PERIODIC SOLUTIONS TO DUFFING'S EQUATION VIA THE HOMOTOPY METHOD

A THESIS

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Chapter 1

Introduction

Consider the (possibly nonlinear) differential equation

$$\ddot{x}(t) + C\dot{x}(t) + g(t, x) = e(t), \tag{1.1}$$

where C is a constant; g is continuous, continuously differentiable with respect to x, and is periodic of period P in the variable t; $e(t)$ is continuous and periodic of period P. We are interested in determining initial conditions that guarantee the solution of this equation to also be periodic of period P. Li and Shen [7] assume that there exist two continuous functions $a(t)$ and $b(t)$ and a positive integer n so that

$$n^2 \leq a(t) \leq \frac{\partial g}{\partial x}(t, x) \leq b(t) \leq (n+1)^2 \tag{1.2}$$

for all values of $t \geq 0$, so that $n^2 < a(t)$, and $b(t) < (n+1)^2$ on a subset of positive Lebesgue measure of the interval $[0, P]$, then they show that there exist unique initial values $x(0) = \alpha^*$ and $\dot{x}(0) = \beta^*$ so that the solution to this initial value problem is periodic of period P and is unique with this property. (The continuity assumption on a and b can be weakened to continuity almost everywhere.) Moreover, there is a constructive proof of this result. That is, starting at any initial conditions $x(0) = \alpha$ and $\dot{x}(0) = \beta$, we can produce a path of initial values starting at (α, β) in the phase plane and terminating at (α^*, β^*), and we can produce a homotopy that continuously deforms the starting solution to the unique periodic solution. We call this the Theorem of Li and Shen throughout this paper. We will discuss both the proof of this theorem and a *Mathematica* implementation.

Chapter 2

The Existence Proof of Li and Shen's Theorem

The idea of the proof is as follows. We write the solution to the initial value problem having $(x(0), \dot{x}(0)) = (\alpha, \beta)$ as $x = x(t, v)$ where $v = (\alpha, \beta)^T$. Define

$$f(v) = (x(P, v), \dot{x}(P, v))^T$$

and set

$$F(v) = v - f(v).$$

Observe that the desired initial conditions for a periodic solution form a fixed point for f and a zero point for F. The continuous dependence on initial conditions theorem in [4] shows that the evaluation of our initial value problem is a continuous function of the initial conditions. Thus, both f and F are C^1 differentiable functions with respect to v. We want to show that F is a homeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$. This will guarantee that F has a unique zero point and thus the theorem follows.

Linearizing equation (1.1), we compute its fundamental solution matrix $Y(t)$ and show that

$$\frac{dF}{dv}(v) = I - Y(P),$$

where I is the 2×2 identity matrix. The bounds given in (1.2) yield a bound on the eigenvalues of the matrix $\frac{dF}{dv}(v)$. This shows that $\frac{dF}{dv}(v)$ is invertible and hence $\|\frac{dF}{dv}(v)^{-1}\|$ exists and is uniformly bounded. By Hadamard's Theorem [11], this is sufficient to imply that F is a homeomorphism.

2.1 Bounding $\|\frac{dF}{dv}(v)^{-1}\|$

Assume we have a solution x to equation (1.1), and a nearby solution x^*. We let $\xi = x(t) - x^*(t)$ and $\ddot{\xi} = \ddot{x}(t) - \ddot{x}^*(t)$. We re-write $\ddot{\xi}$ as a system and consider its fundamental solution matrix $Y(t)$. To obtain this system we have the following

$$\begin{aligned}
\ddot{\xi} &= \ddot{x}(t) - \ddot{x}^*(t) \\
&= -C\dot{x}(t) - g(x,t) + e(t) - (-C\dot{x}^*(t) - g(x^*,t) + e(t)) \\
&= -C(\dot{x}(t) - x^*(t)) - (g(x,t) - g(x^*,t)) \\
&= -C\dot{\xi} - \left(\frac{g(x,t) - g(x^*,t)}{x(t) - x^*(t)}\right)\xi \\
&\approx -C\dot{\xi} - \frac{\partial g}{\partial x}(x,t)\xi.
\end{aligned}$$

Turning this into a system by setting $y = \dot{\xi}$. We have the first variational system

$$\begin{pmatrix} \dot{\xi} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{\partial g}{\partial x}(t,x) & -C \end{pmatrix} \begin{pmatrix} \xi \\ y \end{pmatrix} \tag{2.1}$$

We use the bounds on $\frac{\partial g}{\partial x}$ from the hypothesis of Li and Shen's Theorem to bound the eigenvalues of the fundamental solution matrix of equation (2.1) away from zero. Then we form a second fundamental solution matrix that corresponds to $\frac{dF}{dv}$. Using the bounds on the eigenvalues of the first fundamental solution matrix, we bound the eigenvalues of $\frac{dF}{dv}$ away from zero. We conclude by showing $\|\frac{dF^{-1}}{dv}\|$ exists and is bounded by some constant M.

Computing the fundamental solution matrix of equation (2.1) we have

$$\begin{aligned}
Y(t) &= Exp[\int_0^t \begin{pmatrix} 0 & 1 \\ -\frac{\partial g}{\partial x}(s,x) & -C \end{pmatrix} ds] \\
&= Exp[\begin{pmatrix} 0 & t \\ -\int_0^t \frac{\partial g}{\partial x}(s,x)ds & -Ct \end{pmatrix}].
\end{aligned}$$

We are interested in $Y(P)$. Computing

$$Y(P) = Exp[\begin{pmatrix} 0 & P \\ -\int_0^P \frac{\partial g}{\partial x}(s,x)ds & -CP \end{pmatrix}].$$

The eigenvalues of $Y(P)$ are of the form e^{λ} where λ is an eigenvalue of

$$\begin{pmatrix} 0 & P \\ -\int_0^P \frac{\partial g}{\partial x}(s,x)ds & -CP \end{pmatrix}. \tag{2.2}$$

The eigenvalues of this matrix are

$$\lambda = -CP \pm \sqrt{P^2C^2 - 2P\int_0^P \frac{\partial g}{\partial x}(t,x)\,dt}. \tag{2.3}$$

We now rewrite (1.1) as a system, and compute the partial derivatives with respect to the initial condition v. This gives us a system whose fundamental solution matrix corresponds to $\frac{dF}{dv}$ and also satisfies equation (2.1). By obtaining a bound on (2.3), we will have obtained a bound on $\|\frac{dF}{dv}^{-1}\|$. We begin this process by letting $u = (x,y)^T$ and observing that

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial u(t,v)}{\partial v}\right) &= \frac{\partial}{\partial v}\dot{u}(t,v) \\ &= \frac{\partial}{\partial v}(R(t,u(t,v)) + E(t)) \\ &= \frac{\partial}{\partial v}(R(t,u(t,v))), \end{aligned}$$

where

$$R(t,u(t,v)) = \begin{pmatrix} y \\ -Cy - g(t,x) \end{pmatrix}$$

and

$$E(t) = \begin{pmatrix} 0 \\ e(t) \end{pmatrix}.$$

It follows that

$$\begin{aligned} \frac{\partial}{\partial v}(R(t,u(t,v))) &= \frac{\partial R}{\partial u}\cdot\frac{\partial u}{\partial v} \\ &= \begin{pmatrix} \frac{\partial R_1}{\partial x} & \frac{\partial R_1}{\partial y} \\ \frac{\partial R_2}{\partial x} & \frac{\partial R_2}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial x(t,v)}{\partial \alpha} & \frac{\partial x(t,v)}{\partial \beta} \\ \frac{\partial \dot{x}(t,v)}{\partial \alpha} & \frac{\partial \dot{x}(t,v)}{\partial \beta} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -\frac{\partial g}{\partial x}(t,x) & -C \end{pmatrix} \cdot \begin{pmatrix} \frac{\partial x(t,v)}{\partial \alpha} & \frac{\partial x(t,v)}{\partial \beta} \\ \frac{\partial \dot{x}(t,v)}{\partial \alpha} & \frac{\partial \dot{x}(t,v)}{\partial \beta} \end{pmatrix}. \end{aligned}$$

This shows that $\partial u/\partial v$ formally satisfies the differential equation (2.1), and consequently

$$\frac{\partial u}{\partial v} = \begin{pmatrix} \dfrac{\partial x(t,v)}{\partial \alpha} & \dfrac{\partial x(t,v)}{\partial \beta} \\ \dfrac{\partial \dot{x}(t,v)}{\partial \alpha} & \dfrac{\partial \dot{x}(t,v)}{\partial \beta} \end{pmatrix}$$

is also a fundamental matrix for equation (2.1). Furthermore, when we compute $\dfrac{dF}{dv}(v)$ we have

$$\begin{aligned} \frac{dF}{dv}(v) &= I - \frac{\partial f}{\partial v} \\ &= I - \begin{pmatrix} \dfrac{\partial f}{\partial \alpha} & \dfrac{\partial f}{\partial \beta} \end{pmatrix} \\ &= I - \begin{pmatrix} \dfrac{\partial x(P,v)}{\partial \alpha} & \dfrac{\partial x(P,v)}{\partial \beta} \\ \dfrac{\partial \dot{x}(P,v)}{\partial \alpha} & \dfrac{\partial \dot{x}(P,v)}{\partial \beta} \end{pmatrix} \\ &= I - \frac{\partial u}{\partial v}. \end{aligned} \tag{2.4}$$

Lemma 2.1.1. *Let A be an $n \times n$ matrix. Then λ is an eigenvalue of A if and only if $1-\lambda$ is an eigenvalue of $I-A$.*

Proof. Let A= $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$

and note that $A - I_{n\times n}$= $\begin{pmatrix} a_{11}-1 & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22}-1 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn}-1 \end{pmatrix}$.

Thus λ is an eigenvalue of A if and only if $\lambda = 1+\mu$ where μ is an eigenvalue of $A - I_{n\times n}$. Hence, $\mu = \lambda - 1$, or equivalent $1-\lambda$ is an eigenvalue for $I_{n\times n} - A$

□

Li and Shen [7] obtain a lower bound for all the eigenvalues of $Y(P)$ by using the bounds on $\dfrac{\partial g}{\partial x}$ in the original hypothesis. Then, using Lemma 2.1.1, they find a bound on $|1-\lambda|^2$, where λ is an eigenvalue of $Y(P)$, by taking the minimum of the four positive values

$$\{4sin^2(\sqrt{PA}/2), 4sin^2(\sqrt{2PB}/2), (1-e^{-PC})^2, (1-e^{-A/C})^2\}.$$

The constants A and B are defined as

$$n^2 P < \int_0^P a(t)\,dt = A \leq B = \int_0^P b(t)\,dt < (n+1)^2 P.$$

Since the determinant of a matrix is the product of its eigenvalues, which in our case are always bounded away from zero, and since a 2×2 matrix is invertible if and only if its determinant is non-zero, we have shown that $\frac{dF}{dv}(v)$ is invertible and therefore $\|\frac{dF}{dv}(v)^{-1}\|$ is bounded by some constant M.

2.2 Hadamard's Theorem

We are now able to apply Hadamard's Theorem to $F(v)$.

Theorem 2.2.1 (Hadamard's Theorem [11]). *Assume that* $F : \mathbb{R}^n \to \mathbb{R}^n$ *is continuously differentiable on* $\mathbb{R}^n$ *and that* $\|F'(x)^{-1}\| \leq M < +\infty$ *for all* $x \in \mathbb{R}^n$. *Then* F *is a homeomorphism of* $\mathbb{R}^n$ *onto* $\mathbb{R}^n$.

This shows that $F(v)$ has a unique zero point, $v^* \in \mathbb{R}^2$. Consequently, we have shown there exists a unique periodic solution of period P to equation (1.1).

Chapter 3

The Constructive Proof of Li and Shen's Theorem

3.1 Existence of $\gamma(\delta)$

Now that we know there is a unique point $v^* \in \mathrm{IR}^2$, that satisfies $F(v) = 0$, we would like to find it numerically. To do this we form a path taking us from an initial point $v_0 \in \mathrm{IR}^2$ to the point v^*. This is done by forming the function

$$\Gamma(v, \delta) = F(v) - (1 - \delta)F(v_0), \tag{3.1}$$

and solving the equation

$$\Gamma(v, \delta) = 0 \tag{3.2}$$

for all δ between 0 and 1. When $\delta = 0$, $v = v_0$, and of course $\Gamma(v, 0) = 0$, and when $\delta = 1$, $\Gamma(v, 1) = F(v) = 0$. Thus, by starting at some point $v_0 \in \mathrm{IR}^2$, and $\delta = 0$, we construct a path of v_n's that satisfy $\Gamma(v_n, \delta_n) = 0$, and we reach our desired point v^* when $\delta_n = 1$. Before following this path, we show that the solution to equation 3.2 exists for all δ between 0 and 1. To accomplish this, we turn equation (3.2) into the solution of an initial value problem by differentiating with respect to δ. We consider v as a function of δ and have

$$\begin{aligned} 0 &= d\Gamma(v, \delta) \\ &= \frac{dF}{dv}(v)\frac{dv}{d\delta} + F(v_0) \end{aligned}$$

and since at $\delta = 0$, $v = v_0$ we have

$$\begin{cases} \dfrac{dv}{d\delta} = -\left(\dfrac{dF(v)}{dv}\right)^{-1} F(v_0), \\ v(0) = v_0. \end{cases} \tag{3.3}$$

The standard existence and uniqueness theorems imply there exists a solution for all $\delta \in [0, \epsilon]$. The difficulty lies in the possibility that ϵ may be less than 1. However, Li and Shen [7] argue that by analytic continuation, the solution domain can be extended to include $\epsilon = 1$.

If we define, for each δ, $\gamma(\delta) = v(\delta)$, where $v(\delta)$ is the corresponding value of the solution to (3.3), and thus satisfies equation (3.2), then γ defines a path with $\gamma(0) = v_0$ and $\gamma(1) = v^* = (\alpha^*, \beta^*)^T$. This provides a continuous deformation of $x(t, v_0)$ to $x(t, v^*)$ by considering $x(t, \gamma(\delta))$ as δ ranges from 0 to 1. See Figure 3.1.

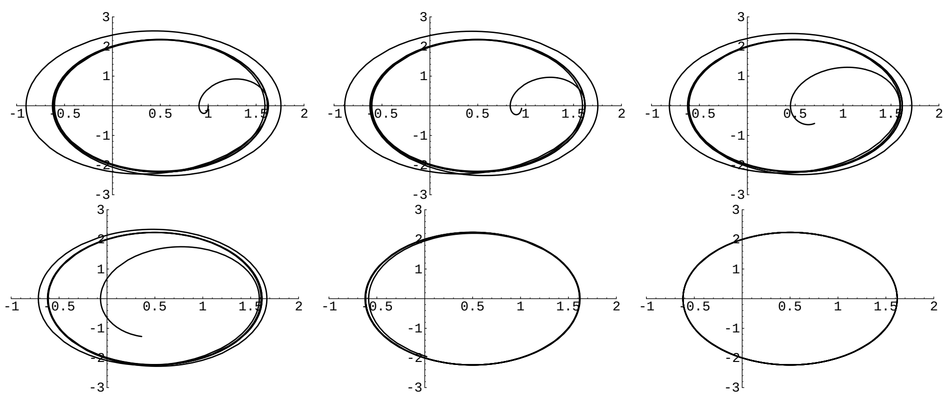

Figure 3.1: Continuous Deformation of $x(t, v_i)$ in the Phase Plane

3.2 Following $\gamma(\delta)$

Now that we know the path $\gamma(\delta)$ exists, we recast (3.3) into a form that we can solve numerically. Instead of differentiating $\Gamma(v, \delta)$ with respect to δ, we re-parameterize $\Gamma(v, \delta)$ with respect to arc length. This turns the initial value problem (3.3) into

$$\begin{cases} d\Gamma(\alpha(s), \beta(s), \delta(s))/ds = 0, \\ \alpha(0) = \alpha_0, \quad \beta(0) = \beta_0, \quad \delta(0) = 0. \end{cases} \tag{3.4}$$

where we have the additional arc length constraint

$$\| \, (\alpha'(s), \beta'(s), \delta'(s)) \, \| = 1. \tag{3.5}$$

We re-write equation (3.4) in matrix form (3.6), and implement a two step solution process.

$$\begin{cases} \begin{pmatrix} 1 - \dfrac{\partial x}{\partial \alpha}(P) & -\dfrac{\partial x}{\partial \beta}(P) & \alpha_0 - x(P, v_0) \\ -\dfrac{\partial \dot{x}}{\partial \alpha}(P) & 1 - \dfrac{\partial \dot{x}}{\partial \beta}(P) & \beta_0 - \dot{x}(P, v_0) \end{pmatrix} \begin{pmatrix} \alpha'(s) \\ \beta'(s) \\ \delta'(s) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\ \alpha(0) = \alpha_0, \quad \beta(0) = \beta_0, \quad \delta(0) = 0. \end{cases} \tag{3.6}$$

The first step of the solution process is to estimate the slope of the tangent line to the curve $\Gamma(v, \delta) = 0$ at a point $\gamma(\delta)$ which is represented by $(\alpha(s), \beta(s), \delta(s))$. When $s = 0$, we begin at $v_0 = \gamma(0)$, which is the initial condition in (3.6). We solve equations (3.6) and (3.5) simultaneously for $\alpha'(0)$, $\beta'(0)$, and $\delta'(0)$, and increment s by setting $s = h$. Then we find a point on the tangent line to $\Gamma(v, \delta)$ at v_0 using Euler's method with step size h. This new point, $\hat{v}$, is

$$(\alpha(0) + h\alpha'(0), \beta(0) + h\beta'(0))$$

and we increment δ by setting

$$\delta(h) = \delta(0) + h\delta'(0).$$

It is unlikely this new point $\hat{v}$ lies on the curve $\Gamma(v, \delta) = 0$, so the second part of our solution process is to use Newton's method starting at $\hat{v}$ to find a new $v(s)$ that satisfies (3.2). We iterate this process until $\delta(s) = 1$, at which point we have reached the end of the path $\gamma(\delta)$, and $v(s)$ will be the unique point $v^* \in \mathbb{R}^2$ that satisfies $F(v^*) = 0$. In Figure 3.2, we give a graphical interpretation of the iterative solution technique in the phase plane.

When following the path $\gamma(\delta)$, Newton's method may fail to converge due to a large Euler's step size. We insure convergence by testing the rate of convergence of Newton's method. If this convergence appears to fail, or is slow, namely if $\|\Gamma(v_{n_{k+1}}, \delta)\| > 10\|\Gamma(v_{n_k}, \delta)\|$, then we reset our Euler's step at half the step size and re-calculate the sequence of points using Newton's method to find v_{n+1}.

On the other hand, when Newton's method does converge to a point on the path $\gamma(\delta)$, we double the current step size used in Euler's method to increase its rate of convergence, especially when the curvature of $\gamma(\delta)$ is small.

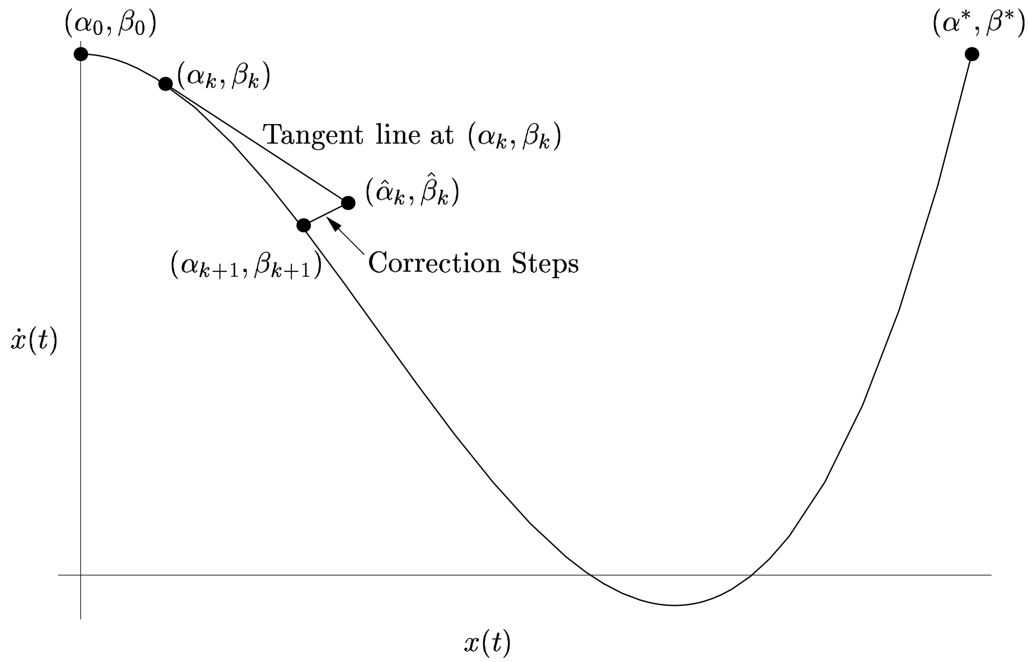

Figure 3.2: Path of Initial Positions in the Phase Plane

Chapter 4

Conclusions

In previous chapters we discussed how to form the "homotopy path" $\gamma(\delta)$. Here we give some numerical output of this algorithm using the two examples in [7] as well as an example that does not satisfy the hypothesis of Li and Shen's Theorem to show this algorithm will find a periodic solution when more than one such solution exists. In each case, we set the accepted tolerance for the convergence of Newton's method to be 1×10^{-12}; to obtain a higher precision solution, one only needs to change the `homepsilon` constant in the program to the desired accuracy. The code used for these calculations is given in the Appendix.

4.1 Linear Equations

We implemented this method to find the periodic solution of period 2π to the following equation with corresponding starting initial values

$$\begin{cases} \ddot{x} + \dot{x} + 2x = 2sin^2(t) + 3sin(2t), \\ x(0) = 1, \qquad \dot{x}(0) = 0. \end{cases} \tag{4.1}$$

It is easy to verify that equation (4.1) satisfies the original hypothesis by observing that $2sin^2(t) + 3sin(2t)$ is of period 2π and noting $\dfrac{\partial g}{\partial x} = 2$ with $1^2 \leq 2 \leq 2^2$. We then run our program and after 47 iterations we obtain the point

$$v^* = \begin{pmatrix} \alpha^* \\ \\ \beta^* \end{pmatrix} = \begin{pmatrix} 1.369 \times 10^{-11} \\ \\ -2.000 \end{pmatrix}.$$

Since (4.1) is a linear equation, we can solve for the closed form solution of this system and verify that v^* yields the unique periodic solution of period 2π. Using the method of undetermined coefficients, we find the general solution to the second order differential equation to be

$$x(t) = c_1 e^{-\frac{1}{2}t} sin(\frac{\sqrt{7}}{2}t) + c_2 e^{-\frac{1}{2}t} cos(\frac{\sqrt{7}}{2}t) + \frac{1}{2} - \frac{1}{2}cos(2t) - sin(2t).$$

Note that $x(0) = c_2$ and $\dot{x}(0) = \frac{\sqrt{7}}{2}c_1 + \frac{1}{2}c_2 - 2$, so if $x(0) = 0$, then $c_2 = 0$, and if $\dot{x}(0) = -2$, then $c_1 = 0$. Under these initial conditions the solution becomes

$$x(t) = \frac{1}{2} - \frac{1}{2}cos(2t) - sin(2t),$$

which has fundamental period of π and thus is of period 2π.

We tested the accuracy of our numerical solution by substituting α^*, β^* into our initial value problem and evaluating x and $\dot{x}$ at zero and the period as well as 5 times the period. Finding the differences we obtained

$$\begin{array}{ll} x(0) - x(2\pi) = 1.393 \times 10^{-11} & x(0) - x(10\pi) = 1.363 \times 10^{-11} \\ \dot{x}(0) - \dot{x}(2\pi) = -5.620 \times 10^{-12} & \dot{x}(0) - \dot{x}(10\pi) = -6.174 \times 10^{-14}. \end{array}$$

Figure 3.1 shows a few iterations of the continuous deformation from $x(t, v_0)$ to $x(t, v^*)$. Figure 4.1 shows the trajectories of the beginning solution and the periodic solution.

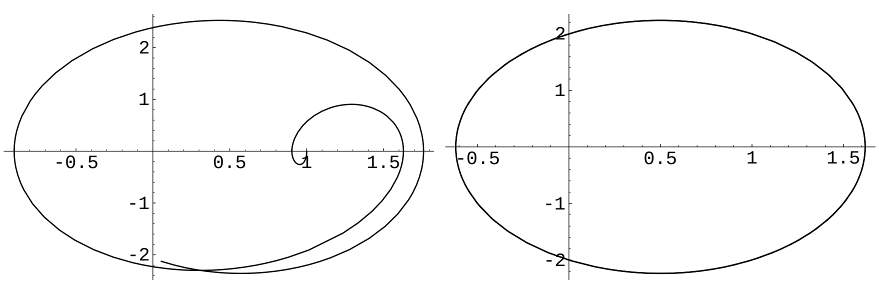

Figure 4.1: $x(t, v_0)$ and $x(t, v^*)$ in the Phase Plane $t = [0, 2\pi]$

4.2 Non-Linear Equations

4.2.1 Li and Shen equation

Next we consider a non-linear equation from [7] satisfying the original hypothesis

$$\begin{cases} \ddot{x} - \dot{x} + 2x + sin(x/\sqrt{2}) = \sqrt{2}\pi + 2sin(t) + cos(cos(\pi/4 - t)), \\ x(0) = 1, \qquad \dot{x}(0) = 0. \end{cases} \tag{4.2}$$

The hypothesis of Li and Shen's theorem is verified by noting $\sqrt{2}\pi + 2sin(t) + cos(cos(\pi/4 - t))$ has period 2π and $\dfrac{\partial g}{\partial x} = 2 + \dfrac{\sqrt{2}}{2} cos(x/\sqrt{2})$ which, as in the previous example is bounded by 1^2 and 2^2. After running our program for 47 iterations we obtained the point

$$v^* = \begin{pmatrix} \alpha^* \\ \\ \beta^* \end{pmatrix} = \begin{pmatrix} 3.221 \\ \\ 1.000 \end{pmatrix}.$$

Which yields the following numerical results

$$\begin{array}{ll} x(0) - x(2\pi) = -3.136 \times 10^{-12} & x(0) - x(10\pi) = -6.116 \times 10^{-7} \\ \dot{x}(0) - \dot{x}(2\pi) = -3.956 \times 10^{-12} & \dot{x}(0) - \dot{x}(10\pi) = 5.237 \times 10^{-7}. \end{array}$$

Figure 4.2 shows a plot of the trajectory of the beginning solution for $v = (1, 0)^T$ along with the trajectory for the periodic solution, for $v = 3.221, 1.00)^T$.

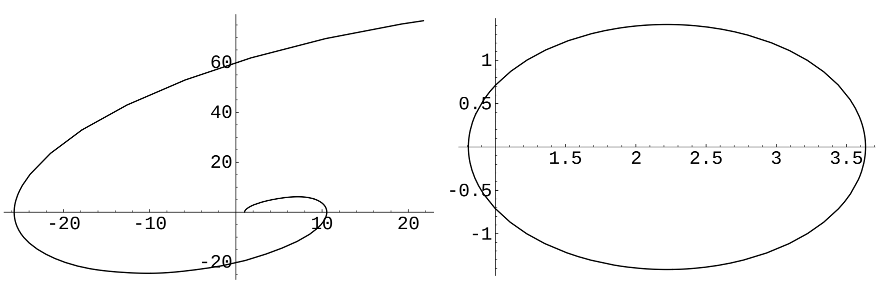

Figure 4.2: $x(t, v_0)$ and $x(t, v^*)$ in the Phase Plane $t = [0, 2\pi]$

4.2.2 A more general equation

Now that we have shown our code works for examples that satisfy the original hypothesis, we show that this method will find initial conditions that yield periodic solutions to a more general class of Duffing's equation. For this we start with

$$\begin{cases} \ddot{x} + x - \dfrac{x^3}{6} = \dfrac{1}{3} cos(3/5t), \\ x(0) = 1, \qquad \dot{x}(0) = 0. \end{cases} \tag{4.3}$$

as our initial value problem. We run our program, and after 50 iterations we obtain the point

$$v^* = \begin{pmatrix} \alpha^* \\ \\ \beta^* \end{pmatrix} = \begin{pmatrix} 0.55079 \\ \\ 2.51454 \times 10^{-12} \end{pmatrix}.$$

Once we test the accuracy and obtain

$$x(0) - x(\frac{10\pi}{3}) = -8.11018 \times 10^{-12} \qquad x(0) - x(\frac{50\pi}{3}) = 3.9968 \times 10^{-15}$$

$$\dot{x}(0) - \dot{x}(\frac{10\pi}{3}) = 7.40786 \times 10^{-12} \qquad \dot{x}(0) - \dot{x}(\frac{50\pi}{3}) = -1.1726 \times 10^{-13}.$$

Figure 4.3 shows the trajectories of the beginning solution and the periodic solution.

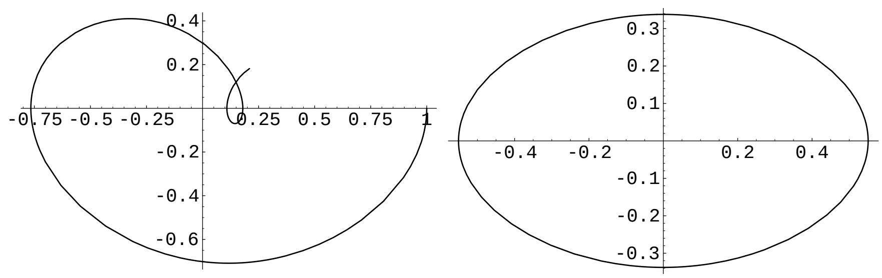

Figure 4.3: $x(t, v_0)$ and $x(t, v^*)$ in the Phase Plane $t = [0, 2\pi]$

4.3 Summary

We found the homotopy method to be a quick and accurate way to find periodic solutions to nonlinear differential equations. The method requires only that the matrix

$$\begin{pmatrix} 1 - \frac{\partial x}{\partial \alpha}(P) & -\frac{\partial x}{\partial \beta}(P) \\ -\frac{\partial \dot{x}}{\partial \alpha}(P) & 1 - \frac{\partial \dot{x}}{\partial \beta}(P) \end{pmatrix} \tag{4.4}$$

be non-singular in order for $F(v)$ to be a local homeomorphism, and hence for a homotopic path to exist from a given initial condition to the initial condition that yields a periodic solution.

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Appendix A

Nonlinear Homotopy Code

A.1 A General Case

```
(*Initialization*)
Clear[A, alpha, beta, gamma, alphadot, betadot, gammadot, n]
epsilon = 1/10^6;
homepsilon = 1/10^12;
h = 1/10;
step = 1/10;
maxsteps = 500;
alpha0 = 1;
beta0 = 0;
gamma0 = 0;
n = 0;
P = 10*Pi/3;
solution0 =
     NDSolve[{x'[t] == y[t],
     y'[t] == -x[t] +x[t]^3/6 +1/3*Cos[3/5*t],
     x[0] == alpha0, y[0] == beta0}, {x, y}, {t,0, P},
     WorkingPrecision -> 26, MaxSteps -> Infinity];
faz = alpha0 - Evaluate[x[P]/. solution0];
fbz = beta0 - Evaluate[y[P]/. solution0];
alpha = alpha0;
beta = beta0;
gamma = gamma0;
u = {alpha, beta, gamma};
Hpu = Hprime[u];
alphadot = Hpu[[2]];
betadot = Hpu[[3]];
gammadot = Hpu[[4]];
If[gammadot < 0, {alphadot = -alphadot, betadot = -betadot, gammadot = -gammadot}];

(*End Initialization*)

(*Predictor Corrector Method*)

alphaold = alpha;
betaold = beta;
gammaold = gamma;
alpha = alphaold + step*alphadot;
beta = betaold + step*betadot;
gamma = gammaold + step*gammadot;
Print["Euler's Prediction : ", alpha, " ", beta, " ", gamma];
Clear[q];
q = Correct[alpha, beta, gamma];
w = q[[2]];
```

```
alpha = w[[1]];
beta = w[[2]];
gamma = w[[3]];
If[q[[1]] == True,
    { step = 2*step, Print["New Point Found: ", alpha, " ", beta, " ", gamma]
    },
    { step = step/2, alpha = alphaold, beta = betaold, gamma = gammaold,
      Print["Step size reduced to ", step]
    }
];



For[n = 1, n < maxsteps && gamma < 1, n++,
    {u = {alpha, beta, gamma},
    Hpu = Hprime[u],
    alphadottest = Hpu[[2]],
    betadottest = Hpu[[3]],
    gammadottest = Hpu[[4]],

    alphaold = alpha,
    betaold = beta,
    gammaold = gamma,
    alpha = alphaold + step*alphadot,
    beta = betaold + step*betadot,
    gamma = gammaold + step*gammadot,
    Print["Euler's Prediction : ", alpha, " ", beta, " ", gamma],

    If[gamma < 1, {
        q = Correct[alpha, beta, gamma],
        w = q[[2]],
        alpha = w[[1]],
        beta = w[[2]],
        gamma = w[[3]],

        If[q[[1]] == True,
          {step = 2*step,
            Print["New Point Found: ", alpha, " ", beta, " ", gamma]}, {step =
               step/2, alpha = alphaold, beta = betaold, gamma = gammaold,
            Print["Step size reduced to ", step]}
          ]
        },
      {step = step/2, alpha = alphaold, beta = betaold, gamma = gammaold}
      ]


    }]
(*End Predictor Corrector Method*)


(*Newton's Correction Method*)
(*   In this method, we recieve an Euler approximation (alpha, beta, gamma),
     and try to correct it with a sequence of Newton's steps towards the homotopic path
*)

Correct[alpha_, beta_, gamma_] := Module[{Hpu, v, Hpv, Hom, test},
    v = {alpha, beta};
    Hpv = Hprime[v][[1]];
```

```
    Hom = Homotopy[v, gamma];
    Homv = {Hom[[1]][[1]][[1]], Hom[[2]][[1]][[1]]};


    (*Newton's Step*)
    w = v - Inner[Times, Inverse[Hpv], Homv];
    Hom = Homotopy[w, gamma];
    Homw = {Hom[[1]][[1]][[1]], Hom[[2]][[1]][[1]]};


    Print["H(v)", Homv];
    Print["H(w)", Homw];
    diff = Sqrt[(Homw[[1]]^2 + Homw[[2]]^2)];
    Print["diff ", diff];

(*      If Newtons step is correcting our approximation i.e. making the homotopy evaluate closer
        to zero, then we continue to make Newton's steps. If, however, Newton's method is not
        returning better approximations i.e. not converging to the homotopic path, we reduce
        our step size and make a new Euler's prediction step
*)

    test = 0;

    While[ diff > homepsilon,
      test = 1;
      Print["Newtons step:  ", w[[1]], " ", w[[2]], " ", gamma];
      v = w;
      Homv = Homw;
      Hpv = Hprime[v][[1]];
      w = v - Inner[Times, Inverse[Hpv], Homv];
      Hom = Homotopy[w, gamma];
      Homw = {Hom[[1]][[1]][[1]], Hom[[2]][[1]][[1]]};
      (*Here we can see how fast Newton's method is converging*)
      Print["H(v)", Homv];
      Print["H(w)", Homw];
      diff = Sqrt[(Homw[[1]]^2 + Homw[[2]]^2)];
      Print["diff ", diff];
    ];


    If[test == 0, u = {False, {alpha, beta, gamma}},
      u = {True, {v[[1]], v[[2]], gamma}}]; u]

(*End of Newtons Correction Method*)



(*Homotopy Evaluation Method*)
(*  Evaluates the value of the F(v)-(1-gamma)F(v0) at a given
values of v and gamma
*)

Homotopy[v_, gam_] := Module[{alpha, beta, gamma, solution, H1, H2},
    alpha = v[[1]];
    beta = v[[2]];
    gamma = gam;

    solution =
```

```
        NDSolve[{x'[t] == y[t],
        y'[t] == -x[t] +x[t]^3/6 +1/3*Cos[3/5*t],
        x[0] == alpha, y[0] == beta}, {x, y}, {t, 0, P},
        WorkingPrecision -> 26, MaxSteps -> Infinity];

    H1 = alpha - Evaluate[x[P] /. solution] - (1 - gamma)*(alpha0 - Evaluate[X[0, P] /. solution0]);
    H2 = beta - Evaluate[y[P] /. solution] - (1 - gamma)*(beta0 - Evaluate[Y[0, P] /. solution0]);

    {H1, H2}
]

(*End of Homotopy Evaluation Method*)


(* Runga Kutta Method *)
(*   Runga Kutta 4 ODE system solver
     Passed:
     f is the rhs of the first system
     g is the rhs of the second system
     a is the time where we begin
     b is the time where we end
     xa is the initial displacement
     ya is the initial velocity
     h is the uniform step size to be used

     Returned
     T is the time matrix which is partitioned in steps of h between a and b
     X is the matrix of x(t) for each time in t
     Y is the matirx y(t) for each time in t
*)

Rk4sys[f_, g_, a_, b_, xa_, ya_, h_] :=
  Module[{T, X, Y, M, f1, g1, f2, g2, f3, g3, f4, g4, i, j},
    M = Ceiling[(b - a)/h];
    ClearAll[T, X, Y];
    Array[T, M + 1];
    Array[X, M + 1];
    Array[Y, M + 1];

    For[i = 1, i <= M, i++,
      {T[i + 1] = i*h;
        X[i + 1] = 0;
        Y[i + 1] = 0;
        }];
    T[1] = 0;
    X[1] = xa;
    Y[1] = ya;

    For[j = 1, j <= M, j++,
      {
        f1 = N[f[T[j], X[j], Y[j]]];
        g1 = N[g[T[j], X[j], Y[j]]];

        f2 = N[f[T[j] + h/2, X[j] + f1*h/2, Y[j] + f1*h/2]];
        g2 = N[g[T[j] + h/2, X[j] + g1*h/2, Y[j] + g1*h/2]];

        f3 = N[f[T[j] + h/2, X[j] + f2*h/2, Y[j] + f2*h/2]];
        g3 = N[g[T[j] + h/2, X[j] + g2*h/2, Y[j] + g2*h/2]];
```

```
        f4 = N[f[T[j] + h, X[j] + f3*h, Y[j] + f3*h]];
        g4 = N[g[T[j] + h, X[j] + g3*h, Y[j] + g3*h]];

        X[j + 1] = X[j] + (f1 + 2*f2 + 2*f3 + f4)*h/6;
        Y[j + 1] = Y[j] + (g1 + 2*g2 + 2*g3 + g4)*h/6;
        }];
    {T, X, Y}
    ]

(* End Runga Kutta Method *)


(* Hprime method *)
(*    Computes H' in terms of the Matrix A and alphadot, betadot and
      gammadot at a given initial condition vector v
*)

Hprime[v_] :=
  Module[{alpha, beta, gamma, solutionaph, solutionap2h, solutionamh,
      solutionam2h, solutionbph, solutionbp2h, solutionbmh, solutionbm2h,
      pxal, pyal, pxbt, pybt, fa, fb, ans, alphadottest, betadottest,
      gammadottest},

    alpha = v[[1]];
    beta = v[[2]];

    f[t_, x_, y_] := y;
    g[t_, x_, y_] :=-x+x^3/6+1/3*Cos[3/5*t];
    a = 0;
    b = P;
    xa = alpha;
    ya = beta;
    h = .01;
    M = Ceiling[(b - a)/h];
    soln = Rk4sys[f, g, a, b, xa, ya, h];
    solnaph = Rk4sys[f, g, a, b, xa + h, ya, h];
    solnap2h = Rk4sys[f, g, a, b, xa + 2*h, ya, h];

    solnbph = Rk4sys[f, g, a, b, xa, ya + h, h];
    solnbp2h = Rk4sys[f, g, a, b, xa, ya + 2h, h];

(* Here we use second order forward differencing to compute our partial derivatives *)

    pxal = Table[{soln[[1]][i],
           ( -3*soln[[2]][i] + 4*solnaph[[2]][i] - solnap2h[[2]][i])/(2*h)}, {i, 1, M}];
    pyal = Table[{soln[[1]][i],
           ( -3*soln[[3]][i] + 4*solnaph[[3]][i] - solnap2h[[3]][i])/(2*h)}, {i, 1, M}];
    pxbt = Table[{soln[[1]][i],
           ( -3*soln[[2]][i] + 4*solnbph[[2]][i] - solnbp2h[[2]][i])/(2*h)}, {i, 1, M}];
    pybt = Table[{soln[[1]][i],
           ( -3*soln[[3]][i] + 4*solnbph[[3]][i] - solnbp2h[[3]][i])/(2*h)}, {i, 1, M}];


(* We re-write these because mathematica's subscript syntax is difficult to follow *)

    pxa = pxal[[M]][[2]];
    pxb = pxbt[[M]][[2]];
```

```
    pya = pyal[[M]][[2]];
    pyb = pybt[[M]][[2]];
    fa = faz[[1]];
    fb = fbz[[1]];

    x1 = 1 - pxa;
    x2 = -pya;
    y1 = -pxb;
    y2 = 1 - pyb;

    A={{x1,y1},{x2,y2}};

(* Here we solve for alphadot, betadot, and gammadot using Cramer's rule*)

    z1 = fa;
    z2 = fb;
    d = x1*y2 - x2*y1;

    gammadot = 1/Sqrt[(z2*y1 - z1*y1 )^2/d^2 + (z1*x2 - z2*x1)^2/d^2 + 1];
    alphadot = gammadot*(z2*y1 - z1*y2)/d;
    betadot = gammadot*(z1*x2 - z2*x1)/d;


     {A, alphadot, betadot, gammadot}
    ]

(*End of Hprime Method*)
```