. The system representation is balanced in the sense of Moore if its grammians are diagonal and equal, as in Eq. (2). For a flexible structure with n components (or $N=2n$ states), the balanced grammian has the following form, see Jonckheere (1984), and Gregory (1984)

$$\Gamma_k \cong diag(\gamma_{ki} I_2), \quad k=1,2, \quad i=1,\ldots,n \tag{26}$$

where I_2 is the unit matrix of order two. Matrix A is almost block-diagonal, with dominant 2x2 blocks on the main diagonal

$$A \cong diag(A_i), \quad i=1,\ldots,n \tag{27}$$

where A_i is given in Eq. (24). Introducing Eqs. (26) and (27) to Eq. (1) gives

$$B_{ki} B_{ki}^T \cong C_{ki}^T C_{ki} \cong -\gamma_{ki}^2 (A_i + A_i^T) = 2\zeta_i \omega_i \gamma_{ki}^2 I_2. \tag{28}$$

For flexible structures the orientation of the Moore balanced coordinates is almost independent of matrices B and C, and the matrix A is almost invariant in balanced coordinates, as in Eq. (27). This can be stated as follows. Let (A_{b1}, B_{b1}, C_{b1}) be the Moore balanced representation of a flexible structure (A, B_1, Cl), let (A_{b2}, B_{b2}, C_{b2}) be the Moore balanced representation of a flexible structure (A, B_2, C_2), and let R be the transformation from the first to the second representation, Denote r_i the i-th column of R, and then let

$$\varepsilon_{bi} = r_i / \| r_i \| \cong e_i, \quad i=1,\ldots \ n \tag{29}$$

which is a direct consequence of the closeness of the balanced and modal representation, shown by Jonckheere (1984), and Gregory (1984).

The results show that in the Moore balanced representation the matrix A and the orientation of the balanced coordinates are almost invariant under input and output locations. In fact, the transformation from the first to the second balanced representation

$$(A_b, B_{b2}, C_{b2}) \cong (A_b, R^{-1}B_{b1}, C_{b1}R) \tag{30a}$$

requires only a re-scaling of the coordinates, and the transformation matrix is diagonally dominant

$$R \cong diag(r_1 I_2,\ r_2 I_2, \ldots\ r_n I_2), \quad r_i = \gamma_{1i}^2/\gamma_{2i}^2 \tag{30b}$$

where γ_{1i}, γ_{2i} are the *i-th* Hankel singular values of the first and the second system, respectively.

Balanced H_∞ controller for a flexible structure. The properties of flexible structures, specified above, are now extended to the H_∞ balanced flexible structures. It was shown by Gawronski (1993) that for flexible structures in the Moore balanced representation, the solutions of the two Riccati equations (1) are diagonally dominant.

Let R_k, $k=1,2,$ be the transformation of (A, B_k, C_k) from the Moore balanced representation to the H_∞ balanced representation, and r_{ki} be the *i-th* column of R_k. Then:

Proposition 7. For flexible structures

$$\varepsilon_{hki} = r_{ki}/\| r_{ki} \| \cong e_i \quad k=1,2, \quad i=1,\ldots,n \tag{31}$$

and the solutions of HCARE and HFARE in Moore balanced coordinates are diagonally dominant

$$S_{\infty c} \cong diag(s_{\infty ci} I_2), \quad S_{\infty c} \cong diag(s_{\infty ei} I_2), \quad i=1,2,\ldots,n \tag{32}$$

Proof. The diagonally dominant solutions of the Riccati equations, Eqs.(6a,b), follow from the properties of flexible structures, Eqs.(26)-(28). Thus the transformation matrices $\mathbf{R_k}$ from open- to closed-loop balanced representation are diagonally dominant.□

The proposition shows that the orientation of the H_∞ balanced coordinates is almost identical to the orientation of the Moore balanced coordinates.

Proposition 8. The open- and closed-loop balanced representations are related as follows:

$$(A_h, B_{h1}, B_{h2}, C_{h1}, C_{h2}) \cong (A_b, R_k^{-1}B_{b1}, R_k^{-1}B_{b2}, C_{b1}R_k, C_{b2}R_k) \tag{33a}$$

where either $k=1$ or $k=2$, and the transformation R_k, is as follows:

$$R_k \cong diag(r_{k1}I_2, r_{k2}I_2, \ldots . r_{kn}I_2), \quad r_{ki} = (s_{\infty ci}/s_{\infty ci})^{1/4} \tag{33b}$$

and the HCARE, HFARE solution M_∞ is diagonally dominant in the Moore balanced coordinates

$$M_\infty \cong diag\ (\mu_{\infty i}I_2), \ \mu_{\infty i} = \sqrt{s_{\infty ci}s_{\infty ci}}, \quad i=1,\ldots,n \tag{33c}$$

Proof It is easy to show that the solutions of HCARE and HFARE are $S_{\infty ch} = R_k^T S_{\infty c} R_k$, $S_{\infty eh} = R_k^{-1} S_{\infty e} R_k^{-T}$, and introducing R_k as in Eq.(33b), one obtains a balanced solution as in Eq. (33c).□

For flexible structures Proposition 2 is extended. Denote $\kappa_i = \gamma_{2i}^2 - \gamma_{1i}^2/\rho^2$, "and note that for flexible structures the balanced Riccati equations (6) can be written as follows, using Eqs.(26)-(28) and (32),

$$\kappa_i \mu_{\infty i}^2 + \mu_{\infty i} - \gamma_{1i}^2 \cong 0, \quad i=1,\ldots,n. \tag{34}$$

The solution of the i-th equation

$$\mu_{\infty i} \cong (-1 \pm \sqrt{1+4\gamma_{1i}^2\kappa_i})/2\kappa_i \tag{35}$$

is real and positive for $\kappa_i > -0.25\gamma_{1i}^2$. From Eq.(35), one obtains $(2\kappa_i\mu_{\infty i}+1)^2 \cong 1+4\gamma_{1i}^2\kappa_i$, or after simplifications $\kappa_i\mu_{\infty i}^2 + \mu_{\infty i} \cong \gamma_{1i}^2$. Thus

$$\mu_{\infty i} \le \gamma_{1i}^2 \qquad \text{for } \kappa_i \ge 0, \tag{36a}$$

$$\mu_{\infty i} > \gamma_{1i}^2 \qquad \text{for } 0 > \kappa_i > -0.25\gamma_{1i}^2 \tag{36b}$$

The above results can be specified for the H_2 systems by setting $\rho^{-1} = 0$. Thus for H_2 controller $\kappa_i \cong \gamma_{2i}^2$, and from Eq. (35), it follows that

$$\mu_{2i} \cong \left(-1 + \sqrt{1 + 4\gamma_{1i}^2\gamma_{2i}^2}\right)/2\gamma_{2i}^2 \tag{37}$$

is the unique positive solution of the balanced H_2 Riccati equations. Thus μ_{2i} is the i-th characteristic value of an H_2 system, a result obtained by Gawronski (1993). Also, from Eqs.(36) one obtains

$$\mu_{2i} \le \mu_{\infty i} \le \gamma_{1i}^2, \quad \text{and} \quad \mu_{21} \le \rho \le \gamma_{11}^2, \qquad \text{for } \kappa_i \ge 0 \tag{38a}$$

$$\mu_{\infty i} > \gamma_{1i}^2, \quad \text{and} \quad \rho > \gamma_{11}^2, \qquad \text{for } 0 > K_i > -0.25\gamma_{1i}^2 \tag{38b}$$

4. REDUCED CONTROLLERS FOR FLEXIBLE STRUCTURES

The order of the central H_∞ controller is equal to the order of the plant, and may be too large for implementation. Order reduction is therefore an important design issue. Although the reduction of a generic H_∞ controller is not a straightforward task, an H_∞ controller for flexible structures inherits special properties useful for the controller reduction purposes.

Reduction index. A reduction index, or an indicator of importance of controller components, is necessary to make reasonable decisions concerning the controller reduction. The following reduction index for an H_∞ controller is introduced:

$$\sigma_{\infty i} = \gamma_{2i}^2 \mu_{\infty i}(1 + \alpha_i^2), \qquad \alpha_i = \gamma_{1i}/\gamma_{2i}\rho. \tag{39}$$

The following properties of α_i are obtained from comparison of κ_i and α_i. Since $\kappa_i=\gamma_{2i}^2(1-\alpha_i^2)$, thus

$$\alpha_i \leq 1 \qquad for \quad \kappa_i \geq 0 \tag{40a}$$

$$1<\alpha_i^2<1+1/4\gamma_{1i}^2\gamma_{2i}^2 \qquad for \quad -1/4\gamma_{1i}^2<\kappa_i<0 \tag{40b}$$

The above choice of reduction index is justified by its following properties.

Reduction index and closed-loop poles. $(A_\infty,B_\infty,C_\infty)$ *is* the state-space representation of the central H_∞ controller, see Glover and Doyle (1988) and Doyle *et al.* (1989), where

$$A_\infty=A+B_2k_c+k_eC_2+\rho^{-2}B_1B_1^TS_{\infty c}, \quad B_\infty=-k_e, \quad C_\infty=k_c \tag{41}$$

and $k_c=-B_2^TS_{\infty c}$, $k_e=-S_oS_{\infty e}C_2^T$, $S_o=(I-\rho^{-2}S_{\infty e}S_{\infty c})^{-1}$. Defining the closed-loop state variable $x_o^T=[x^T\ \varepsilon^T]$, where $\varepsilon=x-\hat{x}$, one obtains the closed-loop balanced state-space equations from Eqs. (4) and (41)

$$\dot{x}_o=A_ox_o+B_ow, \quad z=C_ox_o \tag{42a}$$

where

$$A_o=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}, \quad B_o=\begin{bmatrix}B_1\\ B_1+k_eD_{21}\end{bmatrix}, \quad C_o=\begin{bmatrix}C_1+D_{12}k_c & -D_{12}k_c\end{bmatrix} \tag{42b}$$

$$A_{11}=A+B_2k_c,\ A_{12}=-B_2k_c,\ A_{21}=-\rho^{-2}B_1B_1^TM_\infty,\ A_{22}=A+k_eC_2+\rho^{-2}B_1B_1^TM_\infty \tag{42c}$$

Proposition 9. Suppose that

$$\sigma_{\infty i}\ll 1 \text{ for } i=k+1,\ldots,n, \tag{43}$$

then

$$A_{22i} \cong A_i - 2\sigma_{\infty i} I_2, \tag{44}$$

i.e., the *i*-th pole is shifted by $2\sigma_{\infty i}$ with respect to the open-loop location,

Proof. For flexible structures A is diagonally dominant, and the following components are diagonally dominant

$$B_2 k_c = B_2 B_2^T M_\infty \cong diag(2\zeta_i \omega_i \ \gamma_{2i}^2 \mu_{\infty i}) = diag(2\zeta_i \omega_i \mu_{2i}) \tag{45a}$$

$$k_e C_2 = M_\infty C_2^T C_2 \cong diag(2\zeta_i \omega_i \ \gamma_{2i}^2 \mu_{\infty i}) = diag(2\zeta_i \omega_i \mu_{2i}) \tag{45b}$$

$$\rho^{-2} B_1 B_1^T M_\infty \cong diag(2\zeta_i \omega_i \ \gamma_{1i}^2 \mu_{\infty i}/\rho^2) = diag(2\zeta_i \omega_i \mu_{2i} \alpha_i^2), \tag{45c}$$

thus each of four blocks of A_o is diagonally dominant. If $\sigma_{\infty i} \ll 1$ for $i=k+1,\dots,n$, then the *i*-th diagonal components of A_{12} and A_{21} in the closed-loop matrix A_o (see Eq.(42)) are small for $i=k+1,\dots,n$. Thus for those components the separation principle is valid: gains k_{ci}, k_{ei} are independent. Furthermore, the i-th diagonal block A_{22i} of the matrix A_{22} is as follows

$$A_{22i} \cong A_i - s_{oi} \mu_{\infty i} C_{2i}^T C_{2i} - \rho^{-2} B_{1i} B_{1i}^T \mu_{\infty i} \tag{46}$$

where A_i is given by Eq.(24). For $\sigma_{\infty i} \ll 1$ notice that $s_{oi} \cong 1$, that $\mu_{\infty i} C_{2i}^T C_{2i} \cong 2\zeta_i \omega_i \gamma_{21}^2 \mu_{\infty i} I_2$, and that $\rho^{-2} B_{1i} B_{1i}^T \mu_{\infty i} \cong 2\zeta_i \omega_i \rho^{-2} \gamma_{1i}^2 \mu_{\infty i} I_2 = 2\zeta_i \omega_i \gamma_{21}^2 \mu_{\infty i} \alpha_i^2 I_2$. In consequence, Eq.(46) now becomes Eq.(44). □

The index $\sigma_{\infty i}$ serves as an indicator of importance of the *i*-th balanced component of the H_∞ controller. If $\sigma_{\infty i}$ is small, the i-th component is considered negligible and can be truncated.

Reduction index and the controller performance. Let the vector ε be partitioned as $\varepsilon^T = [\varepsilon_r^T, \ \varepsilon_t^T]$, with ε_r of dimension n_r, c, of dimension n_t, and $n_r + n_t = n$. Let the matrix of the reduction indices be arranged in decreasing order, $\Sigma_\infty = diag(\sigma_{\infty 1}, \dots, \sigma_{\infty n})$, $\sigma_{\infty i} \geq \sigma_{\infty i+1}$, and be divided consistent y with c,

$$\Sigma_\infty = diag(\Sigma_{\infty r},\ \Sigma_{\infty t}), \tag{47}$$

where $\Sigma_{\infty r} = diag(\sigma_{\infty 1}, \ldots, \sigma_{\infty k})$, $\Sigma_{\infty t} = diag(\sigma_{\infty k+1}, \ldots, \sigma_{\infty n})$. Divide the matrix M_∞ according y, $M_\infty = diag(M_{\infty r}, M_{\infty t})$. The closed-loop system representation (A_o, B_o, C_o) is rearranged such that the closed-loop matrices are divided according to the division of c. Hence the closed-loop state is now $x_o^T = [x_r^T\ \varepsilon_t^T]$ and $x_r = [x\ \varepsilon_r]$

$$A_o = \begin{bmatrix} A_{or} & A_{ort} \\ A_{otr} & A_{ot} \end{bmatrix}, \quad B_o = \begin{bmatrix} B_{or} \\ B_{ot} \end{bmatrix}, \quad C_o = \begin{bmatrix} C_{or} & C_{ot} \end{bmatrix} \tag{48}$$

The reduced-order controller representation is (A_{or}, B_{or}, C_{or}), and let the closed-loop system state be denoted by $\bar{x}_r$.

Proposition 10. For the condition of Eq. (43) satisfied the performance of the closed-loop system with the reduced-order controller is almost identical to the full-order controller in the sense that $\|x_r - \bar{x}_r\| \cong 0$.

Proof. It follows from Eq.(45) that for $\sigma_{\infty i} \ll 1$ $(i=k+1,\ldots,n)$, *one obtains* $\|A_{otr}\| \cong \|A_{ort}\| \cong 0$, and the closed-loop block A_{ot} is almost identical to the open-loop block A_t, i.e., $A_{ot} \cong A_t$. In this case, from Eqs. (42) and (48), one obtains

$$\dot{x}_r = A_{or} x_r + A_{ort} \varepsilon_t + B_{or} w \cong A_{or} x_r + B_{or} w = \dot{\bar{x}}_r \tag{49}$$

thus $x_r \cong \bar{x}_r$. □

It is easy to see that for an H_2 system, when $\rho^{-1} = 0$, one gets $\sigma_{\infty i} = \sigma_{2i}$, with

$$\sigma_{2i} = \mu_{2i} \gamma_{2i}^2 \tag{50}$$

as introduced by Gawronski (1993).

5. EXAMPLE

The application of the H_∞ controller to the truss structure shown in Fig. 1 is investigated. For this structure l_1=70 in, l_2=100 in, each truss has a cross-section area of 2 in^2, elastic modulus of 1@ lb/in^2, and mass density of 2 lb sec^2/in^2. The structural model has N=26 states, or $n=13$ components. All inputs and outputs are directed vertically. The disturbance w acts at node $n]$. The output z at node $n2$ is minimized. The controlled inputs u and outputs y are collocated at node $n3$, and the components of Cl and B_1 are 300 at node $n3$; other components are zero.

The system H_∞ characteristic values (solid line), H_2 characteristic values (dashed line), and Hankel singular values (dot-dashed line) are compared in Fig.2, showing that the relationship of Eq. (38) holds. The critical value is ρ=469.

In Fig.3 the H_∞ characteristic values obtained from Eq. (6) are compared with its approximate values from Eq. (35), showing that the approximate values are close to the exact ones. The H_∞ and H_2 reduction indices are shown in Fig.4; this figure shows that they coincide for $\sigma_{\infty i} \ll 1$.

Open- and closed-loop impulse responses are compared in Fig.5. The H_∞ reduction index satisfies the condition in Eq. (43) for $k=8,\ldots,13$, i.e., $\sigma_{\infty k} < 0.$ old. Hence the controller can be reduced to 14 states. Indeed, the controller of order 14 (7 components) is stable, and its performance is almost identical to the full-order controller (closed-loop impulse responses of the full- and reduced-order controllers overlap), while the closed-loop systems with controllers of order 13 (6 components) or less are unstable.

The closeness of the open-loop and closed-loop balanced representations, as well as the modal representation, is estimated with the vectors ε_{h1i}, ε_{h2i}, and ε_{mi}, $i=1,\ldots,13$, as defined in Eqs.(33) and (29). For the truss under consideration, the largest values of these vectors were typically in the range 0.98-0.99, with some in the range 0.9-0.98. The remaining values of the vectors were typically in the range 0-0.01, with some in the range 0.01-0.1, thus they satisfied the conditions in Eqs.(31) and (29). The closeness of the open- and closed-loop balanced representations, as well as modal representation, was also tested with simulations of the H_∞ controller performance in the H_∞ balanced coordinates, in the Moore balanced coordinates, and in the modal coordinates. The results obtained were very close to each other for each set of coordinates, either for the full-order, or the reduced-order controller.

6. CONCLUSIONS

The balanced solution of the H_∞ Riccati equations was found, and its properties derived. Its relationship to H_2 balanced controllers and to the open-loop balanced representation was determined.

Several properties of the H_∞ balanced controllers were derived for flexible structures. The H_∞ characteristic values, their upper and lower bounds, and pole placement were derived from the generic properties of flexible structures. The controller reduction index is introduced as tool for designing a reduced-order H_∞ controller of comparable performance to the full-order controller. It is shown that balanced H_2 controllers are special cases of balanced H_∞ controllers. An example illustrated the properties and the design process of an H_m controller for a flexible structure.

Some of the results are approximate, but the approximation error in most cases is small or negligible. Hence, if implemented correctly, the proposed method can be used as a design tool for reduced-order H_∞ controllers.

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Fig. 1. Truss structure.

Fig.2. H_∞, H_2, and Hankel singular values of the truss.

Fig.3. Exact and approximate H_∞ singular values.

Fig.4. H_∞, H_2 reduction indices.

Fig.5. Open- and closed-loop system responses.

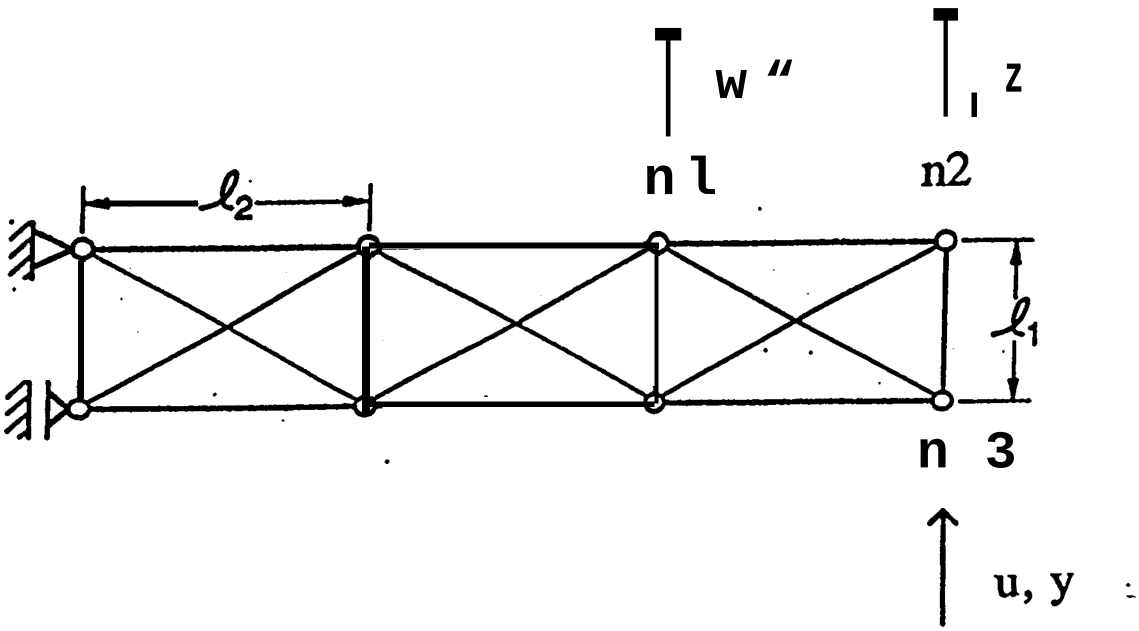
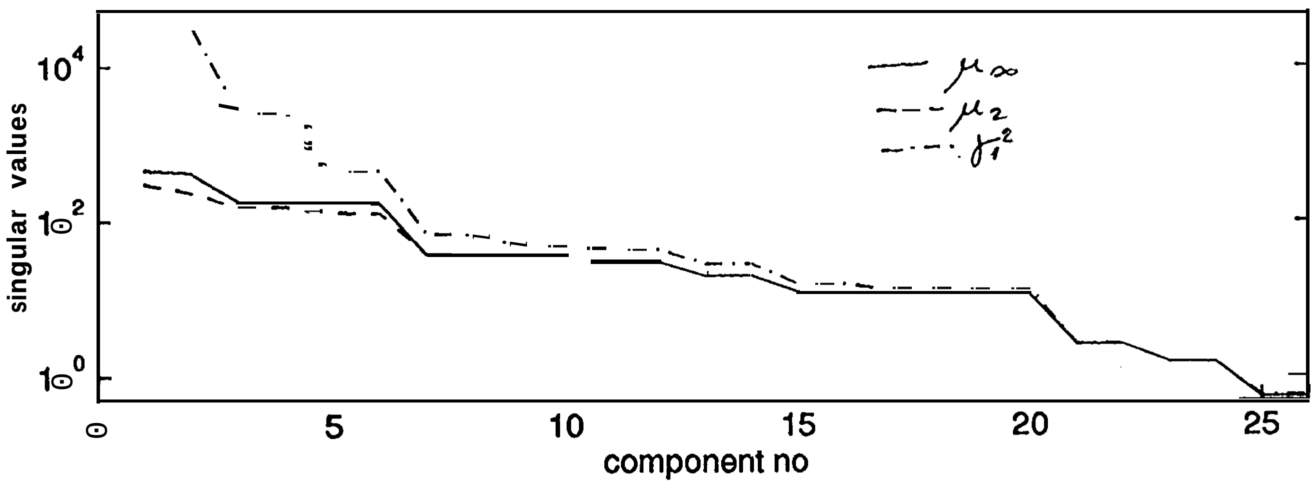
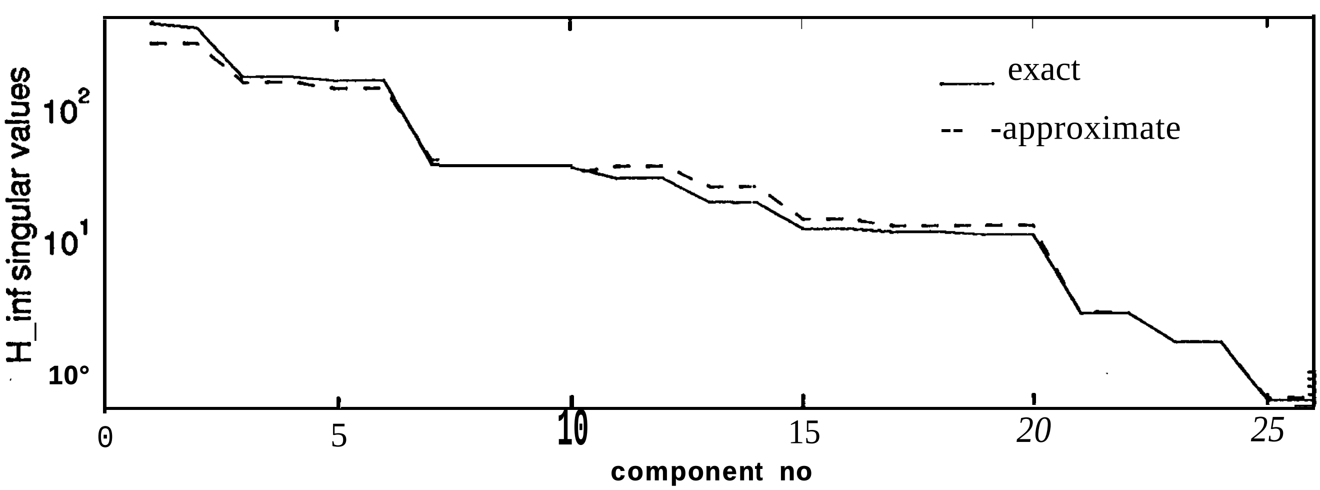


Fig.1



singular values
10^4
10^2
10^0
0
5
10
15
20
25
component no
μ_∞
μ_2
γ_1^2

Fig. 2



exact
-approximate
H_inf singular values
10^2
10^1
10°
0
5
10
15
20
25
component no

Fig. 3

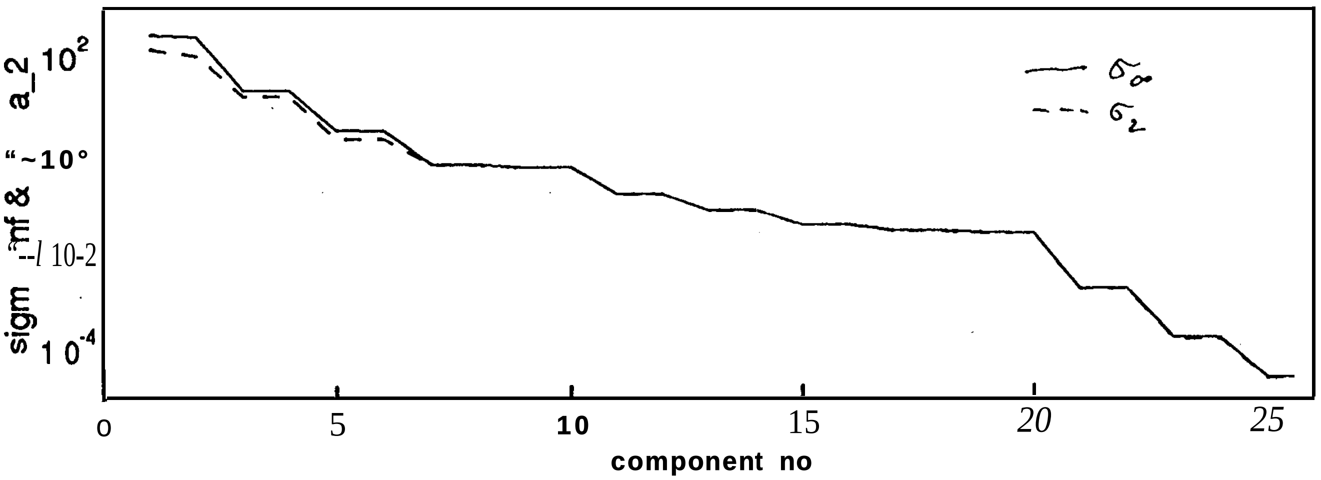
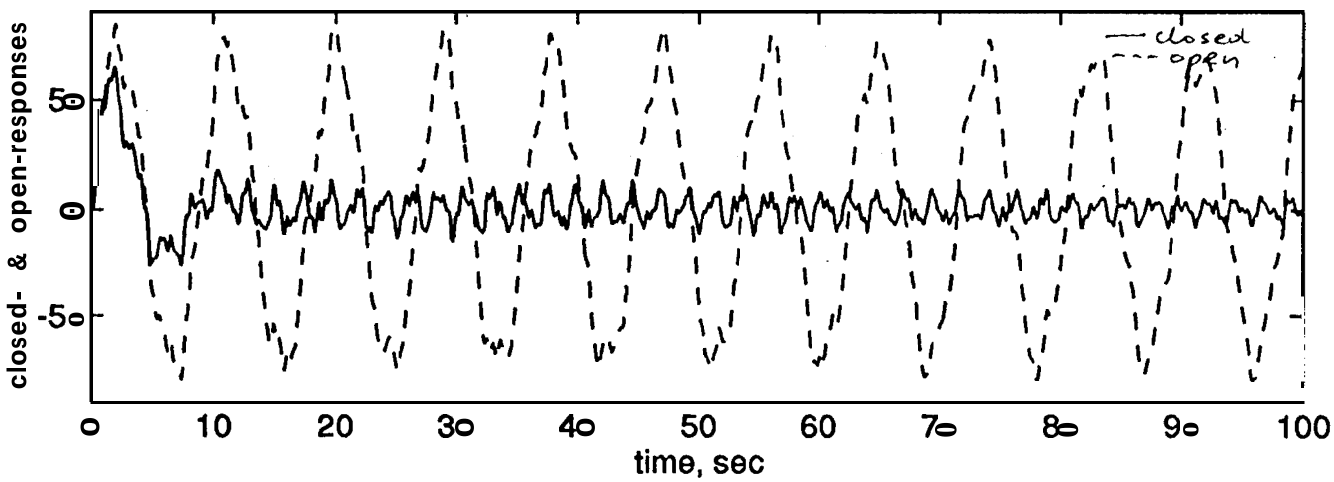


Fig 4



closed- & open-responses
50
0
-50
0
10
20
30
40
50
60
70
80
90
100
time, sec
closed
open

Fig 5