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Part I - Stability Analysis

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Part I - Stability Analysis

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SUMMARY

A compressible linear stability theory is presented for nonparallel three-dimensional boundary-layer flows, taking into account the normal velocity component as well as the streamwise and spanwise variations of the basic flow. The method of multiple scales is used to account for the nonparallelism of the basic flow, and equations are derived for the spatial evolution of the disturbance amplitude and wavenumber. The numerical procedure for obtaining the solution of the nonparallel problem is outlined.

I. INTRODUCTION

For laminar-flow vehicles, a high performance is achieved with respect to range and economy of vehcile operation by reducing the friction drag. The design of such vehicles is strongly influenced by the stability considerations of the boundary layer. To maintain laminar flow with minimum external power, an optimum amount of suction, or cooling (in air), or heating (in water) is required. For the design of swept LFC wings of transonic aircrafts, this optimization process needs accurate computations of the stability characteristics of three-dimensional, compressible boundary-layer flows.

An extensive treatment of the stability theory for compressible flows is given by Mack (1969) for two-dimensional mean flows, where the disturbance can be two or three-dimensional. These stability theories treat the mean flows as quasi-parallel. Some incomplete attempts to account for the nonparallel flow effects by including either the normal or some of the streamwise derivatives of the mean flow were given by Brown (1967), Gunness (1968), and Boehman (1971).

For two-dimensional heated boundary-layer flows, El-Hady (1978) and El-Hady and Nayfeh (1978) introduced a complete nonparallel stability theory to account for the rate of heat transfer between the fluid and the wall. The nonparallel stability results are in better agreement with the heated water experimental data of Strazisar et al (1977) and Strazisar and Reshotko (1978), than the parallel results of Lowell (1974).

Recently, El-Hady and Nayfeh (1979) analyzed the effect of the nonparallelism of the mean flow on the stability characteristics for two-dimensional subsonic and supersonic flows. Results calculated by the nonparallel stability theory are in better agreement with the supersonic experimental data of Laufer and Vrebalovich (1960) and Kendall (1975) than the results calculated by the parallel theory of Mack (1969).

The propagation of three-dimensional disturbances in three-dimensional compressible boundary layers was numerically investigated by Mack (1979) and Lekoudis (1979). Their analysis was for parallel flows over an infinite sweptback wing. Their results show that the effects of compressibility are negligible near the leading and trailing edges (regions of cross-flow type instability). However, away from the leading and trailing edges (regions of T-S type instability), the maximum amplification rate is reduced and the most unstable-wave orientation is considerably changed due to compressibility effects.

In this article, a compressible linear stability theory is presented for three-dimensional disturbance in a nonparallel three-dimensional boundary-layer flows. Section II contains the formulation of the problem. Section III contains the method of solution for the zeroth and first-order problems. The computational procedures are outlined in Section IV.

II. FORMULATION OF THE STABILITY THEORY

We consider the spatial, three-dimensional stability of laminar compressible three-dimensional steady viscous flows to small-amplitude disturbances.

The flow field is described by the Navier-Stokes, energy, and state equations. Lengths, velocities, and time are made dimensionless using a suitable reference length L^*, the freestream velocity U_∞^* and L^*/U_∞^*, respectively. The pressure is made dimensionless using $\rho_\infty^* U_\infty^{*2}$. The temperature, density, specific heats, viscosity, and thermal conductivity are made dimensionless using their corresponding freestream values.

2.1 Disturbance Equations

To study the linear stability of a steady three-dimensional, boundary-layer flow (basic flow), we superpose a small time dependent disturbance on each mean-flow, thermodynamic, and transport quantity. Thus, we let

$$\hat{q}(x,y,z,t) = Q_s(x,y,z) + q(x,y,z,t) \tag{1}$$

where $Q_s(x,y,z)$ is a three-dimensional basic-state quantity and $q(x,y,z,t)$ is a three-dimensional unsteady disturbance quantity. Here, $\hat{q}$ stands for the velocity components (u, v, and w), temperature T, pressure p, density ρ, and viscosity μ. Substituting (1) into equations governing the flow field, subtracting the basic-state quantities, and linearizing the resulting equations in the q's, we obtain the following disturbance equations:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho_s u + \rho U_s) + \frac{\partial}{\partial y}(\rho_s v + \rho V_s) + \frac{\partial}{\partial z}(\rho_s w + \rho W_s) = 0 \tag{2}$$

$$\rho_s\left(\frac{\partial u}{\partial t} + U_s \frac{\partial u}{\partial x} + u \frac{\partial U_s}{\partial x} + V_s \frac{\partial u}{\partial y} + v \frac{\partial U_s}{\partial y} + W_s \frac{\partial u}{\partial z} + w \frac{\partial U_s}{\partial z}\right)$$

$$+ \rho\left(U_s \frac{\partial U_s}{\partial x} + V_s \frac{\partial U_s}{\partial y} + W_s \frac{\partial U_s}{\partial z}\right) = -\frac{\partial p}{\partial x} + \frac{1}{R}\left\{\frac{\partial}{\partial x}\left[\mu_s\left(r \frac{\partial u}{\partial x} + m \frac{\partial v}{\partial y} + m \frac{\partial w}{\partial z}\right)\right.\right.$$

$$+ \mu\left(r \frac{\partial U_s}{\partial x} + m \frac{\partial V_s}{\partial y} + m \frac{\partial W_s}{\partial z}\right)\Big] + \frac{\partial}{\partial y}\left[\mu_s\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + \mu\left(\frac{\partial U_s}{\partial y} + \frac{\partial V_s}{\partial x}\right)\right]$$

$$\left.\left. + \frac{\partial}{\partial z}\left[\mu_s\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} + \mu\left(\frac{\partial W_s}{\partial x} + \frac{\partial U_s}{\partial z}\right)\right]\right\}\right. \tag{3}$$

$$\rho_s\left(\frac{\partial v}{\partial t} + U_s \frac{\partial v}{\partial x} + u \frac{\partial V_s}{\partial x} + V_s \frac{\partial v}{\partial y} + v \frac{\partial V_s}{\partial y} + W_s \frac{\partial v}{\partial z} + w \frac{\partial V_s}{\partial z}\right)$$

$$+ \rho\left(U_s \frac{\partial V_s}{\partial x} + V_s \frac{\partial V_s}{\partial y} + W_s \frac{\partial V_s}{\partial z}\right) = -\frac{\partial p}{\partial y} + \frac{1}{R}\left\{\frac{\partial}{\partial x}\left[\mu_s\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right.\right.$$

$$+ \mu\left(\frac{\partial U_s}{\partial y} + \frac{\partial V_s}{\partial x}\right)\Big] + \frac{\partial}{\partial y}\left[\mu_s\left(m \frac{\partial u}{\partial x} + r \frac{\partial v}{\partial y} + m \frac{\partial w}{\partial z}\right) + \mu\left(r \frac{\partial V_s}{\partial y} + m \frac{\partial U_s}{\partial x}\right.\right.$$

$$\left.\left. + m \frac{\partial W_s}{\partial z} + \frac{\partial}{\partial z}\left[\mu_s\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) + \mu\left(\frac{\partial V_s}{\partial z} + \frac{\partial W_s}{\partial y}\right)\right]\right\}\right. \tag{4}$$

$$\rho_s\left(\frac{\partial w}{\partial t} + U_s \frac{\partial w}{\partial x} + u \frac{\partial W_s}{\partial x} + V_s \frac{\partial w}{\partial y} + v \frac{\partial W_s}{\partial y} + W_s \frac{\partial w}{\partial z} + w \frac{\partial W_s}{\partial z}\right)$$

$$= -\frac{\partial p}{\partial z} + \frac{1}{R}\left\{\frac{\partial}{\partial x}\left[\mu_s\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + \mu\left(\frac{\partial W_s}{\partial x} + \frac{\partial U_s}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu_s\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right.\right.$$

$$+ \mu\left(\frac{\partial V_s}{\partial z} + \frac{\partial W_s}{\partial y}\right)\Big] + \frac{\partial}{\partial z}\left[\mu_s\left(m \frac{\partial u}{\partial x} + m \frac{\partial v}{\partial y} + r \frac{\partial w}{\partial z}\right) + \mu\left(m \frac{\partial U_s}{\partial x} + m \frac{\partial V_s}{\partial y}\right.\right.$$

$$\left.\left. + r \frac{\partial W_s}{\partial z}\right)\Big]\right\} \tag{5}$$

$$\rho_s\left[\frac{\partial T}{\partial t} + u\frac{\partial T_s}{\partial x} + U_s\frac{\partial T}{\partial x} + v\frac{\partial T_s}{\partial y} + V_s\frac{\partial T}{\partial y} + w\frac{\partial T_s}{\partial z} + W_s\frac{\partial T}{\partial z}\right] + \rho\left[U_s\frac{\partial T_s}{\partial x} + V_s\frac{\partial T_s}{\partial y}\right.$$

$$\left. + W_s\frac{\partial T_s}{\partial z}\right] = Ec\left[\frac{\partial p}{\partial t} + u\frac{\partial P_s}{\partial x} + U_s\frac{\partial p}{\partial x} + V_s\frac{\partial p}{\partial y} + w\frac{\partial P_s}{\partial z} + W_s\frac{\partial p}{\partial z}\right.$$

$$\left. + \frac{1}{R}\hat{\phi}\right] + \frac{1}{RPr}\left[\frac{\partial}{\partial x}\left(\mu_s\frac{\partial T}{\partial x} + \mu\frac{\partial T_s}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu_s\frac{\partial T}{\partial y} + \mu\frac{\partial T_s}{\partial y}\right)\right.$$

$$\left. + \frac{\partial}{\partial z}\left(\mu_s\frac{\partial T}{\partial z} + \mu\frac{\partial T_s}{\partial z}\right)\right] \tag{6}$$

and the state equation. Here $R = \rho_e U_e L/\mu_e$ is the Reynolds number, $Pr = C_{P_e}\mu_e/\kappa_e$ is the Prandtl number, $Ec = U_e^2/C_{P_e}T_e$ is the Eckert number, and $\hat{\phi}$, the perturbation dissipation function, is defined as

$$\hat{\phi} = \mu_s\left\{2r\left(\frac{\partial U_s}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial V_s}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial W_s}{\partial z}\frac{\partial w}{\partial z}\right) + 2m\left[\frac{\partial U_s}{\partial x}\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right.\right.$$

$$\left. + \frac{\partial V_s}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + \frac{\partial W_s}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\right] + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\left(\frac{\partial U_s}{\partial y} + \frac{\partial V_s}{\partial x}\right)$$

$$\left. + 2\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\left(\frac{\partial V_s}{\partial z} + \frac{\partial W_s}{\partial z}\right) + 2\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\left(\frac{\partial U_s}{\partial z} + \frac{\partial W_s}{\partial x}\right)\right\}$$

$$+ \mu\left\{r\left[\left(\frac{\partial U_s}{\partial x}\right)^2 + \left(\frac{\partial V_s}{\partial y}\right)^2 + \left(\frac{\partial W_s}{\partial z}\right)^2\right] + 2m\left(\frac{\partial U_s}{\partial x}\frac{\partial V_s}{\partial y} + \frac{\partial U_s}{\partial x}\frac{\partial W_s}{\partial z} + \frac{\partial V_s}{\partial y}\frac{\partial W_s}{\partial z}\right)\right.$$

$$\left. + \left(\frac{\partial U_s}{\partial y} + \frac{\partial V_s}{\partial x}\right)^2 + \left(\frac{\partial V_s}{\partial z} + \frac{\partial W_s}{\partial y}\right)^2 + \left(\frac{\partial V_s}{\partial z} + \frac{\partial W_s}{\partial x}\right)^2\right\} \tag{7}$$

Moreover, r and m are given by

$$r = \frac{2}{3}(\ell + 2),\quad m = \frac{2}{3}(\ell - 1) \tag{8}$$

where ℓ is the ratio of the second to the first viscosity coefficients.

Equations (2)-(6) represent the stability equations for a three-dimensional disturbance in a three-dimensional basic flow.

2.2 Boundary Conditions

The analysis presented here is applicable to cases with suction or blowing as well as cases with uniform or non-uniform wall heating or cooling. The stability problem is completed by the specification of the appropriate initial and boundary conditions. We consider next the boundary conditions.

At the wall, we require the vanishing of the component of the relative velocity of the fluid/solid surface that is parallel to the interface, even in the region of perforations. This is a reasonable assumption provided that the percentage of the permeable area is small and most of the flow there is directed normal to the wall. The normal velocity and thermal boundary conditions at the wall need careful consideration.

For an impermeable wall, both the mean and disturbance velocities normal to the surface must vanish, whereas for a permeable wall this is not the case. The mean normal velocity component is not zero. For different disturbance normal velocities at the wall, Gaponov (1971, 1975) showed a destabilizing effect of a nonzero value for the normal component of the disturbance velocity at the wall. His results are based on the calculation of the neutral stability curves. Such curves are not directly related to transition. Moreover, the boundary condition used by Gaponov is not of a practical application. Of more interest are the results of Lekoudis (1978), who examined the effect of the normal component of the disturbance velocity on the growth rates of disturbances. This boundary condition is given in the form of an admittance calculated for different configurations. He concluded that the condition of zero normal velocity at the wall is a reasonable approximation when the surface permeability is very small.

The thermal boundary condition for the disturbances needs an analysis of the heat conduction problem in the region very close to the wall (e.g., Dunn and Lin, 1955). This analysis results in a thermal boundary condition for the disturbance in the form

$$\frac{\partial T}{\partial y}(0) + g\,T(0) = 0$$

where g is a constant that depends on the disturbance frequency, and the physical properties of the liquid and the adjacent solid wall. This condition holds for very low frequencies because the thermal fluctuations can penetrate large distances into the solid wall. On the other hand, for very high frequencies, the thermal inertia of the solid makes the thermal fluctuations die out in the solid very close to the surface, and the wall remains at the temperature of the basic flow. In this case, the condition $T(0) = 0$ is a very accurate approximation.

In the freestream, it is assumed that all disturbance quantities die out for subsonic disturbances and satisfy a radiation condition for supersonic disturbances. For boundary layers in subsonic flows, disturbances have amplitudes that decay exponentially in the freestream. For boundary layers in supersonic flows, we restrict our analysis to subsonic disturbances, that is to disturbances that move subsonically with respect to the freestream. The amplitudes of these disturbances decay also exponentially in the freestream.

In this analysis, we consider walls of small permeability and subsonic disturbances of sufficiently high frequencies. With these assumptions, the disturbance boundary conditions become

$$u = v = w = T = 0 \text{ at } y = 0 \tag{9}$$

$$u,v,w,T \to 0 \quad \text{as } y \to \infty \tag{10}$$

2.3 Nonparallel-Flow Considerations

We consider weakly nonparallel flows. Thus, to account for the nonparallelism of the mean flow, we require that all mean-flow variables be weak functions of the streamwise and spanwise positions. Moreover, we require that the normal velocity component be small compared with the other velocity components. These assumptions are expressed mathematically by writing the mean-flow variables in the form

$$U_s = U_s(x_1,y,z_1),\ V_s = \epsilon V_s(x_1,y,z_1),\ W_s = W_s(x_1,y,z_1),$$

$$p_s = p_s(x_1,z_1),\ T_s = T_s(x_1,y,z_1) \tag{11}$$

where

$$x_1 = \epsilon x,\ z_1 = \epsilon z \tag{12}$$

and ϵ is a small dimensionless parameter characterizing the nonparallelism of the mean flow. In the parallel-flow approximation, $V_s = 0$ and all variables in Eqs. (11) are independent of x_1 and z_1. The main idea behind the nonparallel-flow analysis is to make a perturbation about the parallel-flow solution (Bouthier, 1973; Gaster, 1974; Nayfeh et al, 1974). The method of multiple scales is used to effect this perturbation. Accordingly, different streamwise scales x, x_1, z, and z_1 are introduced. The fast scales x and z are used to describe the relatively rapid streamwise and spanwise variations of the traveling wave disturbance. The slow scales x_1 and z_1 are used to describe the relatively slow variations of the mean-flow quantities, the disturbance wavenumber, the growth rate, and the amplitude.

III. METHOD OF SOLUTION

To determine an approximate solution to Eqs. (2)-(11), we use the method of multiple scales and seek a first-order expansion for the disturbance variables u, v, w, p, and T in the form of a traveling harmonic wave, that is

$$q(x_1,y,z_1,t;\varepsilon) = \left[q_0(x_1,y,z_1) + \varepsilon q_1(x_1,y,z_1) + \ldots\right]\exp(i\theta) \qquad (13)$$

where the phase function θ is defined by

$$\nabla\theta = \vec{K}_0 = \left[\alpha_0(x_1,z_1),\ \beta_0(x_1,z_1)\right] \qquad (14a)$$

$$\frac{\partial\theta}{\partial t} = -\omega \qquad (14b)$$

Here α_0 and β_0 are the quasi-parallel components of the wavenumber vector $\vec{K}_0$ in the x and z directions and ω is the circular frequency.

For the case of quasi-parallel spatial stability ω is real and α_0 and β_0 are complex, while for the case of quasi-parallel temporal stability, α_0 and β_0 are real and ω is complex. For the general case, α_0, β_0, and ω are complex. The present study is limited to spatial stability so that ω is a known real quantity.

The viscosity disturbance is related to the temperature disturbance by

$$\mu(x_1,y,z_1,t;\varepsilon) = \frac{d\mu_s}{dT_s}\,T(x_1,y,z_1,t;\varepsilon) \qquad (15)$$

while the density disturbance is related to the temperature disturbance by the state equation.

Substituting Eqs. (13)-(15) into Eqs. (2)-(6), transforming the time and spatial derivatives from t, x, and z to θ, x_1, z_1, and equating the coefficients of ε^0 and ε on both sides, we obtain problems describing the q_0 and q_1 disturbance quantities. These problems are referred to as the zeroth and first-order problems. They are given next for the physical problem of air boundary layer treated as perfect gas.

3.1 The Zeroth-Order Problem

Substituting (13)-(15) into (2)-(11) and equating the coefficients of ε^0 on both sides, we obtain

$$L_1(u_0,v_0,w_0,p_0,T_0) = i\rho_s(\alpha_0 u_0 + \beta_0 w_0) + i(\alpha_0 U_s + \beta_0 W_s - \omega)\rho_0$$

$$+ \frac{\partial}{\partial y}(\rho_s v_0) = 0 \qquad (16)$$

$$L_2(u_0, v_0, w_0, p_0, T_0) = \left[i\rho_s(\alpha_0 U_s + \beta_0 W_s - \omega) + \frac{\mu_s}{R}(r\alpha_0^2 + \beta_0^2)\right] u_0$$

$$+ \left(\rho_s \frac{\partial U_s}{\partial y} - \frac{i}{R}\frac{\partial \mu_s}{\partial y}\alpha_0\right)v_0 + \frac{f}{R}\mu_s\alpha_0\beta_0 w_0 + i\alpha_0 p_0 - \frac{T_0}{R}\frac{\partial}{\partial y}\left(\frac{d\mu_s}{dT_s}\frac{\partial U_s}{\partial y}\right)$$

$$- \frac{1}{R}\frac{\partial \mu_s}{\partial y}\frac{\partial u_0}{\partial y} - \frac{if}{R}\mu_s\alpha_0 \frac{\partial v_0}{\partial y} - \frac{1}{R}\frac{d\mu_s}{dT_s}\frac{\partial U_s}{\partial y}\frac{\partial T_0}{\partial y} - \frac{1}{R}\mu_s\frac{\partial^2 u_0}{\partial y^2} = 0 \qquad (17)$$

$$L_3(u_0, v_0, w_0, p_0, T_0) = \left[i\rho_s(\alpha_0 U_s + \beta_0 W_s - \omega) + \frac{\mu_s}{R}(\alpha_0^2 + \beta_0^2) \right] v_0$$
$$- \frac{im}{R}\alpha_0 \frac{\partial \mu_s}{\partial y} u_0 - \frac{im}{R}\beta_0 \frac{\partial \mu_s}{\partial y} w_0 - \frac{i}{R}\frac{d\mu_s}{dT_s}(\alpha_0 \frac{\partial U_s}{\partial y} + \beta_0 \frac{\partial W_s}{\partial y}) T_0$$
$$- \frac{if}{R}\mu_s\alpha_0 \frac{\partial u_0}{\partial y} - \frac{r}{R}\frac{\partial \mu_s}{\partial y}\frac{\partial v_0}{\partial y} - \frac{if}{R}\mu_s\beta_0 \frac{\partial w_0}{\partial y} + \frac{\partial p_0}{\partial y} - \frac{r}{R}\mu_s \frac{\partial^2 v_0}{\partial y^2} = 0 \quad (18)$$

$$L_4(u_0, v_0, w_0, p_0, T_0) = i\rho_s(\alpha_0 U_s + \beta_0 W_s - \omega) + \frac{\mu_s}{R}(\alpha_0^2 + r\beta_0^2) \Big] w_0$$
$$+ \frac{f}{R}\mu_s\alpha_0\beta_0 u_0 + (\rho_s \frac{\partial W_s}{\partial y} - \frac{i}{R}\beta_0 \frac{\partial \mu_s}{\partial y}) v_0 + i\beta_0 p_0 - \frac{1}{R}\frac{\partial}{\partial y}(\frac{d\mu_s}{dT_s}\frac{\partial W_s}{\partial y}) T_0$$
$$- \frac{if}{R}\mu_s\beta_0 \frac{\partial v_0}{\partial y} - \frac{1}{R}\frac{\partial \mu_s}{\partial y}\frac{\partial w_0}{\partial y} - \frac{1}{R}\frac{d\mu_s}{dT_s}\frac{\partial W_s}{\partial y}\frac{\partial T_0}{\partial y} - \frac{\mu s}{R}\frac{\partial^2 w_0}{\partial y^2} = 0 \quad (19)$$

$$L_5(u_0, v_0, w_0, p_0, T_0) = \{ i\rho_s(\alpha_0 U_s + \beta_0 W_s - \omega) - \frac{(\gamma-1)M_e^2}{R}\frac{d\mu_s}{dT_s}$$
$$\left[(\frac{\partial U_s}{\partial y})^2 + (\frac{\partial W_s}{\partial y})^2 \right] + \frac{1}{RPr}\left[\mu_s(\alpha_0^2 + \beta_0^2) - \frac{\partial^2 \mu_s}{\partial y^2} \right\} T_0$$
$$+ \left[\rho_s \frac{\partial T_s}{\partial t} - \frac{2i(\gamma-1)M_e^2}{R}\mu_s(\alpha_0 \frac{\partial U_s}{\partial y} + \beta_0 \frac{\partial W_s}{\partial y}) \right] v_0 - i(\gamma-1)M_e^2(\alpha_0 U_s$$
$$+ \beta_0 W_s - \omega) p_0 - \frac{2}{RPr}\frac{\partial \mu_s}{\partial y}\frac{\partial T_0}{\partial y} - \frac{2(\gamma-1)M_e^2}{R}\mu_s(\frac{\partial U_s}{\partial y}\frac{\partial u_0}{\partial y} + \frac{\partial W_s}{\partial y}\frac{\partial w_0}{\partial y})$$
$$- \frac{\mu_s}{RPr}\frac{\partial^2 T_0}{\partial y^2} = 0 \quad (20)$$

$$u_0 = v_0 = w_0 = T_0 = 0 \text{ at } y = 0 \quad (21)$$

$$u_0, v_0, w_0, T_0 \to 0 \text{ as } y \to \infty \quad (22)$$

The operators L_1-L_5 correspond to the continuity, x-momentum, y-momentum, z-momentum, and energy equations, respectively.

Equations (16)-(22) constitute an eigenvalue problem that is solved numerically. The numerical solution is discussed in the next section, and the solution of Eqs. (16)-(22) can be expressed in the form

$$u_0 = A(x_1,z_1)\,\zeta_1\,(x_1,y,z_1) \tag{23a}$$

$$v_0 = A(x_1,z_1)\,\zeta_3\,(x_1,y,z_1) \tag{23b}$$

$$p_0 = A(x_1,z_1)\,\zeta_4\,(x_1,y,z_1) \tag{23c}$$

$$T_0 = A(x_1,z_1)\,\zeta_5\,(x_1,y,z_1) \tag{23d}$$

$$w_0 = A(x_1,z_1)\,\zeta_7\,(x_1,y,z_1) \tag{23e}$$

The amplitude function $A(x_1,z_1)$ is determined by imposing the solvability condition at the next level of approximation.

3.2 The First-Order Problem

Substituting (13)-(15) into Eqs. (2)-(11) and equating the coefficients of ϵ on both sides, we obtain

$$L_i(u_1,v_1,w_1,P_1,T_1) = I_i \quad \text{for } i = 1,2,\ldots,5 \tag{24}$$

$$u_1 = v_1 = w_1 = T_1 = 0 \quad \text{at } y = 0 \tag{25}$$

$$u_1,v_1,w_1,T_1, \to 0 \quad \text{as } y \to \infty \tag{26}$$

where the operators L_1-L_5 are defined by Eqs. (16)-(20) and I_1-I_5 are the inhomogeneous terms in the continuity, x-momentum, z-momentum, and energy equations, respectively. These inhomogeneous terms reflect the effects of

the streamwise and spanwise variations of the disturbance amplitude, the normal basic-velocity component and the streamwise and spanwise variations of the wavenumber. Using Eq. (23), these inhomogeneous terms are defined by

$$I_1 = -\left[\frac{1}{T_s}\zeta_1 + \frac{\gamma M_e^2 U_s}{T_s}\zeta_4 - \frac{U_s}{T_s^2}\zeta_5\right]\frac{dA}{dx_1} - \left[\frac{\gamma M_e^2 W_s}{T_s}\zeta_4 - \frac{W_s}{T_s^2}\zeta_5 + \frac{1}{T_s}\zeta_7\right]\frac{dA}{dz_1}$$
$$+\left[\frac{1}{T_s^2}\frac{\partial T_s}{\partial x_1}\zeta_1 - \frac{1}{T_s}\frac{\partial \zeta_1}{\partial x_1} - \frac{\gamma M_e^2}{T_s}\left(\frac{\partial U_s}{\partial x_1} + \frac{\partial V_s}{\partial y} + \frac{\partial W_s}{\partial z_1} - \frac{U_s}{T_s}\frac{\partial T_s}{\partial x_1} - \frac{V_s}{T_s}\frac{\partial T_s}{\partial y}\right.\right.$$
$$\left.\left. - \frac{W_s}{T_s}\frac{\partial T_s}{\partial z_1}\right)\zeta_4 - \frac{\gamma M_e^2 U_s}{T_s}\frac{\partial \zeta_4}{\partial x_1} - \frac{\gamma M_e^2 V_s}{T_s}\frac{\partial \zeta_4}{\partial y} - \frac{\gamma M_e^2 W_s}{T_s}\frac{\partial \zeta_4}{\partial z_1} + \frac{1}{T_s^2}\left(\frac{\partial U_s}{\partial x_1} + \frac{\partial V_s}{\partial y}\right.\right.$$
$$\left.\left. + \frac{\partial W_s}{\partial z_1} - 2\frac{U_s}{T_s}\frac{\partial T_s}{\partial x_1} - 2\frac{V_s}{T_s}\frac{\partial T_s}{\partial y} - 2\frac{W_s}{T_s}\frac{\partial T_s}{\partial z_1}\right)\zeta_5 + \frac{U_s}{T_s^2}\frac{\partial \zeta_5}{\partial x_1} + \frac{W_s}{T_s^2}\frac{\partial \zeta_5}{\partial z_1}$$
$$+\frac{V_s}{T_s^2}\frac{\partial \zeta_5}{\partial y} + \frac{1}{T_s^2}\frac{\partial T_s}{\partial z_1}\zeta_7 - \frac{1}{T_s}\frac{\partial \zeta_7}{\partial x_1} - \frac{1}{T_s}\frac{\partial \zeta_7}{\partial z_1}\Bigg] A \qquad (27)$$

$$I_2 = \left[\left(\frac{2ir}{R}\mu_s\alpha_0 - \frac{U_s}{T_s}\right)\zeta_1 + \frac{1}{R}\frac{\partial \mu_s}{\partial y}\zeta_3 + \frac{f}{R}\mu_s\frac{\partial \zeta_3}{\partial y} - \zeta_4 + \frac{if}{R}\mu_s\beta_0\zeta_7\right]\frac{dA}{dx_1}$$
$$+\left[\left(\frac{2i}{R}\mu_s\beta_0 - \frac{W_s}{T_s}\right)\zeta_1 + \frac{if}{R}\mu_s\alpha_0\zeta_7\right]\frac{dA}{dz_1} + \Bigg\{\left[\frac{ir}{R}\frac{\partial}{\partial x_1}(\alpha_0\mu_s) + \frac{i}{R}\frac{\partial}{\partial z_1}(\beta_0\mu_s)\right.$$
$$\left. - \frac{1}{T_s}\frac{\partial U_s}{\partial x_1}\right]\zeta_1 + \left(\frac{2ir}{R}\mu_s\alpha_0 - \frac{U_s}{T_s}\right)\frac{\partial \zeta_1}{\partial x_1} + \left(\frac{2i}{R}\mu_s\beta_0 - \frac{W_s}{T_s}\right)\frac{\partial \zeta_1}{\partial z_1} - \frac{V_s}{T_s}\frac{\partial \zeta_1}{\partial y}$$
$$+\frac{m}{R}\frac{\partial \mu_s}{\partial x_1}\frac{\partial \zeta_3}{\partial y} + \frac{f}{R}\mu_s\frac{\partial}{\partial x_1}\left(\frac{\partial \zeta_3}{\partial y}\right) + \frac{1}{R}\frac{\partial \zeta_5}{\partial y}\frac{\partial \zeta_3}{\partial x_1} - \frac{\gamma M_e^2}{T_s}\left(U_s\frac{\partial U_s}{\partial x_1} + V_s\frac{\partial U_s}{\partial y}\right.$$
$$\left. + W_s\frac{\partial U_s}{\partial z_1}\right)\zeta_4 - \frac{\partial \zeta_4}{\partial x_1} + \left[\frac{i}{R}\alpha_0\frac{d\mu_s}{dT_s}\left(r\frac{\partial U_s}{\partial x_1} + m\frac{\partial V_s}{\partial y} + m\frac{\partial W_s}{\partial z_1}\right) - \frac{1}{T_s^2}\left(U_s\frac{\partial U_s}{\partial x_1}\right.\right.$$
$$\left.\left. + V_s\frac{\partial U_s}{\partial y} + W_s\frac{\partial U_s}{\partial z_1}\right) + \frac{i}{R}\beta_0\frac{d\mu_s}{dT_s}\left(\frac{\partial W_s}{\partial x_1} + \frac{\partial U_s}{\partial z_1}\right)\right]\zeta_5 + \left[\frac{im}{R}\frac{\partial}{\partial x_1}(\beta_0\mu_s) + \frac{i}{R}\frac{\partial}{\partial z_1}(\alpha_0\mu_s)\right.$$
$$\left. - \frac{1}{T_s}\frac{\partial U_s}{\partial z_1}\right]\zeta_7 + \frac{if}{R}\mu_s\beta_0\frac{\partial \zeta_7}{\partial x_1} + \frac{if}{R}\mu_s\alpha_0\frac{\partial \zeta_7}{\partial z_1}\Bigg\} A \qquad (28)$$

$$I_3 = \left[\frac{m}{R}\frac{\partial \mu_s}{\partial y}\zeta_1 + \frac{f}{R}\mu_s\frac{\partial \zeta_1}{\partial y} + \left(\frac{2i}{R}\mu_s\alpha_0 - \frac{U_s}{T_s}\right)\zeta_3 + \frac{1}{R}\frac{d\mu_s}{dT_s}\frac{\partial U_s}{\partial y}\zeta_5\right]\frac{dA}{dx_1}$$

$$+ \left[\left(\frac{2i}{R}\mu_s\beta_0 - \frac{W_s}{T_s}\right)\zeta_3 + \frac{1}{R}\frac{d\mu_s}{dT_s}\frac{\partial W_s}{\partial y}\zeta_5 + \frac{m}{R}\frac{\partial \mu_s}{\partial y}\zeta_7 + \frac{f}{R}\mu_s\frac{\partial \zeta_7}{\partial y}\right]\frac{dA}{dz_1}$$

$$+\left\{\frac{m}{R}\frac{\partial \mu_s}{\partial y}\frac{\partial \zeta_1}{\partial x_1} + \frac{1}{R}\frac{\partial \mu_s}{\partial x_1}\frac{\partial \zeta_1}{\partial y} + \frac{f}{R}\mu_s\frac{\partial}{\partial x_1}\left(\frac{\partial \zeta_1}{\partial y}\right) + \left[\frac{i}{R}\frac{\partial}{\partial x_1}(\alpha_0\mu_s) + \frac{i}{R}\frac{\partial}{\partial z_1}(\beta_0\mu_s)\right.\right.$$

$$\left. - \frac{1}{T_s}\frac{\partial V_s}{\partial y}\right]\zeta_3 - \frac{V_s}{T_s}\frac{\partial \zeta_3}{\partial y} + \left(\frac{2i}{R}\mu_s\alpha_0 - \frac{U_s}{T_s}\right)\frac{\partial \zeta_3}{\partial x_1} + \left(\frac{2i}{R}\mu_s\beta_0 - \frac{W_s}{T_s}\right)\frac{\partial \zeta_3}{\partial z_1}$$

$$+ \frac{1}{R}\left[f\frac{d\mu_s}{dT}\frac{\partial}{\partial x_1}\left(\frac{\partial U_s}{\partial y}\right) + \frac{\partial}{\partial z_1}\left(\frac{\partial W_s}{\partial y}\right) + \frac{\partial U_s}{\partial y}\frac{\partial}{\partial x_1}\left(\frac{d\mu_s}{dT_s}\right) + \frac{\partial W_s}{\partial y}\frac{\partial}{\partial z_1}\left(\frac{d\mu_s}{dT_s}\right) + r\frac{d\mu_s}{dT_s}\frac{\partial^2 V_s}{\partial y^2}\right.$$

$$+ \left(r\frac{\partial V_s}{\partial y} + m\frac{\partial U_s}{\partial x_1} + m\frac{\partial W_s}{\partial z_1}\right)\frac{\partial}{\partial y}\left(\frac{d\mu_s}{dT_s}\right)\Big]\zeta_5 + \frac{1}{R}\frac{d\mu_s}{dT_s}\left(\frac{\partial U_s}{\partial y}\frac{\partial \zeta_5}{\partial x_1} + \frac{\partial W_s}{\partial y}\frac{\partial \zeta_5}{\partial z_1}\right)$$

$$\left. + \frac{1}{R}\frac{d\mu_s}{dT_s}\left(r\frac{\partial V_s}{\partial y} + m\frac{\partial U_s}{\partial x_1} + m\frac{\partial W_s}{\partial z_1}\right)\frac{\partial \zeta_5}{\partial y} + \frac{m}{R}\frac{\partial \mu_s}{\partial y}\frac{\partial \zeta_7}{\partial z_1} + \frac{f}{R}\mu_s\frac{\partial}{\partial z_1}\left(\frac{\partial \zeta_7}{\partial y}\right)\right\}A \qquad (29)$$

$$I_4 = \left[\frac{if}{R}\mu_s\beta_0\zeta_1 + \left(\frac{2i}{R}\mu_s\alpha_0 - \frac{U_s}{T_s}\right)\zeta_7\right]\frac{dA}{dx_1} + \left[\frac{if}{R}\mu_s\alpha_0\zeta_1 + \frac{1}{R}\frac{\partial \mu_s}{\partial y}\zeta_3\right.$$

$$+ \frac{f}{R}\mu_s\frac{\partial \zeta_3}{\partial y} - \zeta_4 + \left(\frac{2ir}{R}\mu_s\beta_0 - \frac{W_s}{T_s}\right)\zeta_7\Big]\frac{dA}{dz_1} + \left\{\left[\frac{im}{R}\frac{\partial}{\partial z_1}(\alpha_0\mu_s)\right.\right.$$

$$\left. + \frac{i}{R}\frac{\partial}{\partial x_1}(\beta_0\mu_s) - \frac{1}{T_s}\frac{\partial W_s}{\partial x_1}\right]\zeta_1 + \frac{if}{R}\mu_s\left(\beta_0\frac{\partial \zeta_1}{\partial x_1} + \alpha_0\frac{\partial \zeta_1}{\partial z_1}\right) + \frac{1}{R}\frac{\partial \mu_s}{\partial y}\frac{\partial \zeta_3}{\partial z_1}$$

$$+ \frac{m}{R}\frac{\partial \mu_s}{\partial z_1}\frac{\partial \zeta_3}{\partial y} + \frac{f}{R}\mu_s\frac{\partial}{\partial z_1}\left(\frac{\partial \zeta_3}{\partial y}\right) - \frac{\gamma M_e^2}{T_s}\left(U_s\frac{\partial W_s}{\partial x_1} + V_s\frac{\partial W_s}{\partial y} + W_s\frac{\partial W_s}{\partial z_1}\right)\zeta_4$$

$$- \frac{\partial \zeta_4}{\partial z_1} + \left[\frac{1}{T_s^2}\left(U_s\frac{\partial W_s}{\partial x_1} + V_s\frac{\partial W_s}{\partial y} + W_s\frac{\partial W_s}{\partial z_1}\right) + \frac{i\alpha_0}{R}\frac{d\mu_s}{dT_s}\left(\frac{\partial W_s}{\partial x_1} + \frac{\partial U_s}{\partial z_1}\right)\right.$$

$$\left. + \frac{i\beta_0}{R}\frac{d\mu_s}{dT_s}\left(m\frac{\partial U_s}{\partial x_1} + m\frac{\partial V_s}{\partial y} + r\frac{\partial W_s}{\partial z_1}\right)\right]\zeta_5 + \left[\frac{i}{R}\frac{\partial}{\partial x_1}(\alpha_0\mu_s) + \frac{ir}{R}\frac{\partial}{\partial z_1}(\beta_0\mu_s)\right]\zeta_7$$

$$\left. + \left(\frac{2i}{R}\mu_s\alpha_0 - \frac{U_s}{T_s}\right)\frac{\partial \zeta_7}{\partial x_1} + \left(\frac{2ir}{R}\mu_s\beta_0 - \frac{W_s}{T_s}\right)\frac{\partial \zeta_7}{\partial z_1} - \frac{V_s}{T_s}\frac{\partial \zeta_7}{\partial y}\right\}A \qquad (30)$$

$$I_5 = [(\gamma - 1)M_e^2(\frac{2}{R}\mu_s \frac{\partial U_s}{\partial y}\zeta_3 + U_s\zeta_4) + (\frac{2i}{RPr}\alpha_0\mu_s - \frac{U_s}{T_s})\zeta_5] \frac{dA}{dx_1}$$

$$+ [(\gamma - 1)M_e^2 (\frac{2}{R}\mu_s \frac{\partial W_s}{\partial y}\zeta_3 + W_s\zeta_4) + (\frac{2i}{RPr} \beta_0\mu_s - \frac{W_s}{T_s})] \frac{dA}{dz_1}$$

$$+ \Bigg\{\{(\gamma - 1)M_e^2[\frac{\partial P_s}{\partial x_1} + \frac{2i}{R}\alpha_0\mu_s (r\frac{\partial U_s}{\partial x_1} + m\frac{\partial V_s}{\partial y} + m\frac{\partial W_s}{\partial z_1}) + \frac{2i}{R} \beta_0\mu_s (\frac{\partial U_s}{\partial z_1} + \frac{\partial W_s}{\partial x_1})]$$

$$- \frac{1}{T_s}\frac{\partial T_s}{\partial x_1}\} \zeta_1 + \frac{2}{R}\mu_s (\gamma - 1)M_e^2 [\frac{\partial U_s}{\partial y}\frac{\partial \zeta_3}{\partial x_1} + \frac{\partial W_s}{\partial y}\frac{\partial \zeta_3}{\partial z_1} (m\frac{\partial U_s}{\partial x_1} + r\frac{\partial V_s}{\partial y}$$

$$+ m\frac{\partial W_s}{\partial z_1})\frac{\partial \zeta_3}{\partial y}] - \frac{\gamma M_e^2}{T_s}(U_s\frac{\partial T_s}{\partial x_1} + V_s\frac{\partial T_s}{\partial y} + W_s\frac{\partial T_s}{\partial z_1})\zeta_4 + (\gamma - 1)M_e^2(U_s\frac{\partial \zeta_4}{\partial x_1}$$

$$+ V_s\frac{\partial \zeta_4}{\partial y} + W_s\frac{\partial \zeta_4}{\partial z_1}) + [\frac{1}{T_s^2}(U_s\frac{\partial T_s}{\partial x_1} + V_s\frac{\partial T_s}{\partial y} + W_s\frac{\partial T_s}{\partial z_1}) + \frac{i}{RPr}(\mu_s\frac{\partial \alpha_0}{\partial x_1}$$

$$+ 2\alpha_0\frac{\partial \mu_s}{\partial x_1}) + \frac{i}{RPr}(\mu_s\frac{\partial \beta_0}{\partial z_1} + 2\beta_0\frac{\partial \mu_s}{\partial z_1})]\zeta_5 + (\frac{2i}{RPr}\alpha_0\mu_s - \frac{U_s}{Ts})\frac{\partial \zeta_5}{\partial x_1}$$

$$+ (\frac{2i}{RPr}\mu_s\beta_0 - \frac{W_s}{T_s})\frac{\partial \zeta_5}{\partial z_1} - \frac{V_s}{T_s}\frac{\partial \zeta_5}{\partial y} + \Bigg\{(\gamma - 1)M_e^2\Big[\frac{\partial P_s}{\partial z_1} + \frac{2i}{R}\alpha_0\mu_s (\frac{\partial U_s}{\partial z_1}$$

$$+ \frac{\partial W_s}{\partial x_1}) + \frac{2i}{R}\beta_0\mu_s (m\frac{\partial U_s}{\partial x_1} + m\frac{\partial V_s}{\partial y} + r\frac{\partial W_s}{\partial z_1})\Big] - \frac{1}{T_s}\frac{\partial T_s}{\partial z_1}\Bigg\}\zeta_7\Bigg\} A \qquad (31)$$

The inhomogeneous Eqs. (24)-(26) have a solution if, and only if, a solvability condition is satisfied. This condition demands the inhomogeneous terms be orthogonal to every solution of the adjoint homogeneous problem. In the next section, we obtain equations for the modulation of the wave amplitude and the wavenumber with position using the solvability condition.

IV. COMPUTATIONAL PROCEDURE

The zeroth-order stability problem defined by Eqs. (16)-(22) is reduced to the solution of the following set of eight first-order differential equations with eight homogeneous boundary conditions

$$\frac{\partial Z_{0i}}{\partial y} - \sum_{j=1}^{8} a_{ij} Z_{0j} = 0 \quad \text{for } i = 1, 2, \ldots, 8 \tag{32}$$

$$Z_{01} = Z_{03} = Z_{05} = Z_{07} = 0 \quad \text{at } y = 0 \tag{33}$$

$$Z_{01}, Z_{03}, Z_{05}, Z_{07} \to 0 \quad \text{as } y \to \infty \tag{34}$$

where

$$Z_{01} = u_0, \quad Z_{02} = \frac{\partial u_0}{\partial y}, \quad Z_{03} = v_0, \quad Z_{04} = P_0,$$

$$Z_{05} = T_0 \quad Z_{06} = \frac{\partial T_0}{\partial y}, \quad Z_{07} = w_0, \quad Z_{08} = \frac{\partial w_0}{\partial y} \tag{35}$$

and the a_{ij} are the elements of 8x8 variable-coefficient matrix. The nonzero elements of this matrix are

$$a_{12} = 1$$

$$a_{21} = \frac{iR}{T_s \mu_s} (\alpha_0 U_s + \beta_0 W_s - \omega) + \alpha_0^2 + \beta_0^2$$

$$a_{22} = -\frac{1}{\mu_s} \frac{\partial \mu_s}{\partial y}$$

$$a_{23} = \frac{R}{T_s \mu_s} \frac{\partial U_s}{\partial y} - i\alpha_0 \left(\frac{1}{\mu_s} \frac{\partial \mu_s}{\partial y} + \frac{f}{T_s} \frac{\partial T_s}{\partial y}\right)$$

$$a_{24} \quad \frac{i\alpha_0 R}{\mu_s} - f\gamma M_e^2 \alpha_0 (\alpha_0 U_s + \beta_0 W_s - \omega)$$

$$a_{25} = \frac{f\alpha_0}{T_s}(\alpha_0 U_s + \beta_0 W_s - \omega) - \frac{1}{\mu_s}\frac{\partial}{\partial y}\left(\frac{d\mu_s}{dT_s}\frac{\partial U_s}{\partial y}\right)$$

$$a_{26} = \frac{1}{\mu_s}\frac{d\mu_s}{dT_s}\frac{\partial U_s}{\partial y}$$

$$a_{31} = -i\alpha_0$$

$$a_{33} = \frac{1}{T_s}\frac{\partial T_s}{\partial y}$$

$$a_{34} = -i\gamma M_e^2(\alpha_0 U_s + \beta_0 W_s - \omega)$$

$$a_{35} = \frac{i}{T_s}(\alpha_0 U_s + \beta_0 W_s - \omega)$$

$$a_{37} = -i\beta_0$$

$$a_{41} = -i\chi\alpha_0\left(\frac{2}{\mu_s}\frac{\partial \mu_s}{\partial y} + \frac{r}{T_s}\frac{\partial T_s}{\partial y}\right)$$

$$a_{42} = -i\chi\alpha_0$$

$$a_{43} = \chi\left[-\alpha_0^2 - \beta_0^2 + \frac{r}{T_s\mu_s}\frac{\partial \mu_s}{\partial y}\frac{\partial T_s}{\partial y} + \frac{r}{T_s}\frac{\partial^2 T_s}{\partial y^2} - \frac{iR}{T_s\mu_s}(\alpha_0 U_s + \beta_0 W_s - \omega)\right]$$

$$a_{44} = -i\chi r\gamma M_e^2\left[\left(\frac{1}{\mu_s}\frac{\partial \mu_s}{\partial y} + \frac{1}{T_s}\frac{\partial T_s}{\partial y}\right)(\alpha_0 U_s + \beta_0 W_s - \omega) + \alpha_0\frac{\partial U_s}{\partial y} + \beta_0\frac{\partial W_s}{\partial y}\right]$$

$$a_{45} = i\chi\left[\left(\frac{1}{\mu_s}\frac{d\mu_s}{dT_s} + \frac{r}{T_s}\right)\left(\alpha_0\frac{\partial U_s}{\partial y} + \beta_0\frac{\partial W_s}{\partial y}\right) + \frac{r}{T_s\mu_s}\frac{\partial \mu_s}{\partial y}(\alpha_0 U_s + \beta_0 W_s - \omega)\right]$$

$$a_{46} = \frac{i\chi r}{T_s}(\alpha_0 U_s + \beta_0 W_s - \omega)$$

$$a_{47} = -i\chi\beta_0\left(\frac{2}{\mu_s}\frac{\partial \mu_s}{\partial y} + \frac{r}{T_s}\frac{\partial T_s}{\partial y}\right)$$

$$a_{48} = - i\chi\beta_0$$

$$a_{56} = 1$$

$$a_{62} = - 2Pr(\gamma - 1)M_e^2 \frac{\partial U_s}{\partial y}$$

$$a_{63} = \frac{RPr}{T_s\mu_s} \frac{\partial T_s}{\partial y} - 2iPr(\gamma - 1)M_e^2(\alpha_0 \frac{\partial U_s}{\partial y} + \beta_0 \frac{\partial W_s}{\partial y})$$

$$a_{64} = \frac{iRPr}{\mu_s} (\gamma - 1)M_e^2 (\alpha_0 U_s + \beta_0 W_s - \omega)$$

$$a_{65} = \frac{iRPr}{T_s\mu_s} (\alpha_0 U_s + \beta_0 W_s - \omega) + \alpha_0^2 + \beta_0^2 - (\gamma-1)M_e^2 \frac{Pr}{\mu_s} \frac{d\mu_s}{dT_s} \left[(\frac{\partial U_s}{\partial y})^2 + (\frac{\partial W_s}{\partial y})^2 \right] - \frac{1}{\mu_s} \frac{\partial^2 \mu_s}{\partial y^2}$$

$$a_{66} = - \frac{2}{\mu_s} \frac{\partial \mu_s}{\partial y}$$

$$a_{68} = - 2Pr(\gamma - 1)M_e^2 \frac{\partial W_s}{\partial y}$$

$$a_{83} = - i\beta_0 (\frac{1}{\mu_s} \frac{\partial \mu_s}{\partial y} + \frac{f}{T_s} \frac{\partial T_s}{\partial y}) + \frac{R}{\mu_s T_s} \frac{\partial W_s}{\partial y}$$

$$a_{84} = \frac{iR\beta_0}{\mu_s} - f\beta_0\gamma M_e^2 (\alpha_0 U_s + \beta_0 W_s - \omega)$$

$$a_{85} = \frac{f\beta_0}{T_s} (\alpha_0 U_s + \beta_0 W_s - \omega) - \frac{1}{\mu_s} \frac{\partial}{\partial y} (\frac{d\mu_s}{dT_s} \frac{\partial W_s}{\partial y})$$

$$a_{86} = - \frac{1}{\mu_s} \frac{d\mu_s}{dT_s} \frac{\partial W_s}{\partial y}$$

$$a_{87} = \frac{iR}{T_s\mu_s} (\alpha_0 U_s + \beta_0 W_s - \omega) + \alpha_0^2 + \beta_0^2$$

$$a_{88} = -\frac{1}{\mu_s}\frac{\partial \mu_s}{\partial y}$$

where

$$\chi = 1/\left[\frac{R}{\mu_s} + ir\gamma M_e^2(\alpha_0 U_s + \beta_0 W_s - \omega)\right]$$

Equations (32)-(34) constitute an eigenvalue problem and it has nontrivial solutions only for certain combinations of the parameters α_0, β_0, ω and R.

4.1 Eigenvalues and Vectors

Outside the boundary layer (at $y = y_e > \delta$, where δ is the boundary-layer thickness), the mean-flow quantities are independent of y and the nonzero elements of the coefficient matrix $[a_{ij}]$ given by Eq. (32) are constants. They become

$$\hat{a}_{12} = 1,\ \hat{a}_{21} = iR(\alpha_0 + C\beta_0 - \omega) + \alpha_0^2 + \beta_0^2,$$

$$\hat{a}_{24} = iR\alpha_0 - f\gamma M_e^2\alpha_0(\alpha_0 + C\beta_0 - \omega),$$

$$\hat{a}_{25} = f\alpha_0(\alpha_0 + C\beta_0 - \omega),\ \hat{a}_{31} = -i\alpha_0,\ \hat{a}_{34} = -i\gamma M_e^2(\alpha_0 + C\beta_0 - \omega),$$

$$\hat{a}_{35} = i(\alpha_0 + C\beta_0 - \omega),\ \hat{a}_{37} = -i\beta_0,\ \hat{a}_{42} = -i\chi\alpha_0,$$

$$\hat{a}_{43} = -\chi\left[iR(\alpha_0 + C\beta_0 - \omega) + \alpha_0^2 + \beta_0^2\right],\ \hat{a}_{46} = i\chi r(\alpha_0 + C\beta_0 - \omega),$$

$$\hat{a}_{48} = -i\chi\beta_0,\ \hat{a}_{56} = 1,\ \hat{a}_{64} = -iRPr(\gamma - 1)M_e^2(\alpha_0 + C\beta_0 - \omega),$$

$$\hat{a}_{65} = iRPr\,(\alpha_0 + C\beta_0 - \omega) + \alpha_0^2 + \beta_0^2,\ \hat{a}_{84} = iR\beta_0 - f\gamma M_e^2\beta_0(\alpha_0 + C\beta_0 - \omega),$$

$$\hat{a}_{85} = f\beta_0(\alpha_0 + C\beta_0 - \omega),\ \hat{a}_{87} = \hat{a}_{21}$$

where

$$\chi = 1/\left[R + ir\gamma M_e^2(\alpha_0 + C\beta_0 - \omega)\right] \text{ , and } C = W_s \text{ at } y = y_e$$

Equations (32) with a constant coefficient matrix $[\hat{a}_{ij}]$ permit a solution that can be expressed in the general form

$$Z_{0_i} = \sum_{j=1}^{8} \Lambda_{ij} c_j \exp(\lambda_j y) \text{ for } i = 1,2,\ldots,8 \tag{36}$$

where the λ_j are eigenvalues of the matrix $[\hat{a}_{ij}]$, the Λ_{ij} are the elements of the corresponding eight eigenvectors, and the c_j are arbitrary constants.

The values λ_j and Λ_{ij} can be derived analytically by rewriting the eight first-order equations (32) with constant coefficients as four second-order equations in the form

$$\frac{d^2 J_i}{dy^2} - \sum_{j=1}^{4} b_{ij} J_j = 0 \text{ for } i = 1,2,..,4 \tag{37}$$

where

$$J_1 = Z_{01},\ J_2 = Z_{04},\ J_3 = Z_{05},\ J_4 = Z_{07} \tag{38}$$

and the coefficients b_{ij} are given by

$$b_{11} = \hat{a}_{21},\ b_{12} = \hat{a}_{24},\quad b_{13} = \hat{a}_{25},$$

$$b_{22} = \hat{a}_{24}\hat{a}_{42} + \hat{a}_{34}\hat{a}_{43} + \hat{a}_{46}\hat{a}_{64} + \hat{a}_{48}\hat{a}_{84},$$

$$b_{23} = \hat{a}_{25}\hat{a}_{42} + \hat{a}_{35}\hat{a}_{43} + \hat{a}_{46}\hat{a}_{65} + \hat{a}_{48}\hat{a}_{85},$$

$$b_{32} = \hat{a}_{64},\ b_{33} = \hat{a}_{65},\quad b_{42} = \hat{a}_{84},\ b_{43} = \hat{a}_{85},\ b_{44} = \hat{a}_{21}$$

The solution of Eqs. (37) has the form

$$J_i = \sum_{j=1}^{8} B_{ij} d_j \exp(\lambda_j y) \quad \text{for } i = 1,2,\ldots,4 \tag{39}$$

where the λ_j are the same as the eigenvalues of Eqs. (32) with constant coefficients, the B_{ij} are the elements of the corresponding eight eigenvectors, and the d_j are arbitrary constants. From the characteristic determinant, it follows that the eigenvalues are

$$\lambda_{1,5} = \mp (b_{11})^{1/2} \tag{40}$$

$$\lambda_{2,6} = \mp \left\{ \frac{1}{2}(b_{22} + b_{33}) + \left[\frac{1}{4}(b_{22} - b_{33})^2 + b_{23}b_{32} \right]^{1/2} \right\}^{1/2} \tag{41}$$

$$\lambda_{3,7} = \mp \left\{ \frac{1}{2}(b_{22} + b_{33}) - \left[\frac{1}{4}(b_{22} - b_{33})^2 + b_{23}b_{32} \right]^{1/2} \right\}^{1/2} \tag{42}$$

$$\lambda_{4,8} = \mp (b_{11})^{1/2} \tag{43}$$

The B_{ij} can be obtained from the solution of the characteristic equation. They are given by

$$B_{ij} = 1, \; B_{2j} = 0, \; B_{3j} = 0, \; b_{4j} = 0$$

$$\text{for } j = 1,5 \tag{44}$$

$$B_{1j} = \frac{(\lambda_j^2 - \hat{a}_{65})\hat{a}_{24} + \hat{a}_{25}\hat{a}_{64}}{(\hat{a}_{21} - \lambda_j^2)}, \quad B_{2j} = \hat{a}_{65} - \lambda_j^2,$$

$$B_{3j} = -\hat{a}_{64}, \quad B_{4j} = \frac{\hat{a}_{64}\hat{a}_{85} + (\lambda_j^2 - \hat{a}_{65})\hat{a}_{84}}{(\hat{a}_{21} - \lambda_j^2)}$$

$$\text{for } j = 2,3,6,7 \tag{45}$$

and

$$B_{1_j} = 0,\ B_{2_j} = 0,\ B_{3_j} = 0,\ B_{4_j} = 1$$

$$\text{for } j = 4,8 \tag{46}$$

The Λ_{ij} are related to the B_{ij} by Eqs. (36)-(39); they are

$$\Lambda_{1_j} = 1,\ \Lambda_{2_j} = \lambda_j,\ \Lambda_{3_j} = (\hat{a}_{31}B_{1_j} + \hat{a}_{34}B_{2_j} + \hat{a}_{35}B_{3_j} + \hat{a}_{37}B_{4_j})/\lambda_j B_{1_j},$$

$$\Lambda_{4_j} = B_{2_j}/B_{1_j},\ \Lambda_{5_j} = B_{3_j}/B_{1_j},\ \Lambda_{6_j} = \lambda_j B_{3_j}/B_{1_j},\ \Lambda_{7_j} = B_{4_j}/B_{1_j},$$

$$\Lambda_{8_j} = (\hat{a}_{84}B_{2_j} + \hat{a}_{85}B_{3_j} + \hat{a}_{87}B_{4_j})/\lambda_j B_{1_j} \tag{47}$$

These eigenvectors are normalized such that

$$Z_{0_1} = \sum_{j=1}^{8} c_j \exp(\lambda_j y) \quad \text{at } y = y_e$$

4.2 Boundary Conditions

The boundary conditions at infinity (34) demand the constants c_5, c_6, c_7, and c_8 be zero. To set up these boundary conditions for numerical solution, we first solve Eqs. (36) for the $c_j \exp(\lambda_j y)$ and obtain

$$c_j \exp(\lambda_j y) = \sum_{i=1}^{8} f_{ij} Z_{0_i} \quad \text{for } j = 1,2,\ldots 8 \tag{48}$$

where the matrix $[f_{ij}]$ is the inverse of $[\Lambda_{ij}]$. Setting $c_5=c_6=c_7=c_8=0$ in Eq. (48) leads to

$$\sum_{i=1}^{8} e_{ij} Z_{0_i} = 0 \text{ for } j = 5,6,7,\text{and } 8 \text{ at } y = y_e \tag{49}$$

Equations (49) replace the boundary conditions Eqs. (34).

The boundary conditions at the wall (33) can be set up for numerical solution by writing them in the form

$$\sum_{j=1}^{8} e_{ij} Z_{1j} = 0 \text{ for } i = 1,2,\ldots 8 \tag{50}$$

where the e_{ij} are the elements of an 8 x 8 matrix with only four nonzero elements.

4.3 Integration and Orthonormalization

For the spatial stability problem, we assign values to ω and R and two relations among α_{0_r}, α_{0_i}, β_{0_r}, and β_{0_i}, where $\alpha_0 = \alpha_{0_r} + i\alpha_{0_i}$ and $\beta_0 = \beta_{0_r} + i\beta_{0_i}$. Then, we guess the remaining two relations. We determine the f_{ij} in Eqs. (49) and use this boundary condition to construct a linear combination of the general solution given by Eqs. (36). As $y \to \infty$, the four growing solutions in Eqs. (36) are eliminated. A variable step size algorithm developed by Scott and Watts (1977), based on the Runge-Kutta-Fehlburg fifth-order formulas, is used to integrate Eqs. (32) from $y = y_e$ to the wall. A straightforward integration fails to produce four linearly independent solutions because of the buildup of parasitic errors among the different solutions. To overcome this difficulty, the integrator used is coupled with an orthonormalization test that is based on the modified Gram-Schmidt procedure.

Since testing for independence after each integration step is expensive, we use a modified algorithm (Darlow et al, 1977) and choose a preselected set of points where orthonormalization is performed. These

points are assigned a priori by using information about the points where orthonormalization is needed.

At the wall, the values of the linearly independent solution vectors are linearly combined to satisfy all but one of the wall boundary conditions. The last wall boundary condition can only be satisfied by this combined solution when the exact remaining relations among α_{0_r}, α_{0_i}, β_{0_r}, and β_{0_i} have been found. A Newton-Raphson procedure is used to determine these relations. With the eigenvalue determined to within the desired accuracy, the eigenfunctions can be recovered using the stored solution vectors. This solution can be expressed in the form

$$Z_{0_i} = A(x_1, z_1)\, \zeta_i\, (x_1, y, z_1) \text{ for } i = 1,2,\ldots,8 \tag{51}$$

4.4 Solvability Condition and the Adjoint

With the solution of the zeroth-order problem given by (51), the first-order problem becomes

$$\frac{\partial Z_{1i}}{\partial y} - \sum_{j=1}^{8} a_{ij} Z_{1j} = G_i \frac{\partial A}{\partial x_1} + E_i \frac{\partial A}{\partial z_1} + D_i A \text{ for } i = 1,2,\ldots 8 \tag{52}$$

$$Z_{11} = Z_{13} = Z_{15} = Z_{17} = 0 \text{ at } y = 0 \tag{53}$$

$$Z_{11},\ Z_{13},\ Z_{15},\ Z_{17} \to 0 \quad \text{as } y \to \infty \tag{54}$$

where the G_i, E_i, and D_i are known functions of the ζ_i, α_0, β_0, and mean-flow quantities; they are defined by

$$G_1 \quad \frac{\partial A}{\partial x_1} + E_1 \frac{\partial A}{\partial z_1} + D_1 A = 0 \tag{55a}$$

$$G_2 \quad \frac{\partial A}{\partial x_1} + E_2 \frac{\partial A}{\partial z_1} + D_2 A = - if\alpha_0 T_s I_1 - \frac{R}{\mu_s} I_2 \tag{55b}$$

$$G_3 \quad \frac{\partial A}{\partial x_1} + E_3 \frac{\partial A}{\partial z_1} + D_3 A = T_s I_1 \tag{55c}$$

$$G_4 \quad \frac{\partial A}{\partial x_1} + E_4 \frac{\partial A}{\partial z_1} + D_4 A = r\chi\left(\frac{T_s}{\mu_s}\frac{\partial \mu_s}{\partial y} + 2\frac{\partial T_s}{\partial y}\right) I_1 + r\chi T_s \frac{\partial}{\partial y} I_1 + \frac{R\chi}{\mu_s} I_3 \tag{55d}$$

$$G_5 \quad \frac{\partial A}{\partial x_1} + E_5 \frac{\partial A}{\partial z_1} + D_5 A = 0 \tag{55e}$$

$$G_6 \quad \frac{\partial A}{\partial x_1} + E_6 \frac{\partial A}{\partial z_1} + D_6 A = - \frac{RPr}{\mu_s} I_5 \tag{55f}$$

$$G_7 \quad \frac{\partial A}{\partial x_1} + E_7 \frac{\partial A}{\partial z_1} + D_7 A = 0 \tag{55g}$$

$$G_8 \quad \frac{\partial A}{\partial x_1} + E_8 \frac{\partial A}{\partial z_1} + D_8 A = - if\beta_0 T_s I_1 - \frac{R}{\mu_s} I_4 \tag{55h}$$

where I_1 - I_5 are defined by Eqs. (27)-(30).

Since the homogeneous parts of (52)-(54) are the same as (32)-(34) and since the latter have a nontrivial solution, the inhomogeneous Eqs. (52)-(54) have a solution if, and only if, a solvability condition is satisfied. In this case, the solvability condition is

$$\int_0^\infty \sum_{i=1}^{8} \left[G_i \frac{\partial A}{\partial x_1} + E_i \frac{\partial A}{\partial z_1} + D_i A\right] \hat{W}_i dy = 0 \tag{56}$$

where the $\hat{W}_i(x_1, y, z_1)$ are solutions of the adjoint homogeneous problem corresponding to the same eigenvalue. Thus, they are solutions of

$$\frac{\partial \hat{W}_i}{\partial y} + \sum_{j=1}^{8} a_{ji} \hat{W}_j = 0 \text{ for } i = 1,2,\ldots,8 \tag{57}$$

$$\hat{W}_2 = \hat{W}_4 = \hat{W}_6 = \hat{W}_8 = 0 \quad \text{at } y = 0 \tag{58}$$

$$\hat{W}_2, \hat{W}_4, \hat{W}_6, \hat{W}_8 \to 0 \quad \text{as } y \to \infty \tag{59}$$

We solve the adjoint problem, Eqs. (57)-(59), following the same numerical procedure used to solve the zeroth-order problem. Outside the boundary layer (at $y = y_e$), Eqs. (57), has constant coefficients and its solution can be written in the form

$$\hat{W}_i = \sum_{j=1}^{8} \Lambda^*_{ij} c^*_j \exp(\lambda_j y) \quad \text{for} \quad i = 1,2,\ldots,8 \tag{60}$$

where the λ_j are the same as those for the zeroth-order homogeneous problem, Eqs. (32)-(34), but the Λ^*_{ij} are different from the Λ_{ij}. The Λ^*_{ij} can be obtained analytically in the same way we obtained the Λ_{ij}. The Λ^*_{ij} components are given by

$$\Lambda^*_{1j} = 1, \ \Lambda^*_{2j} = -(B^*_{1j} + \hat{a}_{42}B^*_{2j})/\lambda_j B^*_{1j}, \ \Lambda^*_{3j} = -\hat{a}_{43}B^*_{2j}/\lambda_j B^*_{1j},$$

$$\Lambda^*_{4j} = B^*_{2j}/B^*_{1j}, \ \Lambda^*_{5j} = B^*_{3j}/B^*_{1j}, \ \Lambda^*_{6j} = -(\hat{a}_{46}B^*_{2j} + B^*_{3j})/\lambda_j B^*_{1j},$$

$$\Lambda^*_{7j} = B^*_{4j}/B^*_{1j}, \ \Lambda^*_{8j} = -(\hat{a}_{48}B^*_{2j} + B^*_{4j})/\lambda_j B^*_{1j} \tag{61}$$

where

$$B^*_{1j} = 1, \ B^*_{2j} = \frac{(\lambda_j^2 - \hat{a}_{65})\hat{a}_{24} + \hat{a}_{25}\hat{a}_{64}}{(\hat{a}_{65} - \lambda_j^2)(b_{22} - \lambda_j^2) - \hat{a}_{64}b_{23}},$$

$$B^*_{3j} = \frac{\hat{a}_{24}b_{23} - (b_{22} - \lambda_j^2)\hat{a}_{25}}{(\hat{a}_{65} - \lambda_j^2)(b_{22} - \lambda_j^2) - \hat{a}_{64}b_{23}}, \ B^*_{4j} = 0$$

$$\text{for } j = 1,5 \tag{62}$$

$$B_{1_j}^* = 0,\ B_{2_j}^* = -\hat{a}_{64}, B_{3_j}^* = b_{22}-\lambda_j^2,\ B_{4_j}^* = 0$$

$$\text{for } j = 2,3,6,7 \tag{63}$$

and

$$B_{1_j}^* = 0,\ B_{2_j}^* = \frac{(\lambda_j^2-\hat{a}_{65})\hat{a}_{84} + \hat{a}_{85}\hat{a}_{64}}{(\hat{a}_{65}-\lambda_j^2)(b_{22}-\lambda_j^2)-\hat{a}_{64}b_{23}},$$

$$B_{3_j}^* = \frac{\hat{a}_{84}b_{23}-(b_{22}-\lambda_j^2)\hat{a}_{85}}{(\hat{a}_{65}-\lambda_j^2)(b_{22}-\lambda_j^2)-\hat{a}_{64}b_{23}},\ B_{4_j}^* = 1$$

$$\text{for } j = 4,8 \tag{64}$$

In solving the adjoint problem, the eigenvalue relations we found before are used in one integration to produce the adjoint solution. The solution of the adjoint problem can provide an independent check on the eigenvalues obtained earlier. Moreover, solving the adjoint problem provides an easier and accurate way of calculating the group velocity instead of the approximate and lengthy finite difference techniques.

4.5 Amplitude and Wavenumber Equations

Substituting for G_i, E_i, and D_i from Eqs. (55) in the solvability condition (56), we obtain the following equation for the modulation of the wave amplitude A with position

$$Q_1 \frac{\partial A}{\partial x_1} + Q_2 \frac{\partial A}{\partial z_1} = H_1 A \tag{65}$$

where

$$Q_1 = \int_0^\infty \sum_{i=1}^{8} G_i \hat{W}_i dy \tag{66}$$

$$Q_2 = \int_0^\infty \sum_{i=1}^{8} E_i \hat{W}_i dy \tag{67}$$

$$H_1 = \int_0^\infty \sum_{i=1}^{8} D_i \hat{W}_i dy \tag{68}$$

Here, Q_1 and Q_2 are proportional to the components of the group velocity

$(\frac{\partial \omega}{\partial \alpha_0}, \frac{\partial \omega}{\alpha \beta_0})$

To determine H_1, we need to evaluate $\partial \alpha_0/\partial x_1$, $\partial \alpha_0/\partial z_1$, $\partial \beta_0/\partial x_1$, $\partial \beta_0/\partial z_1$, $\partial \zeta_i/\partial x_1$, and $\partial \zeta_i/\partial z_1$. To accomplish this, we replace Z_i by ζ_i in (32)-(34), differentiate the result with respect to x_1, and obtain

$$\frac{\partial}{\partial y}(\frac{\partial \zeta_i}{\partial x_1}) - \sum_{j=1}^{8} a_{ij}(\frac{\partial \zeta_j}{\partial x_1}) = G_i \frac{\partial \alpha_0}{\partial x_1} + E_i \frac{\partial \beta_0}{\partial x_1} + S_{xi} \text{ for } i = 1,2,.,8 \tag{69}$$

$$\frac{\partial \zeta_1}{\partial x_1} = \frac{\partial \zeta_3}{\partial x_1} = \frac{\partial \zeta_5}{\partial x_1} = \frac{\partial \zeta_7}{\partial x_1} = 0 \text{ at } y = 0 \tag{70}$$

$$\frac{\partial \zeta_1}{\partial x_1}, \frac{\partial \zeta_3}{\partial x_1}, \frac{\partial \zeta_5}{\partial x_1}, \frac{\partial \zeta_7}{\partial x_1} \to 0, \text{ as } y \to \infty \tag{71}$$

Similarly, differentiation of (32)-(34) with respect to z_1 yields

$$\frac{\partial}{\partial y}(\frac{\partial \zeta_i}{\partial z_1}) - \sum_{j=1}^{8} a_{ij}(\frac{\partial \zeta_j}{\partial z_1}) = G_i \frac{\partial \alpha_0}{\partial z_1} + E_i \frac{\partial \beta_0}{\partial z_1} + S_{zi} \text{ for } i = 1,2,\ldots,8 \tag{72}$$

$$\frac{\partial \zeta_1}{\partial z_1} = \frac{\partial \zeta_3}{\partial z_1} = \frac{\partial \zeta_5}{\partial z_1} = \frac{\partial \zeta_7}{\partial z_1} = 0 \quad \text{at } y = 0 \tag{73}$$

$$\frac{\partial \zeta_1}{\partial z_1}, \frac{\partial \zeta_3}{\partial z_1}, \frac{\partial \zeta_5}{\partial z_1}, \frac{\partial \zeta_7}{\partial z_1} \to 0 \text{ as } y \to \infty \tag{74}$$

Here G_i, E_i, and S_i are known functions of ζ_i, α_0, β_0, and the basic-flow quantities; they are given by

$$G_i = \sum_{j=1}^{8} \zeta_j \frac{\partial a_{ij}}{\partial \alpha_0}\Big|_{\beta_0,\, x_1} = \sum_{j=1}^{8} \zeta_j \frac{\partial a_{ij}}{\partial \alpha_0}\Big|_{\beta_0, z_1} \quad \text{for } i = 1,2,..,8 \tag{75}$$

$$E_i = \sum_{j=1}^{8} \zeta_j \frac{\partial a_{ij}}{\partial \beta_0}\Big|_{\alpha_0, x_1} = \sum_{j=1}^{8} \zeta_j \frac{\partial a_{ij}}{\partial \beta_0}\Big|_{\alpha_0, z_1} \quad \text{for } i = 1,2,..,8 \tag{76}$$

$$S_{xi} = \sum_{j=1}^{8} \zeta_j \frac{\partial a_{ij}}{\partial x_1}\Big|_{\alpha_0,\beta_0}, \quad S_{zi} = \sum_{j=1}^{8} \zeta_j \frac{\partial a_{ij}}{\partial z_1}\Big|_{\alpha_0,\beta_0} \quad \text{for } i = 1,2,..,8 \tag{77}$$

Again, applying the solvability conditions to (69)-(71) and (72)-(74), we obtain equations for the modulation of the wavenumber with position

$$Q_1 \frac{\partial \alpha_0}{\partial x_1} + Q_2 \frac{\partial \alpha_0}{\partial z_1} = H_2 \tag{78}$$

$$Q_1 \frac{\partial \beta_0}{\partial x_1} + Q_2 \frac{\partial \beta_0}{\partial z_1} = H_3 \tag{79}$$

where Q_1 and Q_2 are given in (66) and (67) and H_2 and H_3 are given by

$$H_2 = i \int_0^{\infty} \sum_{j=1}^{8} S_{xj} \hat{W}_j dy \tag{80}$$

$$H_3 = i \int_0^{\infty} \sum_{j=1}^{8} S_{zj} \hat{W}_j dy \tag{81}$$

The quantities H_1, H_2, H_3, Q_1 and Q_2 in Eqs. (65), (78), and (79) are slowly varying functions of x and z. For a parallel mean flow, the H's vanish and the Q's are constant. Nayfeh and Padhye (1979) derived equations

similar to (65), (78) and (79) for incompressible nonparallel three-dimensional flows.

In the spatial theory α_0 and β_0 are complex and ω is real. We define a real wavenumber vector of magnitude k_0 and direction ψ according to

$$\vec{k}_0 = (\alpha_{0r}, \beta_{0r}), \qquad \psi = \tan^{-1}(\beta_{0r}/\alpha_{0r}) \tag{82}$$

and a real spatial amplification-rate vector of magnitude σ_0 and direction $\bar{\psi}$ according to

$$\vec{\sigma}_0 = (\alpha_{0i}, \beta_{0i}), \qquad \bar{\psi} = \tan^{-1}(\beta_{0i}/\alpha_{0i}) \tag{83}$$

The solution of the eigenvalue problem, gives the complex dispersion relation

$$\omega = \omega(\vec{k}_0, \sigma_0, x, z) \tag{84}$$

For fixed ω, x, and z, there are four real parameters, k_0, ψ, σ, and $\bar{\psi}$. Two of them can be determined from the eigenvalue calculation.

In general, the direction of the wave propagation ψ is different from the direction of the wave amplification $\bar{\psi}$. The propagation angle ψ can be used as an input parameter, while the question of determining the direction of the amplification $\bar{\psi}$ is still open. Mack (1977) chose the direction given by the real part of the group velocity angle to be the direction of amplification. He showed that for two-dimensional basic flows, the direction of the group velocity deviates by a few degrees from the streamwise direction. This deviation decreases as the Mach number increases.

Lekoudis (1979) and Runyan and George (1979) chose the direction of amplification to be the direction of the local potential flow. An amplification direction given by the real ratio of the complex group velocities was derived by Nayfeh (1979). For a parallel mean flow, the amplification direction is given by the real ratio of $\partial\alpha_0/\partial\beta_0$, which was derived by Cebeci and Stewartson (1979) and Nayfeh (1979). For a monocromatic wave generated by a source oscillating at frequency ω at $x = 0$ to penetrate large values of x and z, the ray equation

$$\frac{dz}{dx} = \frac{Q_2}{Q_1} = \text{a real quantity} \tag{85}$$

defines the direction of the wave amplification for the physical problem of real x and z. The wave amplitude and wavenumber will vary along the ray as

$$A = A_0 \exp\left(\int H_1 \, d\xi \right) \tag{86}$$

$$\alpha_0 = \int H_2 \, d\xi \tag{87}$$

$$\beta_0 = \int H_3 \, d\xi \tag{88}$$

Equations (86)-(88) are derived from Eqs. (65), (78), and (79) by using

$$\frac{dx_1}{d\xi} = Q_1 \quad \text{and} \quad \frac{dz_1}{d\xi} = Q_2 \tag{89}$$

Using Eqs. (14), and (86)-(88) in Eq. (51), we obtain

$$Z_{0i} = A_0 \zeta_i(x_1, y, z_1) \exp\left[i \int (\alpha_0 + \beta_0 \frac{Q_2}{Q_1} - i\varepsilon \frac{H_1}{Q_1}) \, dx - i\omega t \right] + O(\varepsilon) \tag{90}$$

where Z_{0i} is related to the disturbance variables by (35), and the constant A_0 is determined from the initial conditions. It is clear from (90) that, in addition to the dependence of the eigensolution on x_1 and z_1, the amplification of the disturbance is a function of the normal distance from the wall.

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Final Report for Part I

16. Abstract

A compressible linear stability theory is presented for nonparallel three-dimensional boundary-layer flows, taking into account the normal velocity component as well as the streamwise and spanwise variations of the basic flow. The method of multiple scales is used to account for the nonparallelism of the basic flow, and equations are derived for the spatial evolution of the disturbance amplitude and wavenumber. The numerical procedure for obtaining the solution of the nonparallel problem is outlined.

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