Computational Determination of Orlov Volumes

S.D. Metzler, J.E. Bowsher and R.J. Jaszczak

Abstract— **Objective: Orlov has derived the sufficient conditions for adequate two-dimensional projection sampling of a three-dimensional density function in order to reconstruct that density function. This condition may be represented as a curve of vantage angles on a unit sphere of directions. Orlov's condition states that the density function can be unambiguously reconstructed if the curve of vantage angles intersects all great circles on the sphere. The set of points for which Orlov's condition is met may be termed the Orlov volume. Although this volume can be intuitively determined for simple sampling orbits, there is no known algorithm in the literature for determining the volume for the generalized case. Further, the Orlov volume principle may be applied to converging and diverging collimators, in addition to parallel-hole collimators. Methods: We consider a voxelized representation of the volume inside an orbit of a given collimator type. We then construct a digitized version of the vantage points of the voxel for a given camera orbit. We then determine if any great circles can exist on the Orlov sphere without intersecting the vantage curve. Results: We have implemented this algorithm in C++ using Object-Oriented programming techniques. The algorithm considers generic collimator types, of which we have currently implemented slant-hole, parallel-hole and pinhole collimators. Other collimator types can be added without modification to the algorithm. Multiple orbits can be simultaneously considered. Multiple collimator types can also be simultaneously considered. The output is the voxelized volume that meets Orlov's condition. Conclusions: The algorithm has successfully determined the Orlov volume in cases that are easily verified intuitively. It has been used to study the more complex scenarios of pinhole collimators following spiral orbits and simultaneous acquisition of parallel-hole and slant-hole collimators. This technique may be useful for studying sufficient orbits and understanding sampling artifacts.**

Keywords— **Orlov, Reconstruction, Sufficient Sampling**

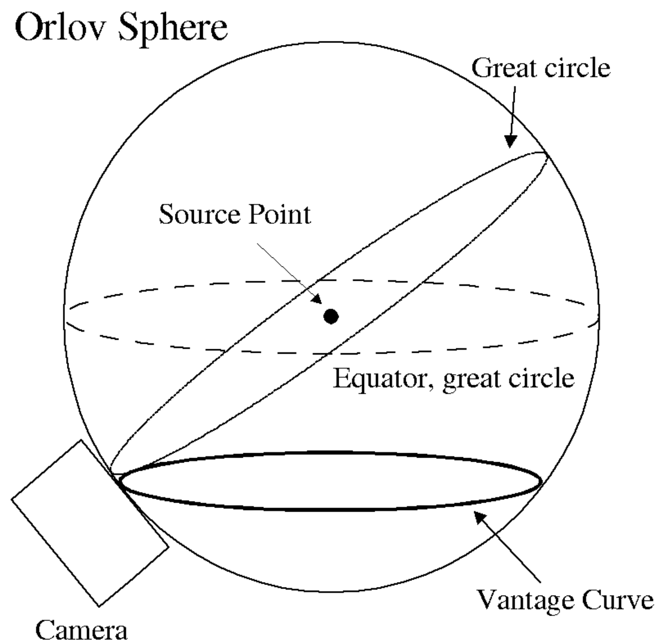

Fig. 1. Orlov Sphere and Vantage Curve for Slant-hole Collimator. The figure depicts the vantage curve for a slant-hole collimator as it observes a source point at the center of the sphere. The camera follows a circular orbit that coincides with the vantage curve. It is possible to draw great circles on this sphere that do not intersect the vantage curve. This vantage curve is the same as the vantage curve of a tilted parallel-hole collimator following the same orbit.

I. Introduction

In his work with electron microscopy, Orlov derived the sufficient-sampling condition for three-dimensional reconstruction from projection data [1]. He stated his condition geometrically: the curve of vantage angles on a unit sphere of directions must "have points in common with any arc of a great circle [1]." If this condition is met, the sampled density function can be unambiguously reconstructed.

Two Orlov spheres are shown in Figures 1 and 2. Figure 1 shows the vantage curve for a slant-hole or tilted parallel-hole collimator following an orbit that coincides with the vantage curve. It is possible to draw great circles on the Orlov sphere that do not intersect the vantage curve. The projection data derived from this orbit would be insufficient to unambiguously reconstruct the source. In Figure 2, the vantage curve for a parallel-hole collimator is shown. The collimator is not tilted. The vantage curve intersects all great circles on the sphere.

Tuy and Smith realized that Orlov's condition is met by a limited set of points that will be referred to in this paper as the Orlov volume [2,3]. It has been observed that reconstruction artifacts occur in regions outside the Orlov volume [4].

The Orlov volume can be determined intuitively in the cases of simple collimator types, such as parallel-hole, following simple circular orbits. The symmetry of the sampling makes the Orlov volume cylindrical in this case. However, there is no known algorithm in the literature for determining the algorithm for the generalized case of non-parallel-hole collimators and non-circular orbits.

A method has been developed to computationally determine the Orlov volume for any set of collimators following any set of orbits. Herein this method will be described.

II. Algorithm

A voxelized representation of the volume to evaluate is created. Then a set of collimator models is constructed to simulate the positions, orientations and spatial extents of the collimators. For example, a single parallel-hole collimator following an orbit that includes m projection views would be represented by m collimator models. Two collimators with m and n projection views, respectively, would

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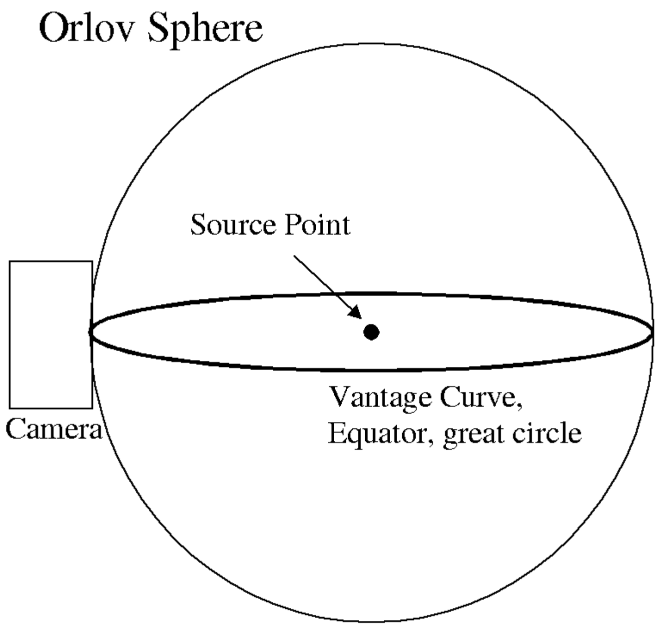

Fig. 2. Orlov Sphere and Vantage Curve for Parallel-hole Collimator. The figure depicts the vantage curve for the parallel-hole collimator as it observes a source point at the center of the sphere. The camera follows a circular orbit that coincides with the the equator of the sphere. The vantage curve coincides with the equator. All great circles on this sphere intersect the vantage curve.

be represented by $m+n$ collimator models. For each voxel, a digitized version of the vantage points of the voxel are determined from the set of collimator models. The digitized vantage curve is then evaluated to determine if any great circles can exist on the Orlov sphere without intersecting the vantage curve.

This algorithm has been implemented in C++ using Object-Oriented programming techniques. When started, the program allocates and initializes a boolean matrix representing the voxelized volume to consider. It then reads one or several orbit files and constructs a set of collimator representations. All detector representations obey an abstract interface that determines if a given voxel is within the field-of-view of the collimator and the vantage angles for that voxel.

The use of an abstract interface makes it possible to model multiple collimator types simultaneously and extend the program to consider new collimator types without any change to the algorithm. The new collimator simply needs to implement this interface. Multiple collimators can be used because each projection view in the orbit file or files results in the construction of a new collimator representation. All the representations are stored and considered together when evaluating whether a voxel meets Orlov's criteria.

Each voxel is fully evaluated by considering the vantage curve generated by all the projection views before the next voxel is considered. This reduces the required memory considerably. For each projection view, it is determined if the voxel in question is seen by the detector at that view. If it is seen, the vantage angles are recorded to make a vantage curve.

After all the projection views have been considered for a voxel, the vantage curve is evaluated to determine if any great circles can exist on the sphere without intersecting the vantage curve. The parameterization of a generic great circle (Fig. 3) can be found by considering that the points with the minimum and maximum z values are

$$\begin{aligned} \vec{r}_{\max} &= (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \\ \vec{r}_{\min} &= (-\sin\theta\cos\phi, -\sin\theta\sin\phi, -\cos\theta). \end{aligned} \tag{1}$$

A normal to the circle can be parameterized as

$$\begin{aligned} \hat{N} &= \left(\sin\left(\theta+\frac{\pi}{2}\right)\cos\phi, \sin\left(\theta+\frac{\pi}{2}\right)\sin\phi, \cos\left(\theta+\frac{\pi}{2}\right)\right) \\ &= (\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta). \end{aligned} \tag{2}$$

The normal can be used to determine the basis vector orthogonal to $\vec{r}_{\max}$, $\vec{r}_{\min}$, and $\hat{N}$.

$$\begin{aligned} \hat{b} &= \vec{r}_{\max} \times \hat{N} \\ &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \end{vmatrix} \\ &= -\hat{x}\sin\phi + \hat{y}\cos\phi \end{aligned} \tag{3}$$

Any point on the great circle can be parameterized as

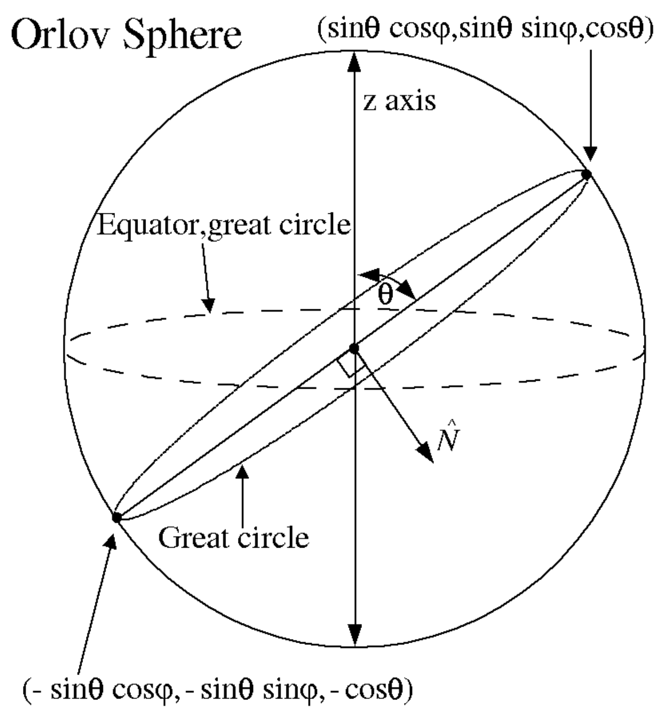

Fig. 3. Generic Great Circle on an Orlov Sphere. The point $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ has the maximum value of z on this curve. The point $(-\sin\theta\cos\phi, -\sin\theta\sin\phi, -\cos\theta)$ has the minimal value of z on this curve. The normal to the plane of this great circle is $\hat{N}$.

$$\vec{r} = \alpha\vec{r}_{\max} + \beta\hat{b}. \quad (4)$$

Since $|\vec{r}| = 1$, $|\vec{r}_{\max}| = 1$, $|\hat{b}| = 1$, and $\vec{r}_{\max}$ and $\hat{b}$ are orthogonal,

$$\alpha^2 + \beta^2 = 1. \quad (5)$$

Letting $\alpha = \sin\gamma$ and $\beta = \cos\gamma$,

$$\vec{r} = (\sin\gamma\sin\theta\cos\phi - \cos\gamma\sin\phi, \sin\gamma\sin\theta\sin\phi + \cos\gamma\cos\phi, \sin\gamma\cos\theta), \quad (6)$$

where θ and ϕ are the coordinates of the point of maximum z on the curve and γ parameterizes the curve. Equation 6 is used to evaluate if any great circles can exist on the sphere without intersecting the vantage curve.

III. Results

The program has been tested using parallel-hole collimators, slant-hole collimators, pinhole collimators and combinations of the above. Example volumes are described below.

A. Parallel-hole Collimators with Circular Orbits

Parallel-hole collimators with circular orbits have been evaluated and give cylindrical Orlov volumes (Fig. 4), as expected. The radius of the cylindrical Orlov volume in this case is larger than the collimator's radius of rotation (ROR). Clinically, there generally would not be activity outside the ROR since the gamma camera passes through that region, but, as the program indicates, that region is sufficiently sampled. This is because the Orlov volume of a parallel-hole collimator following a circular orbit depends only on the detector dimensions and not the orbit dimensions. That volume is cylindrical with extent given by $\pi w^2 d/4$, where w is the width of the detector in the plane of the orbit and d is the depth of the detector normal to the plane of the orbit.

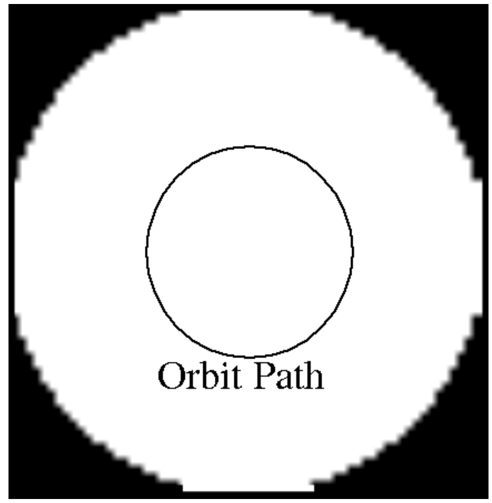

Fig. 4. Orlov Volume for Parallel-hole Collimator with a Circular Orbit. The voxelized volume meeting the Orlov condition for a circular orbit of an untilted parallel-hole collimator is shown in white. The collimator's orbit is shown by the superimposed circle.

B. Pinhole Collimators

The algorithm can also be used to determine sufficiently sampled volumes for converging-beam and diverging-beam collimators. The pinhole collimator has been tested with a circular orbit, giving a circular slice as the Orlov volume. This was expected, because all voxels that are off-axis have insufficient data. The pinhole-collimator has also been tested with a spiral orbit (Fig. 5) and gives a nearly cylindrical volume. The defects in the cylinder are at the ends, as expected because of the strong dependence on the initial orientation of the camera.

C. Combined Parallel-Hole and Tilted Parallel-Hole

Combined parallel-hole and tilted parallel-hole have been used to understand Orlov volumes in conjunction with breast-imaging research using vertical-axis-of-rotation (VAOR) orbits [4]. A digital phantom with a single pendulous breast was used to study the utility of parallel-hole acquisition around the torso using a standard horizontal-axis-of-rotation (HAOR) orbit versus tilted parallel-hole acquisition around the breast using a VAOR orbit. The VAOR orbit allows the camera to be positioned nearer the breast to improve spatial resolution, but the HAOR orbit gives better Orlov sampling. Fig. 6 shows the Orlov volume for a tilted parallel-hole acquisition with two additional arcs. Notice that the VAOR orbit does not yield a volume that includes the entire breast. Fig. 7 shows the Orlov volume for the tilted parallel-hole acquisition combined with a parallel-hole orbit extending around the posterior side of the patient. The VAOR orbit gives a much larger Orlov volume.

IV. Discussion

Orlov's condition assumes continuous angular sampling and infinitesimal sampling bins. Both of these conditions

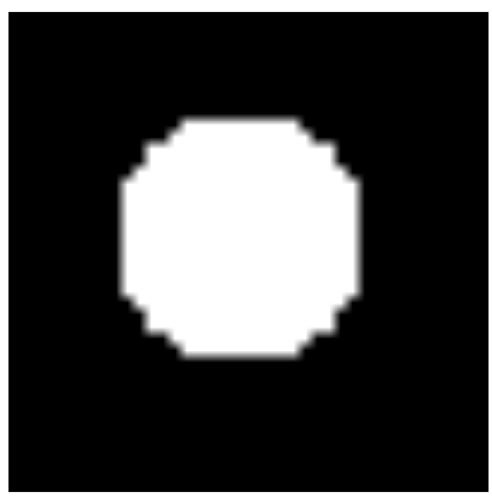

Fig. 5. Orlov Volume for Helical Pinhole-Collimator Orbit. A slice of the voxelized volume meeting the Orlov condition for a helical orbit of a pinhole collimator is shown in white. The volume is nearly cylindrical, except for defects at the ends (not shown) where the valid volume depends on the initial angle of acquisition.

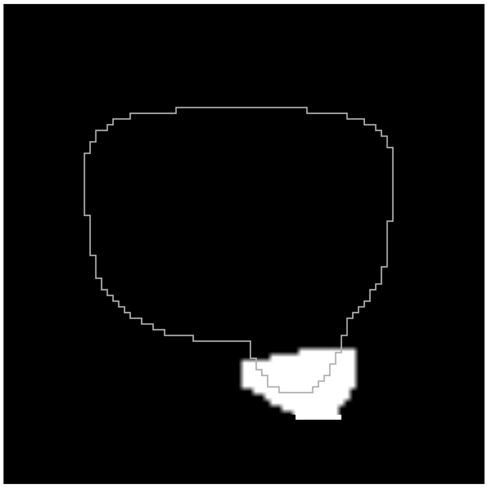

Fig. 6. Orlov Volume for Vertical-Axis-of-Rotation Tilted Parallel-Hole Collimator Orbit. A parallel-hole collimator was used to sample regions in and near the breast. The tilted acquisition was combined with additional arcs near the sternum and lateral side of the breast. A slice of the volume meeting the Orlov condition is shown as white. The outline of the digital phantom is also shown.

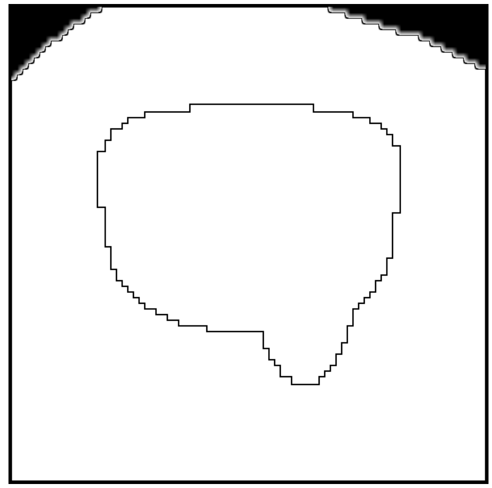

Fig. 7. Orlov Volume for Combined HAOR-VAOR Orbits. The same orbit described for Fig. 6 was supplemented with a standard HAOR orbit around the patient. The additional orbit greatly increases the size of the sufficiently sampled volume. The outline of the digital phantom is also shown.

are invalid in realistic acquisitions. By Nyquist's theorem, there are limitations in reconstruction resolution due to discrete sampling. The discrete nature of the sampling merges the concepts of Nyquist frequency and the Orlov volume. This work represents an important future development.

The current algorithm has successfully determined the Orlov volume in cases that are easily verified intuitively. It has been used to study the more complex scenarios of pinhole collimators following spiral orbits and simultaneous acquisition of parallel-hole and slant-hole collimators. Further, it has improved understanding of artifacts found in VAOR acquisitions of the breast. This technique may be useful for studying sufficient orbits and understanding sampling artifacts for complex projection acquisitions.

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