A Table of Analytical Discrete Fourier Transforms

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Abstract

While most people rely on numercal methods (most notably the fast Fourier transform) for computing discrete Fourier transforms (DFTs), there it is still an occasional need to have analytical DFTs close at hand. Such a table of analytical DFTs is provided in this paper, along with comments and observations, in the belief that it will serve as a useful resource or teaching aid for Fourier practioners.

1 Introduction

The table of discrete Fourier transforms (DFTs) that comprises most of this paper was assembled using analytical methods. The table is not exhaustive, since a good symbolic package or additional patience could certainly produce a few more DFT pairs. However it does include many commonly encountered input sequences.

A few words of explanation are in order. We assume that a function f is sampled on the interval $[-A/2, A/2]$ at N equally spaced points to produce the sequence f_n, where $n = -N/2+1 : N/2$. Of utmost importance is the fact that, when extended, the sequence f_n is N-periodic. This means that if the A-periodic extension of f is continuous at $x = \pm A/2$, then

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$f_{\pm N/2} = f(\pm A/2)$. On the other hand, if the A-periodic extension of f is not continuous at $x = \pm A/2$, then $f_{\pm N/2}$ must be defined as the average value of $f(\pm A/2)$. Similarly, at *any* point of discontinuity in $[-A/2, A/2]$, the sequence of samples f_n must be assigned the average of the function values across the discontinuity. This proviso is given the name *AVED: average values at endpoints and discontinuities*, and is noted in the table.

We can now define the DFT. The forward DFT is given by

$$F_k = \frac{1}{N} \sum_{n=-\frac{N}{2}+1}^{\frac{N}{2}} f_n \omega_N^{-nk}$$

for $k = -N/2+1 : N/2$, where i is the imaginary unit and $\omega_N \equiv e^{i2\pi/N}$, which defines another N-periodic seqeunce F_k. The choice of using a centered interval is somewhat arbitrary. Because of the periodicity of f_n and F_k, the DFT can be defined on any N consecutive points. For the record, the corresponding inverse DFT is given by

$$f_n = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} F_k \omega_N^{nk}$$

for $n = -N/2+1 : N/2$. The table also lists the Fourier coefficients of each input function on the interval $[-A/2, A/2]$,

$$c_k = \frac{1}{A} \int_{-\frac{A}{2}}^{\frac{A}{2}} f(x) e^{-i2\pi kx/A}\, dx \quad \text{for} \quad k = 0, \pm 1, \pm 2, \ldots,$$

in order to compare F_k with c_k.

Each entry of the table is arranged as follows.

Discrete input name	$f_n,\ n \in \mathcal{N}$	
Graph of f_n	$F_k,\ k \in \mathcal{N}$	Graph of F_k
	$\lvert c_k - F_k \rvert,\ k \in \mathcal{N}$	
Continuum input name	$f(x),\ x \in I$	
Graph of $f(x)$	$c_k,\ k \in \mathbf{Z}$	Graph of $\lvert c_k - F_k \rvert$
	Comments	$\max \lvert c_k - F_k \rvert$

The first column has two boxes. The upper box gives the name of the input, below which are graphs of the real and imaginary parts of the discrete input sequence. The lower box contains the name of the continuum input, and the corresponding continuum input graphs. The middle column has six boxes containing, in order from top to bottom, the formula of the input sequence f_n; the analytic N-point DFT output F_k; a measure of the error $|c_k - F_k|$; the formula of the continuum input function $f(x)$; the formula for the Fourier coefficients c_k; an entry for comments, perhaps the most important of which is the AVED warning. This means that *average values at endpoints and discontinuities* must be used if the correct DFT is to be computed. The third column consists of two boxes. The upper box displays graphically the real and imaginary parts of the DFT. The lower box gives the maximum error $\max|c_k - F_k|$, and displays graphically the error $|c_k - F_k|$ for a small (24-point) example.

A few comments and observations on the entries of the table might be useful.

- **Symmetry.** The well-known symmetries of the DFT and the Fourier coefficients are evident in the entries of the table: if f_n is a real, even sequence ($f_{-n} = f_n$), then the resulting DFT sequence is also real and even; if f_n is a real, odd sequence ($f_{-n} = -f_n$), then the resulting DFT sequence is pure imaginary and odd.

- **Exact DFTs.** The DFT is exact, meaning it reproduces the first N Fourier coefficients exactly, in several entries of the table. As shown in cases 1–4, if the input sequence is periodic and does not consist of frequencies greater than $\omega_{max} = N/A$ cycles per unit length (or time), then the DFT is exact. This condition is just the Nyquist sampling condition.

- **Errors.** The table gives estimates of the difference $|c_k - F_k|$ which should be interpreted in an asymptotic sense. If $|c_k - F_k|$ is given as CN^{-p}, the meaning is that

$$\lim_{N\to\infty} \frac{|c_k - F_k|}{N^p} = C,$$

where $0 < C < \infty$ is a constant. These estimates are obtained by a liberal use of Taylor series for large N. This asymptotic measure of error conforms with the pointwise errors in the DFT which can be obtained from the Poisson summation formula. Roughly speaking, if the periodic extension of f has $p-1$ continuous derivatives, and $f^{(p)}$ is bounded and piecewise monotone, then

$$|c_k - F_k| \leq \frac{C}{N^{p+1}} \quad \text{for} \quad k = -\frac{N}{2}+1 : \frac{N}{2},$$

where C is a constant *for all* N [1]. The case $p=0$ corresponds to functions f whose periodic extension is only piecewise continuous. A similar bound holds for DFT approximations of the Fourier transform of compactly supported functions.

For example, cases 5, 6, 6c, 7, 9, 10, 10a, 11, 13, 14, and 15b of the table are piecewise continuous functions ($p=0$), and all show asymptotic errors of the form CN^{-1} or CkN^{-2}. The latter term means that for the low frequency coefficients ($|k| \sim O(1)$), the error can decrease as CN^{-2}; however, for the high frequency coefficients ($|k| \sim O(N/2)$), the error decreases as CN^{-1}. In cases 6b, 6d, 7, 8, 12, and 15a, the derivative of the input function is piecewise continuous ($p=1$), and the errors decrease as CN^{-2}. For input functions with higher degrees of smoothness, analytical DFTs are difficult to compute, however numerical experiments confirm the error estimates for larger values of p.

2 The Table of DFTs

The following notational conventions hold throughout the *Table of DFTs*:

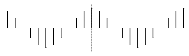
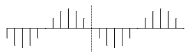
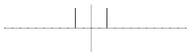
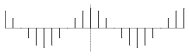
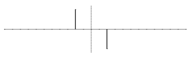
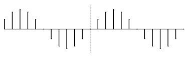
$$\mathcal{N} = \left\{-\frac{N}{2}+1, \ldots, \frac{N}{2}\right\}, \quad \mathcal{I} = \left[-\frac{A}{2}, \frac{A}{2}\right], \quad \mathbf{Z} = \{0, \pm 1, \pm 2, \ldots\};$$

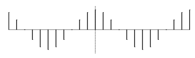
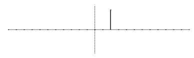
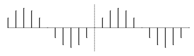
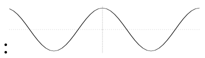
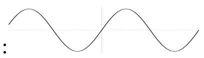
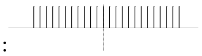
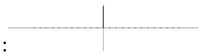
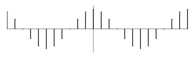
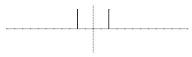
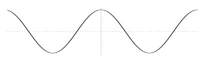
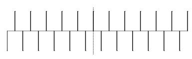
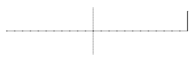
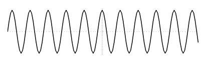
$$\omega_N = e^{i\frac{2\pi}{N}}, \quad \theta_k = \frac{2\pi k}{N}, \quad \delta(k) = \begin{cases} 1 & \text{if} \quad k=0, \\ 0 & \text{if} \quad k \neq 0; \end{cases}$$

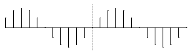
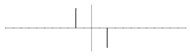
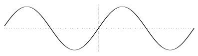
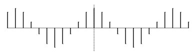
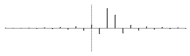
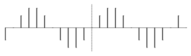
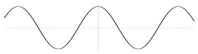
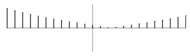
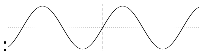
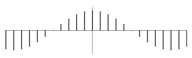
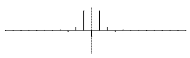
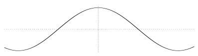
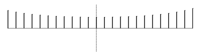
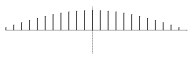
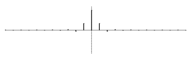
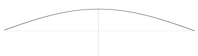
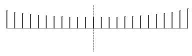
$$\hat{\delta}_N(k) = \begin{cases} 1 & \text{if} \quad k = 0 \text{ or a multiple of } N, \\ 0 & \text{otherwise.} \end{cases}$$

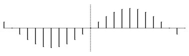
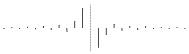
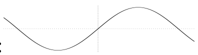
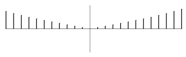
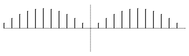
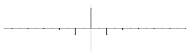
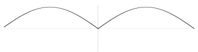
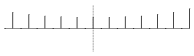
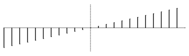
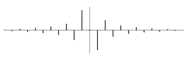
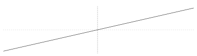
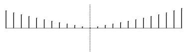
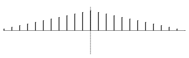
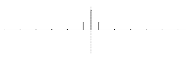
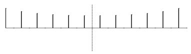
AVED = average values at endpoints and discontinuities,
C is a constant independent of k and N.

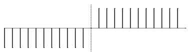
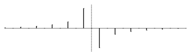
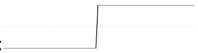
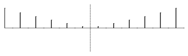
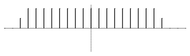
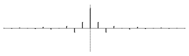
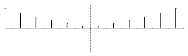
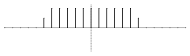
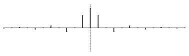
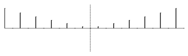
TABLE OF DFTS

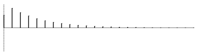
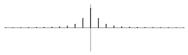
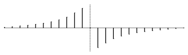
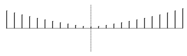
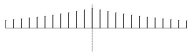
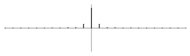
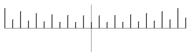
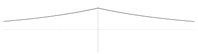
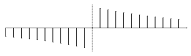
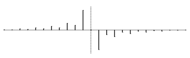
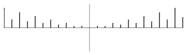
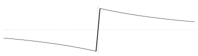
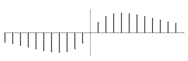
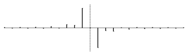
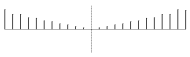
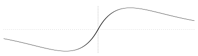
Discrete input name	$f_n,\ n \in \mathcal{N}$	
Graph of f_n	$F_k,\ k \in \mathcal{N}$	Graph of F_k
	$\lvert c_k - F_k \rvert,\ k \in \mathcal{N}$	
Continuum input name	$f(x),\ x \in I$	
Graph of $f(x)$	$c_k,\ k \in \mathbf{Z}$	Graph of $\lvert c_k - F_k \rvert$
	Comments	$\max \lvert c_k - F_k \rvert$
1. Impulse	$\delta(n - n_0)$	F_k:
$\mathcal{R}$: 	$\frac{1}{N}\omega_N^{-n_0 k}$	$\mathcal{R}$: 
$\mathcal{I}$:	Exact	$\mathcal{I}$: 
1. None	–	
	–	
	$n_0 \in \mathcal{N}$	
2a. Paired impulses	$\frac{1}{2}(\delta(n - n_0) + \delta(n + n_0))$	F_k:
$\mathcal{R}$: 	$\frac{1}{N}\cos(\frac{2\pi n_0 k}{N})$	$\mathcal{R}$: 
$\mathcal{I}$:	Exact	$\mathcal{I}$:
2a. None	–	
	–	
	$n_0 \in \mathcal{N}$	
2b. Paired impulses	$\frac{1}{2}(\delta(n + n_0) - \delta(n - n_0))$	F_k:
$\mathcal{R}$: 	$\frac{i}{N}\sin(\frac{2\pi n_0 k}{N})$	$\mathcal{R}$:
$\mathcal{I}$:	Exact	$\mathcal{I}$: 
2b. None	–	
	–	
	$n_0 \in \mathcal{N}$	

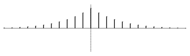
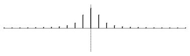
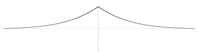
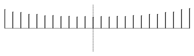
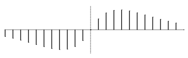
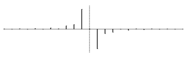
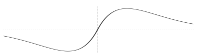
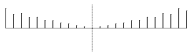
3. Complex harmonic	$\omega_N^{nk_0}$	F_k and c_k:
$\mathcal{R}$: 	$\hat{\delta}_N(k-k_0)$ (periodic)	$\mathcal{R}$: 
$\mathcal{I}$: 	Exact	$\mathcal{I}$:
3. Complex harmonic	$e^{i\frac{2\pi k_0 x}{A}}$	
$\mathcal{R}$: 	$\delta(k-k_0)$	
$\mathcal{I}$: 	$k_0 \in \mathbf{Z}$	$\max\|c_k - F_k\| = 0$
3a. Constant	1	F_k and c_k:
$\mathcal{R}$: 	$\hat{\delta}_N(k)$ (periodic)	$\mathcal{R}$: 
$\mathcal{I}$:	Exact	$\mathcal{I}$:
3a. Constant	1	
$\mathcal{R}$: 	$\delta(k)$	
$\mathcal{I}$:	Case 3: $k_0 = 0$	$\max\|c_k - F_k\| = 0$
4a. Cosine harmonic	$\cos(\frac{2\pi k_0 n}{N})$	F_k and c_k:
$\mathcal{R}$: 	$\frac{1}{2}(\hat{\delta}_N(k-k_0) + \hat{\delta}_N(k+k_0))$	$\mathcal{R}$: 
$\mathcal{I}$:	Exact	$\mathcal{I}$:
4a. Cosine harmonic	$\cos(\frac{2\pi k_0 x}{A})$	
$\mathcal{R}$: 	$\frac{1}{2}(\delta(k-k_0) + \delta(k+k_0))$	
$\mathcal{I}$:	$k_0 \in \mathbf{Z}$	$\max\|c_k - F_k\| = 0$
4b. Critical mode	$\cos(\pi n) = (-1)^n$	F_k and c_k:
$\mathcal{R}$: 	$\hat{\delta}_N(k-\frac{N}{2})$	$\mathcal{R}$: 
$\mathcal{I}$:	Exact	$\mathcal{I}$:
4b. Critical mode	$\cos(\frac{\pi N x}{A})$	
$\mathcal{R}$: 	$\frac{1}{2}(\delta(k-\frac{N}{2}) + \delta(k+\frac{N}{2}))$	
$\mathcal{I}$:	Case 4a: $k_0 = \frac{N}{2}$	$\max\|c_k - F_k\| = 0$

4c. Sine harmonic	$\sin(\frac{2\pi k_0 n}{N})$	F_k and c_k:
$\mathcal{R}$: 	$\frac{i}{2}(\hat{\delta}_N(k+k_0)-\hat{\delta}_N(k-k_0))$	$\mathcal{R}$:
$\mathcal{I}$:	Exact	$\mathcal{I}$: 
4c. Sine harmonic	$\sin(\frac{2\pi k_0 x}{A})$	
$\mathcal{R}$: 	$\frac{i}{2}(\delta(k+k_0)-\delta(k-k_0))$	
$\mathcal{I}$:	$k_0 \in \mathbf{Z}$	$\max\lvert c_k-F_k\rvert=0$
5. Complex wave	$\omega_N^{nk_0}$	F_k and c_k:
$\mathcal{R}$: 	$\frac{\sin(\pi(k-k_0))\sin(2\pi(k-k_0)/N)}{2N\sin^2(\pi(k-k_0)/N)}$	$\mathcal{R}$: 
$\mathcal{I}$: 	$\sim C(k-k_0)N^{-2}$	$\mathcal{I}$:
5. Complex wave	$e^{i\frac{2\pi k_0 x}{A}}$	$\lvert c_k-F_k\rvert,\quad k_0=2.4$
$\mathcal{R}$: 	$\frac{\sin(\pi(k-k_0))}{\pi(k-k_0)}$	
$\mathcal{I}$: 	AVED, $k_0 \notin \mathbf{Z}$	$\max \approx 10^{-2},\ N=24$
6a. Cosine	$\cos(\frac{\pi k_0 n}{N})$	F_k and c_k:
$\mathcal{R}$: 	$\frac{\cos(\pi k)\sin(\pi k_0/2)}{4N} \times \left(\frac{\sin\theta_+}{\sin^2\theta_+/2}-\frac{\sin\theta_-}{\sin^2\theta_-/2}\right)$	$\mathcal{R}$: 
$\mathcal{I}$:	$\sim Ck_0N^{-2}$	$\mathcal{I}$:
6a. Cosine	$\cos(\frac{\pi k_0 x}{A})$	$\lvert c_k-F_k\rvert,\quad k_0=2.4$
$\mathcal{R}$: 	$-\frac{2k_0\cos(\pi k)\sin(\pi k_0/2)}{\pi(4k^2-k_0^2)}$	
$\mathcal{I}$:	$k_0 \notin \mathbf{Z}, \theta_\pm = \frac{\pi(2k\pm k_0)}{N}$	$\max \approx 10^{-3},\ N=24$
6b. Half cosine	$\cos(\frac{\pi n}{N})$	F_k and c_k:
$\mathcal{R}$: 	$\frac{\cos(\pi k)}{4N}(\frac{\sin\theta_+}{\sin^2\theta_+/2}-\frac{\sin\theta_-}{\sin^2\theta_-/2})$	$\mathcal{R}$: 
$\mathcal{I}$:	$\sim CN^{-2}$	$\mathcal{I}$:
6b. Half cosine	$\cos(\pi x)$ on $(-\frac{1}{2},\frac{1}{2}]$	$\lvert c_k-F_k\rvert$
$\mathcal{R}$: 	$-\frac{2\cos(\pi k)}{\pi(4k^2-1)}$	
$\mathcal{I}$:	Case 6a: $\theta_\pm=\frac{\pi(2k\pm 1)}{N}, k_0=1, A=1$	$\max \approx 10^{-3},\ N=24$

6c. Sine	$\sin(\frac{\pi k_0 n}{N})$	F_k and c_k:
$\mathcal{R}$: 	$i\frac{\cos(\pi k)\sin(\pi k_0/2)}{4N} \times \left(\frac{\sin\theta_+}{\sin^2\theta_+/2}+\frac{\sin\theta_-}{\sin^2\theta_-/2}\right)$	$\mathcal{R}$:
$\mathcal{I}$:	$\sim CkN^{-2}$	$\mathcal{I}$: 
6c. Sine	$\sin(\frac{\pi k_0 x}{A})$	$\lvert c_k - F_k\rvert,\ k_0 = 2.4$
$\mathcal{R}$: 	$i\frac{4k\cos(\pi k)\sin(\pi k_0/2)}{\pi(4k^2-k_0^2)}$	
$\mathcal{I}$:	AVED, $k_0 \notin \mathbf{Z}, \theta_\pm = \frac{\pi(2k\pm k_0)}{N}$	max $\approx 10^{-2}$, $N = 24$
6d. Even sine	$\lvert\sin(2\pi n/N)\rvert$	F_k and c_k:
$\mathcal{R}$: 	$\frac{1+\cos(\pi k)}{4N}\left(\frac{\sin\theta_+}{\sin^2\theta_+/2}-\frac{\sin\theta_-}{\sin^2\theta_-/2}\right)$	$\mathcal{R}$: 
$\mathcal{I}$:	$\sim CN^{-2}$	$\mathcal{I}$:
6d. Even sine	$\lvert\sin(2\pi x/A)\rvert$	$\lvert c_k - F_k\rvert$
$\mathcal{R}$: 	$\frac{1}{\pi}\frac{1+\cos(\pi k)}{1-k^2}$	
$\mathcal{I}$:	$\theta_\pm = \frac{2\pi}{N}(k\pm 1), c_{\pm1} = F_{\pm1} = 0$	max $\approx 10^{-3}$, $N = 24$
7. Linear	n/N	F_k and c_k:
$\mathcal{R}$: 	$F_0 = 0, F_k = \frac{i\cos(\pi k)\sin\theta_k}{4N\sin^2\theta_k/2}$	$\mathcal{R}$:
$\mathcal{I}$:	$\sim CkN^{-2}$	$\mathcal{I}$: 
7. Linear	x/A	$\lvert c_k - F_k\rvert$
$\mathcal{R}$: 	$c_0 = 0, c_k = \frac{i\cos(\pi k)}{2\pi k}$	
$\mathcal{I}$:	AVED, $f_{-\frac{N}{2}} = 0$	max $\approx 10^{-2}$, $N = 24$
8. Triangular wave	$1 - 2\lvert n\rvert/N$	F_k and c_k:
$\mathcal{R}$: 	$F_0 = \frac{1}{2}, F_k = \frac{1-\cos(\pi k)}{N^2\sin^2\theta_k/2}$	$\mathcal{R}$: 
$\mathcal{I}$:	$\sim CN^{-2}$	$\mathcal{I}$:
8. Triangular wave	$1 - 2\lvert x\rvert/A$	$\lvert c_k - F_k\rvert$
$\mathcal{R}$: 	$c_0 = \frac{1}{2}, c_k = \frac{1-\cos(\pi k)}{\pi^2 k^2}$	
$\mathcal{I}$:		max $\approx 10^{-3}$, $N = 24$

9. Rectangular wave	$\begin{cases} -1, & -N/2 < n < 0 \\ 1, & 0 < n < N/2 \end{cases}$	F_k and c_k:
$\mathcal{R}$: 	$F_0 = 0, F_k = i\frac{(\cos(\pi k)-1)\sin\theta_k}{2N\sin^2\theta_k/2}$	$\mathcal{R}$:
$\mathcal{I}$:	$\sim CkN^{-2}$	$\mathcal{I}$: 
9. Rectangular wave	$\begin{cases} -1, & -A/2 < x < 0 \\ 1, & 0 < x < A/2 \end{cases}$	$\lvert c_k - F_k \rvert$
$\mathcal{R}$: 	$c_0 = 0, c_k = \frac{i(\cos(\pi k)-1)}{\pi k}$	
$\mathcal{I}$:	AVED, $f_n = 0$ for $n = 0, \pm N/2$	max $\approx 10^{-2}$, $N = 24$
10. Square pulse	$\begin{cases} 1, & \lvert n \rvert < M/2 \\ 0, & M/2 < \lvert n \rvert < N/2 \end{cases}$	F_k and c_k:
$\mathcal{R}$: 	$F_0 = \frac{M}{N}, F_k = \frac{\sin(\pi kM/N)\sin\theta_k}{2N\sin^2\theta_k/2}$	$\mathcal{R}$: 
$\mathcal{I}$:	$\sim CkN^{-2}$	$\mathcal{I}$:
10. Square pulse	$\begin{cases} 1, & \lvert x \rvert < a/2 \\ 0, & a/2 < \lvert x \rvert < A/2 \end{cases}$	$\lvert c_k - F_k \rvert$
$\mathcal{R}$: 	$c_0 = \frac{a}{A}, c_k = \frac{\sin(\pi ka/A)}{\pi k}$	
$\mathcal{I}$:	AVED, $0 < \frac{a}{A} = \frac{M}{N} < 1$	max $\approx 10^{-2}$, $N = 24$
10a. Square pulse	$\begin{cases} 1, & \lvert n \rvert < N/4 \\ 0, & N/4 < \lvert n \rvert < N/2 \end{cases}$	F_k and c_k:
$\mathcal{R}$: 	$F_0 = \frac{1}{2}, F_k = \frac{\sin(\pi k/2)\sin\theta_k}{2N\sin^2\theta_k/2}$	$\mathcal{R}$: 
$\mathcal{I}$:	$\sim CkN^{-2}$	$\mathcal{I}$:
10a. Square pulse	$\begin{cases} 1, & \lvert x \rvert < 1/2 \\ 0, & 1/2 < \lvert x \rvert < 1 \end{cases}$	$\lvert c_k - F_k \rvert$
$\mathcal{R}$: 	$c_0 = \frac{1}{2}, c_k = \frac{\sin(\pi k/2)}{\pi k}$	
$\mathcal{I}$:	AVED, Case 10: $A = 2a = 1$	max $\approx 10^{-2}$, $N = 24$

11. Exponential	$e^{-aAn/N},\ 0 \le n \le N-1$	F_k and c_k:
$\mathcal{R}$: 	$\frac{\sigma(1-e_N^2)-i2\sigma e_N \sin\theta_k}{2N(1-2e_N \cos\theta_k+e_N^2)}$	$\mathcal{R}$: 
$\mathcal{I}$:	$\sim CN^{-1}$	$\mathcal{I}$: 
11. Exponential	$e^{-ax},\ 0 < x < A$	$\lvert c_k - F_k\rvert,\ a=2,\ A=3$ 
$\mathcal{R}$:	$\frac{\sigma(aA-i2\pi k)}{a^2A^2+4\pi^2k^2}$	
$\mathcal{I}$:	AVED, $\sigma = 1-e^{-aA}, e_N = e^{-aA/N}$	$\max \approx 10^{-2},\ N=24$
12. Even exponential	$e^{-aA\lvert n\rvert/N}$	F_k and c_k:
$\mathcal{R}$: 	$\frac{(1-e_N^2)(1-e_2\cos(\pi k))}{N(1-2e_N \cos\theta_k+e_N^2)}$	$\mathcal{R}$: 
$\mathcal{I}$:	$\sim CN^{-2}$	$\mathcal{I}$:
12. Even exponential	$e^{-a\lvert x\rvert},\ \lvert x\rvert < A/2$	$\lvert c_k - F_k\rvert,\ a=2,\ A=1$ 
$\mathcal{R}$: 	$\frac{2aA(1-e_2\cos(\pi k))}{a^2A^2+4\pi^2k^2}$	
$\mathcal{I}$:	$e_2 = e^{-aA/2}, e_N = e^{-aA/N}$	$\max \approx 10^{-3},\ N=24$
13. Odd exponential	$(n/\lvert n\rvert)e^{-aA\lvert n\rvert/N}$	F_k and c_k:
$\mathcal{R}$: 	$i\frac{2(e_N\sin\theta_k)(e_2\cos(\pi k)-1)}{N(1-2e_N\cos\theta_k+e_N^2)}$	$\mathcal{R}$:
$\mathcal{I}$:	$\sim CN^{-1}$	$\mathcal{I}$: 
13. Odd exponential	$(x/\lvert x\rvert)e^{-a\lvert x\rvert},\ \lvert x\rvert < A/2$	$\lvert c_k - F_k\rvert,\ a=2,\ A=1$ 
$\mathcal{R}$: 	$i\frac{4\pi k(e_2\cos(\pi k)-1)}{a^2A^2+4\pi^2k^2}$	
$\mathcal{I}$:	AVED, $e_2 = e^{-aA/2}, e_N = e^{-aA/N}$	$\max \approx 10^{-2},\ N=24$
14. Linear/exponential	$(nA/N)e^{-aA\lvert n\rvert/N}$	F_k and c_k:
$\mathcal{R}$: 	$i\frac{2A\sin\theta_k(\sigma_k(e_N^3-e_N)+\cdots}{N^2(e_N^4-4\cos\theta_k(e_N^3+e_N)+\cdots}$ $\frac{\cdots+Ne_2e_N\cos(\pi k)(1-e_N\cos\theta_k))}{\cdots+2e_N^2(\cos(2\theta_k)+2)+1)}$	$\mathcal{R}$:
$\mathcal{I}$:	$\sim CN^{-1}$	$\mathcal{I}$: 
14. Linear/exponential	$xe^{-a\lvert x\rvert}$	$\lvert c_k - F_k\rvert,\ a=2,\ A=3$ 
$\mathcal{R}$: 	$i\frac{2\pi kA}{d}(e_2\cos(\pi k)+$ $\frac{4aA}{d}(e_2\cos(\pi k)-1))$	
$\mathcal{I}$:	$\sigma_k = 1+(\frac{N}{2}-1)e_2\cos(\pi k)$ $e_2 = e^{-aA/2}, e_N = e^{-aA/N}$ $d = a^2A^2+4\pi^2k^2$, AVED	$\max \approx 10^{-3},\ N=24$

15a. Cosine/exponential $\mathcal{R}$:  $\mathcal{I}$:	$e^{-aA\lvert n\rvert/N}\cos(\pi n/N)$ $\frac{(1-e_N^2)+2e_2e_N\cos(\pi k)\sin\theta_+}{2N(1-2e_N\cos\theta_++e_N^2)} + \frac{(1-e_N^2)-2e_2e_N\cos(\pi k)\sin\theta_-}{2N(1-2e_N\cos\theta_-+e_N^2)}$ $\sim CN^{-2}$	F_k and c_k: $\mathcal{R}$:  $\mathcal{I}$:
15a. Cosine/exponential $\mathcal{R}$:  $\mathcal{I}$:	$e^{-a\lvert x\rvert}\cos(\pi x/A)$ $\frac{e_2k_+\cos(\pi k)+a}{A(k_+^2+a^2)} - \frac{e_2k_-\cos(\pi k)-a}{A(k_-^2+a^2)}$ $e_2 = e^{-aA/2}$, $e_N = e^{-aA/N}$ $\theta_\pm = \frac{\pi}{N}(2k\pm 1)$, $k_\pm = \frac{\pi}{A}(2k\pm 1)$ AVED	$\lvert c_k - F_k\rvert,\ a=2,\ A=3$  max $\approx 10^{-3},\ N=24$
15b. Sine/exponential $\mathcal{R}$:  $\mathcal{I}$:	$e^{-aA\lvert n\rvert/N}\sin(\pi n/N)$ $i\left(\frac{1-e_N\cos\theta_++e_2e_N\cos(\pi k)\sin\theta_+}{N(1-2e_N\cos\theta_++e_N^2)} - \frac{1-e_N\cos\theta_--e_2e_N\cos(\pi k)\sin\theta_-}{N(1-2e_N\cos\theta_-+e_N^2)}\right)$ $\sim CN^{-1}$	F_k and c_k: $\mathcal{R}$: $\mathcal{I}$: 
15b. Sine/exponential $\mathcal{R}$:  $\mathcal{I}$:	$e^{-a\lvert x\rvert}\sin(\pi x/A)$ $\frac{ie_2k_+\sin((2k+1)\pi/2))+a}{A(k_+^2+a^2)} - \frac{ie_2k_-\sin((2k-1)\pi/2))+a}{A(k_-^2+a^2)}$ $e_2 = e^{-aA/2}$, $e_N = e^{-aA/N}$ $\theta_\pm = \frac{\pi}{N}(2k\pm 1)$, $k_\pm = \frac{\pi}{A}(2k\pm 1)$ AVED	$\lvert c_k - F_k\rvert,\ a=2,\ A=3$  max $\approx 10^{-3},\ N=24$

References

[1] W.L. Briggs, V.E. Henson, *The DFT: An Owner's Manual for the Discrete Fourier Transform*, SIAM Publications, 1995.