GAUGE-INVARIANT POISSON BRACKETS FOR CHROMOHYDRODYNAMICS

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Noncanonical hamiltonian structures are presented both for Yang–Mills Vlasov plasmas, and for ideal fluids interacting with Yang–Mills fields. These hamiltonian structures are given Lie-algebraic interpretations.

A problem of some theoretical interest is to describe the hamiltonian structure of a fluid which is coupled self-consistently to a nonabelian gauge field. For short we call such a theory CHD, chromohydrodynamics. This theory is the nonabelian extension of plasma physics.

Here we give the Poisson brackets for a Yang–Mills Vlasov plasma and for a fluid interacting with a self-consistent Yang–Mills field. We also give the Lie-algebraic interpretation of these Poisson brackets.

Consider the following single particle Poisson bracket between functions of $\boldsymbol{x}$, $\boldsymbol{p}$, and g.

$$\{J,K\}_1 = \frac{\partial J}{\partial \boldsymbol{p}} \cdot \frac{\partial K}{\partial \boldsymbol{x}} - \frac{\partial K}{\partial \boldsymbol{p}} \cdot \frac{\partial J}{\partial \boldsymbol{x}} + \left\langle g, \left[\frac{\partial J}{\partial g}, \frac{\partial K}{\partial g}\right]\right\rangle . \quad (1)$$

This is the direct sum of the canonical bracket for the coordinates $\boldsymbol{x}$ and momentum components $\boldsymbol{p}$ of the particle together with a Kirillov bracket [1] for its charge g. The charge belongs to the dual $\mathfrak{g}^*$ of some Lie algebra $\mathfrak{g}$, hence $\partial J/\partial g$ and $\partial K/\partial g$ as well as their commutator $[\partial J/\partial g, \partial K/\partial g]$ all belong to the algebra itself, so the pairing $\langle g, [\partial J/\partial g, \partial K/\partial g]\rangle$ is a scalar. The Jacobi identity for the Kirillov bracket follows from the Jacobi identity for the Lie algebra $\mathfrak{g}$.

For the single-particle hamiltonian,

$$H_1 = \tfrac{1}{2}(p - \langle g, A(x,t)\rangle)^2 - \langle g, A_0(x,t)\rangle , \quad (2)$$

one may derive, from Hamilton's equations,

$$\ddot{x}_i = -\langle g, E_i\rangle - \langle g, \dot{x}_j B_{ji}\rangle , \quad (3)$$

which is the Yang–Mills analogue of the Lorentz force; the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ are defined in terms of the potentials A and A_0 by

$$E_i = \partial A_i/\partial t - \nabla_i A_0 + [A_i, A_0] ,$$

$$B_{ij} = \nabla_j A_i - \nabla_i A_j + [A_i, A_j] . \quad (4)$$

The Poisson bracket for a Vlasov equation in the single particle phase space is simple to define; for any two functionals $\mathcal{J}[f]$, $\mathcal{K}[f]$ depending on the distribution function f on phase space, we take

$$\{\mathcal{J}[f], \mathcal{K}[f]\}_f = \int f \left\{\frac{\delta \mathcal{J}}{\delta f}, \frac{\delta \mathcal{K}}{\delta f}\right\}_1 \mathrm{d}^N x \, \mathrm{d}^N p \, \mathrm{d}^D g . \quad (5)$$

Here N is the dimension of space, D the dimension of the algebra $\mathfrak{g}$. The Jacobi identity for this bracket $\{\ ,\ \}_f$ follows from that for the single particle bracket $\{\ ,\ \}_1$.

The hamiltonian structure of the Yang–Mills Vlasov plasma is the direct sum of this structure with a canonical structure for the fields:

$$\frac{\partial f}{\partial t} + \left\{\frac{\delta \mathcal{H}}{\delta f}, f\right\}_1 = 0 ,$$

$$\frac{\partial A}{\partial t} = \frac{\delta \mathcal{H}}{\delta {}^*E} , \quad \frac{\partial {}^*E}{\partial t} = -\frac{\delta \mathcal{H}}{\delta A} , \quad \frac{\delta \mathcal{H}}{\delta A_0} = 0 . \quad (6)$$

Remark: in the case when $\mathfrak{g}$ is abelian, this hamiltonian structure reduces to that of Marsden and Weinstein [2]. Here the field ${}^*\boldsymbol{E}$, canonically conjugate to $\boldsymbol{A}$, belongs to $\mathfrak{g}^*$, and may be thought of as the transpose of $\boldsymbol{E}$ (in

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a matrix representation). The last equation, a constraint which is compatible with the equations of motion, arises from the gauge symmetry of the system; in fact it is just Gauss's law. The hamiltonian for a nonrelativistic plasma is

$$\mathcal{H} = \int [\tfrac{1}{2}(p - \langle g, A\rangle)^2 - \langle g, A_0\rangle] f \, d^N x \, d^N p \, d^D g$$
$$+ \int \{\tfrac{1}{2}\langle {}^*E_i, E_i\rangle + \langle {}^*E_i, (\nabla_i A_0 - [A_i, A_0])\rangle$$
$$+ \tfrac{1}{4}\langle {}^*B_{ij}, B_{ij}\rangle\} \, d^N x \,, \qquad (7)$$

from which the hamiltonian structure (6) produces the Yang–Mills Vlasov equations.

One passes to the barotropic fluid limit by considering the moments:

$$\rho = \int f \, d^N p \, d^D g \,, \quad M = \int f p \, d^N p \, d^D g \,,$$
$$G = \int f g \, d^N p \, d^D g \,, \qquad (8)$$

and then considering the "cold plasma" limit, where f is determined by these moments alone. The hamiltonian structure (6) in these variables restricts to

$$\partial_t \begin{pmatrix} \rho \\ G \\ M_i \end{pmatrix} = - \begin{pmatrix} 0 & 0 & \nabla_j \rho \\ 0 & -\mathrm{ad}^* G & \nabla_j G \\ \rho \nabla_i & G \nabla_i & \nabla_j M_i + M_j \nabla_i \end{pmatrix} \begin{pmatrix} \delta H/\delta\rho \\ \delta H/\delta G \\ \delta H/\delta M_j \end{pmatrix}, \qquad (9)$$

where H is the cold-plasma limit of (7) and the G–G term in the middle is to be read as

$$-\left(\mathrm{ad}^* \frac{\delta H}{\delta G}\right) G = -G_a \gamma^a_{bc} \frac{\delta H}{\delta G_b} e^c \,, \qquad (10)$$

where γ^a_{bc} are the structure constants of the algebra $\mathfrak{g}$ in a basis with elements e_a, e^c are elements of the dual basis, and $G = G_a e^a$, $G_a \in C^\infty(\mathrm{R}^N)$.

The full hamiltonian structure is the direct sum of (9) with the canonical structure for $^*E$ and A. In order to describe motion of a barotropic fluid, one takes the following hamiltonian:

$$H = \int [(M - \langle G, A\rangle)^2/(2\rho) - \langle G, A_0\rangle$$
$$+ U(\rho) + \tfrac{1}{2}\langle {}^*E_i, E_i\rangle + \tfrac{1}{4}\langle {}^*B_{ij}, B_{ij}\rangle$$
$$+ \langle {}^*E_j, \nabla_j A_0 + [A_0, A_j]\rangle \, d^N x \,, \qquad (11)$$

which is, apart from the internal energy term $U(\rho)$, the restriction of the hamiltonian (7) to a cold plasma. This hamiltonian together with the structure (9) plus the canonical part for $^*E$ and A produces the motion equation for a barotropic fluid which is driven by a Yang–Mills Lorentz force density:

$$\rho\left(\frac{\partial v_j}{\partial t} + v_i \nabla_i v_j\right) + \rho \nabla_j \frac{\partial U(\rho)}{\partial \rho} = -\langle G, E_j + v_i B_{ij}\rangle \,, \qquad (12)$$

where the velocity v_j is given by

$$v_j = \delta H/\delta M_j \,. \qquad (13)$$

For a fluid whose internal energy depends also on entropy density σ, one adds to the hamiltonian structure (9) terms which are analogous to those in ρ, namely

$$\partial\sigma/\partial t = -\nabla_j \sigma \delta H/\delta M_j \,, \qquad (14)$$

and one adds to $\partial M_i/\partial t$ a term $(\sigma \nabla_i \delta H/\delta\sigma)$ as well. All of the CHD equations may then be computed readily from their hamiltonian structure, see e.g., ref. [6].

In the hamiltonian structure for CHD, the noncanonical part depends linearly upon the fluid variables and therefore can be interpreted [3] as a Lie algebra.

Let $\hat{\mathfrak{g}}$ denote a Lie algebra of smooth functions on R^n with values in $\mathfrak{g}$. The Lie algebra $\mathcal{D}(\mathrm{R}^n)$ of vector fields on R^n acts naturally on $C^\infty(\mathrm{R}^n)$ and on $\hat{\mathfrak{g}}$. Let L be a Lie algebra (semidirect product):

$$L = \mathcal{D}(\mathrm{R}^n) \circledS [\hat{\mathfrak{g}} \oplus C^\infty(\mathrm{R}^n) \oplus C^\infty(\mathrm{R}^n)] \,. \qquad (15)$$

Then the natural Poisson bracket on the dual space L^* of L coincides (up to a minus sign) with the noncanonical part of the CHD bracket described above, provided one denotes the dual coordinates to L as follows: M_i is dual to $\partial/\partial x_i$; G^a is dual to $C^\infty(\mathrm{R}^n) \otimes e_a$; ρ and σ are dual to the first and second summand $C^\infty(\mathrm{R}^n)$, respectively.

For further explanations and other applications of these methods see, e.g. refs. [4–6].

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References

[1] V. Guillemin and S. Sternberg, Ann. Phys. 127 (1980) 226.
[2] J.E. Marsden and A. Weinstein, Physica 4D (1982) 394.
[3] B. Kupershmidt, On dual spaces of differential Lie algebras, Lie algebras and related topics, Rutgers 1981 Proceedings, Lecture note in mathematics (Springer, Berlin) to be published.
[4] D.D. Holm and B. Kupershmidt, Canonical and non-canonical hamiltonian formulations of ideal magnetohydrodynamics, J. Math. Phys., to be published; also available as Los Alamos report LA-UR-82-124.
[5] D.D. Holm and B. Kupershmidt, Poisson brackets and Clebsch representations for magnetohydrodynamics, multifluid plasmas, and elasticity, Physica D, to be published.
[6] J. Gibbons, D.D. Holm and B. Kupershmidt, Hamiltonian structure of classical chromohydrodynamics, Physica D, to be published.