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On the Turnover of Business Firms and Business Managers

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ABSTRACT

This paper develops a model of small business failure and sale that is motivated by recent evidence from the small business sector. The evidence consists of findings concerning how the failure and sale of businesses vary with the age of the business and with the tenure of the manager. This evidence motivates two key features of the model, the first being a match between the manager and the business, the second being characteristics of businesses that survive beyond the current match. The parameters of the model are estimated, and the properties of this parametric model are studied. This analysis results in a simple characterization of the workings of the small business sector.

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1. Introduction

This paper develops a model of small business failure and sale that is motivated by recent evidence from the small business sector. The parameters of the model are then estimated, and the properties of this parametric model are studied. This analysis results in a simple characterization of the workings of the small business sector.

The evidence that inspired the model below consists of findings concerning how the failure and sale of small businesses vary with the age of the business and with the tenure of the current manager of the business (see Holmes and Schmitz 1993). Two findings are of particular note in motivating the form of the model. First, examining small businesses of the same age, the probability that a business fails, and the probability that a business is sold, are both initially decreasing in the tenure of the manager at the business. At some point, the discontinuance rates begin to increase in tenure. Hence, the probability of a job separation by the business manager, which occurs if the business is discontinued or sold, is initially decreasing in the tenure at the job (holding fixed the age of the job). Second, examining businesses whose managers have the same tenure at their business, the probability that a business fails is decreasing in the age of the business.

The first finding, that job separations are negatively related to job tenure, has been documented many times in other contexts.

A natural way to capture this phenomenon in a model is by introducing the concept of a job “match.” This has been done by, among others, Jovanovic (1979). We follow in this tradition by assuming that there is some underlying match between each business and each manager. The second finding, that business age is related to business failure and business sale even after controlling for managerial tenure, suggests that there is more to the business, or job, than how well the individual is suited to the job. It indicates that there are characteristics of businesses that are separate from managers. One such characteristic is the location of the business. We incorporate this into the model by assuming that, in addition to the match between the business and manager, each business has a quality that is independent of the manager that is operating the business.

Briefly, the model works as follows. Individuals enter the economy each period by either starting or purchasing a business. If they start a business, they draw a business quality and a match to that business. If they purchase a business, only the match needs to be determined. Business quality has already been determined for such a business. In each period after acquiring the business, the individual decides to manage the business or to separate from the business. If a separation occurs, the individual either discontinues or sells the business.

The parameters of the model are estimated with data drawn from the Characteristics of Business Owners survey. This survey—described in the next section—was a survey of the small business sector. The estimation techniques employed below are in the spirit of Pakes (1986), among others (see Eckstein and Wolpin 1989 for a survey of these methods and for other references).

The characterization of the small business sector implied by the estimated model is as follows. The probability of starting a "good" business is small. Those individuals that continue to manage a business they have started, therefore, typically do so because they have "good" matches. Those individuals with bad matches quickly leave their business, most often by closing the business, but sometimes by selling the business, particularly if the business is high quality. For these high quality businesses, there is a "high" return to finding an individual that is a good match to the business. Businesses that have been sold tend to be of higher quality than businesses that have not been sold. Because there is a high return to finding owners that are good matches for good businesses, and because new owners are just as likely to have bad matches as previous owners, the model implies that businesses that are sold have higher subsequent sales rates than do businesses that have not been sold.

Turning to related research, this paper is most closely associated with those papers that have constructed models of the evolution of business populations. Among these papers are those developed by Jovanovic (1982), Pakes and Ericson (1988), and Hopenhayn (1992). Each of these papers has sought to develop simple characterizations about the workings of a particular business population. An important distinction between the models presented in these papers and that developed below is that in these papers businesses have only a single quality dimension. In the model presented below, businesses have two quality dimensions. Both these dimension are motivated by the evidence mentioned above.

Given the central role that this recent evidence plays in motivating the model, it is worthwhile discussing in what sense the evidence is new. We keep this discussion brief since these issues are addressed in Holmes and Schmitz (1993). In the industrial organization literature there are a number of studies that examine the relationship between business turnover and business age (see, for example, Dunne, Roberts and Samuelson 1989, Evans 1989, and Pakes and Ericson 1988). Seldom, if ever, do these studies have information on the tenure of the manager of the business.

In the labor economics literature measures of job tenure are readily available. It has long been recognized that there is a negative cross-sectional relationship between job tenure and job separations. It has also been known that to interpret these findings as evidence of a matching process between workers and jobs (or some other process that is specific to the worker and job) requires some care. For example, the cross-sectional relationship may be due to a heterogeneous population where people differ by their propensity to stay at a job. There is another type of heterogeneity, one not tied to the individual. Jobs or positions may differ in quality. Some jobs may have a greater propensity to survive because, for example, firms have different survival probabilities. Because of this, proxies have been sought for individual and job heterogeneity. What is different about the evidence presented below is that there is direct information on the job itself, that is, on the age of the job (or, the age of the small business).

The remainder of the paper proceeds as follows. In the next section we review the evidence that motivates the model. We do this by introducing the Characteristics of Business Owners survey and by describing some of the results from Holmes and Schmitz (1993). The model is introduced in Section 3. Section 4 contains some analysis of the model. The estimation of the model is presented in Section 5. The simple characterization of the workings of the small business sector that is implied by the estimated model is developed in Section 6.

2. The Characteristics of Business Owners Survey

The 1982 Characteristics of Business Owners (CBO) survey was a Census Bureau survey drawn from the universe of "small" business tax returns filed in 1982. These tax returns include proprietorship, partnership and subchapter-S corporation tax returns. In this universe of small businesses, the owner of the business is typically the manager of the business as well. Indeed, 80 percent of the businesses have no employees. Hence, in this paper we assume that the owner and manager are the same person, and use the terms interchangeably. The survey consisted of 25 questions. Some questions pertained to the business (such as its age), some to the manager (such as his or her age). We have used this survey to document patterns of business turnover in this population. This research is presented in Holmes and Schmitz (1993). In this section we highlight some of the results from this research.

When constructing the CBO survey, the Census drew samples from five different subpopulations of the population of business tax returns corresponding to five different demographic groups (women, blacks, hispanics, other minority, and nonminority white male). As discussed in Holmes and Schmitz (1993), the turnover patterns across these groups are remarkably similar. We focus here on the nonminority white male sample since it represents by far the largest underlying universe of businesses.

The CBO survey included a number of retrospective questions which allow us to construct histories of businesses and managers as of 1982. In particular, we can classify each business into one of 27 categories defined by the age of the business, the tenure of the manager at the business, and the founder status of the manager, that is, whether the manager had started the business or not. These 27 categories are given in Table 1A. The survey groups businesses into one of six business age (as of 1982) categories: 0 years, 1–2 years, 3–6 years, 7–12 years, 13–22 years, and 23+ years. Tenure of the manager at each business is grouped into the same year groupings as business age: 0, 1–2, 3–6, 7–12, 13–22, 23+. Note that the tenure of a founder of a business is equal to the age of the business. From the survey we know that the vast majority of nonfounders acquired their businesses by purchasing the business (rather than through inheritance, for example). Finally, let us emphasize that the groupings in Table 1A are the groupings which appeared on the survey instrument; we had no choice in how to group years.[1]

Table 1A provides the distribution of the 15,737 observations in the nonminority male sample over the 27 categories. Note that only a small fraction of recently established firms are, not surprisingly, nonfounder firms. However, about one-half of all firms that are 23 years of age or older are nonfounder firms.

Because the CBO survey about the 1982 business was mailed in 1986, and because there was a question about the status of the business in 1986, we are able to classify each business into one of three business turnover categories. We classify a business as "discontinued" if the business is no longer operating as of the survey date in mid-1986. Those businesses that are operating are classified into one of two groups. A business is classified as "kept" if the individual who owned the business in 1982 still owns the business as of the survey date. A business is classified as "sold" if the business is under different ownership as of the survey date. Tables 1B and 1C report the proportion of firms in each cell of Table 1A that were discontinued and sold, respective-

[1]The actual survey question regarding business age asked "what year was the business established?" The choices were 1982, 1980–82, 1976–79, and so on. These year established groupings correspond to businesses of age 0, 1–2, 3–6, and so on. The question regarding managerial tenure asked "What year did you acquire the business?" The choices were the same as those for the year established question, that is, 1982, 1980–82, and so on.

ly.[2] Note that some business age and managerial tenure categories in Table 1A have been combined in Tables 1B and 1C to satisfy Census Bureau disclosure requirements.

We have a number of points to make about Tables 1B and 1C. First, examining small businesses of the same age, the probability that the business fails, and the probability that the business is sold, are both initially decreasing in the tenure of the manager at the business (here we are examining nonfounders, reading from left to right in a row of Table 1B or 1C). At some point, the discontinuance rate begins to increase in tenure. This same pattern holds as we vary the tenure of founders (here we are reading down the first column in Table 1B). Second, examining businesses whose managers have the same tenure at their business, the probability that the business fails is typically decreasing in the age of the business (here we are again examining nonfounders, reading from top to bottom in a column of Table 1B). The largest drop in failure rates occurs "early." There are two "transitions" in which this pattern does not hold (here the increases in failure rates are slight, from 17 to 19 in one case, from 9 to 10 in the other). As mentioned in the introduction, these two patterns suggest constructing a model with two dimensions over which selection occurs: a match dimension and a business quality dimension.

[2]These tables, and all the analysis that follows, do not use the sample weights (the data was stratified by industry and state). For all the tables we have constructed of the sort of Tables 1B and 1C, and in all the model estimates that we have calculated, it made virtually no difference whether the sampling weights were used or not.

Finally, we mention two more patterns in the tables that will be discussed frequently below. Examining businesses of the same age, businesses owned by nonfounders with 0–2 years of tenure have higher discontinuance rates than businesses owned by their founders, except for the very oldest businesses (those of 23+ years). For example, 59 percent of the youngest firms owned by nonfounders were closed as compared to 46 percent for businesses owned by founders. For businesses 3–6 years old, the figures are 38 and 26 percent. The second pattern we note is that a similar relationship holds for sale rates as can be seen in Table 1C. For the very youngest businesses, nonfounders have sale rates of 7 percent, founders 3 percent.

In Holmes and Schmitz (1993), we document the statistical significance and robustness of these patterns. We find that the same patterns hold in analogous cross tabulations for the other four demographic panels (each demographic panel has approximately the same number of observations). The patterns also hold in regression analysis where we control for a number of factors, such as industry, business size, and characteristics of the manager, including age, education, demographic group, and previous business ownership experience.

3. The Model

The model is an overlapping generations economy in which individuals are infinitely lived. Each period a new cohort of individuals of age zero enters the economy. Individuals initially enter the “business” sector of the economy. A fraction e of those entering the sector start a new business; the remaining fraction, 1 ! e, enter by purchasing a business. For now, think of e as determined exogenously. After an initial period in the business sector, individuals decide each period whether to stay in that sector or to leave (permanently) to pursue an outside option.

Individuals are endowed with a unit of labor each period. As mentioned, during their initial period in the economy individuals must use the endowment to manage their business. This management process yields output. Following this initial period, at age one, the person can use their labor endowment in one of two ways. The person can once again manage the business or instead pursue an outside option. If the person pursues the outside option then the individual works at that task in all future periods. If the outside option is pursued, the person either discontinues or sells the business (depending on its value). If the person chooses to stay in the business sector at age one, then at age two the person again has two choices: manage the business or pursue the outside option, leaving the business sector for good. The person continues to face this choice as long as he remains in the business sector.

As will be made clear below, the only market that operates at each date is the market for businesses. In this market, the demand for businesses arises from those individuals entering the economy that purchase businesses. The supply of businesses arises from those persons, age one and greater, who decide to pursue the outside option at that date.

A. Specifics of the Model

We begin by describing the output produced if a business is managed. If a person uses their labor endowment to manage a business in period t, then output q_t is the sum of a "match" quality component q_t^M and a "business" quality component q_t^B, that is, $q_t = q_t^M + q_t^B$. The match quality component q_t^M is specific to a particular individual running a particular business; if another individual were to manage the business he would have a different q_t^M. On the other hand, the business quality component q_t^B is the same regardless of who manages the business. Greenwald (1979) and Jovanovic (1982b) have considered technologies with an analogous decomposition of productivity.

Most previous analyses of selection have assumed that underlying quality is fixed but unknown to the individual. The individual learns about quality through time by observing output. For example, in Jovanovic (1982a), q_t^M represents the assessment of underlying match quality; it varies through time as the prior distribution on the variable is updated. We employ a technically simpler device in this paper. We assume that quality (both match and business) is known to all. However, we assume that there are temporary shocks to both match and business quality so that these variables change through time. These temporary shocks insure that the selection process occurs gradually over time.

More formally, match quality q_t^M is assumed to be the sum of a permanent component μ and a transient component x_t, so that $q_t^M = \mu + x_t$. Similarly, business quality q_t^B is the sum of a permanent component $\$$ and a transient component y_t, so that $q_t^B = \$ + y_t$. Hence, we can write total output as the sum $q_t = (\mu + x_t) + (\$ + y_t)$. We next describe how each of these four components is determined.

The permanent match component μ is determined when an individual becomes the owner of a business. Hence, a permanent match is determined when a business is started and each time a business is purchased by a new owner. For simplicity it is assumed that μ takes on two values, μ_L (low) and μ_H (high), with $\mu_L < \mu_H$. Let 8 denote the probability of drawing a good match. In some versions of the model we assume that the probability of drawing a good match depends on whether the business is being started or purchased. For these cases, we let 8_{NF} denote the probability that an individual purchasing a business (a “nonfounder”) draws μ_H. Analogously, let 8_F denote the probability that a person starting a business (a “founder”) draws μ_H.

Permanent business quality $\$$ is determined when a business is established. For simplicity we assume that $\$$ takes on two values, $\$_L$ (low) and $\$_H$ (high), with $\$_L < \$_H$. Let $>$ denote the probability an individual starting a business draws a good business.

In some versions of the model we assume that the probability of drawing $\$_H$ depends on the μ that the founder draws. For these cases, we let $>_\mu$ be denote the probability of drawing a good business $\$_H$ conditioned upon drawing permanent match μ.

Given these conventions, it is easy to calculate the probability that a founder draws match μ and business quality $\$$: letting

$\phi_{\mu\beta}$ denote this probability, we have $\phi_{\mu_H\beta_H} = \lambda_F\pi_{\mu_H}$, $\phi_{\mu_L\beta_H} = (1-\lambda_F)\pi_{\mu_L}$, and so forth.

The temporary match and business quality variables, x_t and y_t, are assumed to be continuous random variables with infinite support $(-\infty,\infty)$. Let $f(\cdot)$ be the continuous density and $F(\cdot)$ the distribution function for x_t, and define $g(\cdot)$ and $G(\cdot)$ similarly for y_t. We assume both variables have a mean equal to zero. Each variable is distributed independently over time within a given business; each variable is also distributed independently across businesses at a point in time. Finally, x_t and y_t are distributed independently of each other. A simple example of a low realization of x_t would be the following. Suppose a manager's home situation changes in such a way that he desires to be home more frequently. Perhaps his spouse has become sick. If his business is one where he must travel often, and can not be operated out of the home, then the manager is temporarily a bad match with his business. However, the bad match is transient because the spouse is expected to recover next period. A low realization of y_t would occur if road construction made access by consumers to the business temporarily difficult.

Note that since the random variables x and y have infinite support, output in any period can be negative. Hence, we interpret the return to managing a business as including both physical units of the consumption good as well as the utility (or disutility) derived from managing the business. Returning to the above example where a manager's spouse had become sick, if the manager were to operate the business in the period that the spouse was sick, the manager would face additional stress which corresponds in this setup to a low x_{t}.

The only alternative to managing a business is to pursue the outside opportunity. If the person chooses to leave the business sector at any age, age one or greater, then the person receives output of w that period (and in all subsequent periods as well).

Before proceeding we digress briefly to discuss the issue of entrepreneurial ability. As it stands now, ability is suppressed in the model. One way to add entrepreneurial ability is as follows. Let O index ability, with larger O's meaning more ability. Suppose ability has an “additive” effect in both the business sector and in the outside option, that is, assume output is $q_t = \mu + x_t + \$ + y_t + O$ and that the return in the outside option is $w + O$. Adding ability in this manner, in which ability adds equally to the return in the current business and the outside option, does not change any of the analysis below.

A more general point that this discussion highlights is that there is an asymmetry between individuals and businesses. Both businesses and individuals have characteristics that survive beyond a current match. However, businesses are often closed when a current match is broken. Individuals continue to work after a match is broken. This fundamental difference between individuals and businesses is captured in the current formulation of the model.

Returning to the development of the model, individuals are assumed to be risk neutral. Hence, their objective is to maximize the expected sum of discounted output. The discount factor is *.

The only remaining detail is to describe the assumptions about entry into the economy. In each period t, a new cohort of M_t individuals of age zero are born into the economy. We assume the number of newly entering individuals grows at the constant rate of (, i.e., $M_t = (1+()M_{t!1}$. As mentioned, we assume that an exogenous fraction e of these newly entering individuals start businesses. The remaining fraction (1!e) purchase previously existing businesses.[3]

B. Individual Behavior

Consider the problem at date t of an individual of age 1, or greater, who has not pursued the outside opportunity (and hence still owns the business he bought or started when he was age zero). After observing x_t and y_t, and knowing μ and $, the individual makes the following choices: he can keep and manage the business in the period or pursue the outside option. If he pursues the outside option, then he either sells the business to another individual or discontinues the business. We will refer to these actions as

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[3]It would be straightforward to enrich the model to endogenize the process determining how many individuals start firms instead of purchase. For example, we could assume that there is a fixed cost to starting firms. Equilibrium would require that individuals be indifferent between being founders or nonfounders. This condition would determine the equilibrium fraction e. For any version of the model with a fixed e, there exists a level of fixed costs such that this is the equilibrium level of e in a model where e is determined endogenously.

"keep," "sell," and "discontinue," denoting them by the letters "K," "S," and "D."

Let the maximum discounted value of output to the individual from behaving optimally be denoted as $v_{\mu\beta}(x,y)$. Note that we do not index $v(\cdot)$ by time since below we focus on steady-states of the economy. In addition, let $v^a_{\mu\beta}(x,y)$ be the maximum discounted value of output from selecting action a in the current period, $a \in \{K,S,D\}$, and behaving optimally thereafter.

We begin by calculating the return to discontinuing the business in the current period and behaving optimally thereafter. Since we assume free disposal, this return is the discounted value of earnings the individual obtains from working in the outside sector this period and every period thereafter. Recalling that the outside opportunity provides a payment of w each period,[4] the value of discontinuing is then

$$(3.1) \qquad v^D_{\mu\beta}(x,y) = \frac{w}{1 - \delta}.$$

Consider next the return to selling a business. If an individual sells his business the return consists of the proceeds of the sale plus the discounted stream of returns from the outside

[4]It is possible to make the returns to the alternative opportunity endogenous by modeling the alternative opportunity as starting or buying another business within the economy. In this case the M_t individuals acquiring new or established firms would also include individuals who previously owned a business. Empirically, individuals often leave one business to enter another business (we have stressed this in our previous work, Holmes and Schmitz 1990). In the CBO data, however, we have no information as to the current activities of the individuals who sold or discontinued their firms so we have modeled this process as simply as possible.

sector. The price of a business depends upon both the permanent component β and the transient component y of business quality. Given we examine steady-states, the price of a business does not depend on time. Define b_β to be the price of a business with permanent quality β and transient quality y = 0 (price is denominated in units of current output). Note that if two businesses have the same β but one business has a y which is one unit greater than the other business, then it will sell for one output unit more in equilibrium. This follows from the fact that y is purely transitory. Hence, given b_β, the price of a β quality business with nonzero y is $(b_\beta+y)$. Given free disposal of businesses, a necessary condition for a business to be sold is that $b_\beta + y \geq 0$. Hence, the return to selling a business equals

$$v^S_{\mu\beta}(x,y) = b_\beta + y + \frac{w}{1-\delta}. \tag{3.2}$$

We next calculate the return to keeping and managing the firm in the current period. This return is given by

$$v^K_{\mu\beta}(x,y) = (\mu+\beta+x+y) + \delta \cdot Ev_{\mu\beta}, \tag{3.3}$$

where

$$Ev_{\mu\beta} \equiv \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} v_{\mu\beta}(x,y) \cdot f(x) \cdot g(y) \cdot dy \cdot dx \tag{3.4}$$

is the expected future return conditioned on the values of μ and β. The first term of (3.3) equals output in the current period. The second term is the discounted expected future value.

The maximum value to the individual is the maximum of the return over the three actions, that is,

$$(3.5) \qquad v_{\mu\$}(x,y) = \max\{v^{K}_{\mu\$}(x,y), v^{S}_{\mu\$}(x,y), v^{D}_{\mu\$}(x,y)\}.$$

Two "cutoff" points are crucial in characterizing the optimal policy of the individual. Let $\hat{x}_{\mu\$}$ be the level of x at which the individual is indifferent between selling the firm and keeping the firm. This is obtained by setting (3.2) equal to (3.3), and then solving for x. Note that the solution does not depend on y. Let $\hat{y}_{\mu\$}$ be the level of y such that the individual is indifferent between selling and discontinuing the business. Sale is preferable to discontinuance if and only if the sale price is positive. Hence, $\hat{y}_{\mu\$}$ depends only on \$, $\hat{y}_{\mu\$} = \hat{y}_{\$} = !b_{\$}$.

The pair $(\hat{x}_{\mu\$}, \hat{y}_{\$})$ defines three regions as illustrated in figure 1. These regions give the optimal policy as a function of x and y. The region between "sell" and "keep" is separated by a vertical line because as y is increased by one unit the return to "sell" and the return to "keep" both increase by one unit; hence, the relative return to these actions remains unchanged. The region between "sell" and "discontinue" is separated by a horizontal line because a decrease in the transient match component x is irrelevant in this region since the individual is leaving in either case. Finally, the "keep" and "discontinuance" regions are separated by a line with slope !1 since in this region the firm is not being sold and only the sum of x and y is important.

From figure 1, it is easy to describe how to calculate the probability that an individual keeps, sells and discontinues the business in the current period conditioned on μ and \$ (but not conditioned on x and y). This is done by integrating the joint density for x and y over the appropriate x and y regions in figure 1. Let $p^K_{\mu\beta}$, $p^D_{\mu\beta}$, and $p^S_{\mu\beta}$ denote these probabilities that the business is kept, discontinued and sold in the current period.

C. Equilibrium

Before defining a competitive equilibrium, we need more notation. Let $a_{\mu\beta}(x,y) \in \{K,S,D\}$ denote the optimal action given x, y, μ, and β. Then a competitive equilibrium is a list $\{a(\cdot),v(\cdot),b_L,b_H\}$ that satisfies:

i) $v_{\mu\beta}(x,y)$ solves (3.5)

$$a_{\mu\beta}(x,y) = \underset{K,S,D}{\arg\max}\{v^K_{\mu\beta}(x,y),v^S_{\mu\beta}(x,y),v^D_{\mu\beta}(x,y)\}.$$

ii) $$b_{\beta_H} - b_{\beta_L} = \beta_H - \beta_L + (1-\lambda_{NF})\delta\left(Ev_{\mu_L\beta_H} - Ev_{\mu_L\beta_L}\right) + \lambda_{NF}\delta\left(Ev_{\mu_H\beta_H} - Ev_{\mu_H\beta_L}\right).$$

iii) $Supply_t(b_L,b_H) = (1-e)M_t$.

The first condition insures that individuals behave optimally. Condition (ii) states that the price differential between high and low permanent quality businesses must be such that individuals purchasing businesses are indifferent between the two qualities. The left-hand side of (ii) is the price premium that has to be paid in order to obtain permanent high quality instead of low. The three terms on the right-hand side are the benefits from doing so. The first term $\beta_H - \beta_L$ is the additional output in the current period from having β_H instead of β_L. The second term is the probability $1 - \lambda_{NF}$ of drawing a low permanent match times the difference in discounted expected value between β_H and β_L conditional on drawing a low permanent match. The third term corresponds to the event that the individual draws high permanent match quality.

Condition (iii) is that supply of businesses equal the demand for businesses. The supply of businesses is denoted by $Supply_t(b_L,$-$b_H)$. Construction of supply involve calculations that involve the actions $a_{\mu\$}(x,y)$. The procedure is discussed in Appendix A. Here, we briefly describe the procedure. For a given price vector (b_L,b_H), and a given pair (μ,$), we calculate the point $(\hat{x}_{\mu\$},\hat{y}_{\$})$ in figure 1 and the resulting probabilities of “sell,” “discontinue,” and “keep.” Next, if we knew the number (or more formally the measure) of businesses with the pair (μ,$), we could calculate the number of these businesses that are put up for sale by multiplying the number of businesses by the probability of sale. Total supply would then consist of adding four numbers: the number of businesses of each type (μ,$) that are put up for sale. We can calculate the number of businesses of type (μ,$) at the beginning of a representative period using (1) the flow of new firms into the economy that has occurred in previous periods and (2) the turnover probabilities that these firms have faced.

A steady-state of the model is a competitive equilibrium in which prices do not change through time. Since we did not include time subscripts in our definition of competitive equilibrium above, the definition is implicitly one of steady-state equilibrium.

We can prove that a steady-state equilibrium exists under the parameter restriction (1!e) # e/(1+(). Under this condition the number of firms purchased in the current period does not exceed the number of firms started in the previous period. This condition is satisfied by a large margin in our data set.

4. Turnover in the Model

What business turnover patterns can this model produce? More precisely, recalling the business turnover patterns documented in Tables 1B and 1C for the CBO survey, we ask: What types of tables of this sort can the model produce? There are two reasons for addressing this question. First, this analysis will provide some understanding of the model and its properties. Second, in anticipation of the estimation that follows, the analysis provides some indication of the parameter values that will give the model its best “fit” to the CBO survey data. For example, the analysis shows that a version of the model with only a business quality dimension fails to capture some important turnover patterns in the data.

How, then, can we construct versions of Tables 1B and 1C from the model economy? It is useful to break this construction into two steps. First, in section A, we show how the turnover probabilities $p^K_{\mu\$}$, $p^D_{\mu\$}$, and $p^S_{\mu\$}$ vary with μ and $\$$. With these results, if it were known what fraction of businesses in each cell of Tables 1B and 1C were of type (μ,$\$$), then the turnover patterns could be directly obtained. Hence, the second step, in section B and section C, is to calculate the fraction of businesses in each cell that are of type (μ,$\$$).

A. How Turnover Probabilities Depend on μ and \$

Here we discuss how the probability of being kept, sold or discontinued varies with μ and \$. As discussed earlier, given the pair ($\hat{x}_{\mu\$}, \hat{y}_{\$}$) illustrated in figure 1, these probabilities can be calculated by integrating the joint density of x and y over the appropriate regions in (x,y) space. Hence, the task is to determine how a change in μ or \$ shifts the point ($\hat{x}_{\mu\$}, \hat{y}_{\$}$). Formal proofs of the results given in this section are in Appendix B.

Consider first the effect on the pair ($\hat{x}_{\mu\$}, \hat{y}_{\$}$) of increasing μ with \$ fixed. Recall that $\hat{x}_{\mu\$}$ is the point where the individual is indifferent between keeping the business or selling it. An increase in match quality μ raises the return to keeping the business but has no effect on the return to selling. Hence, if a low match individual is willing to keep rather than sell his firm, a high match individual, all else the same, will prefer to keep rather than sell, i.e., $\hat{x}_{\mu_H\$} < \hat{x}_{\mu_L\$}$. An increase in μ has no effect on the cutoff $\hat{y}_{\$}$ between selling and discontinuing because the match is broken in either case. Therefore, if we increase μ, with \$ fixed, it shifts the pair ($\hat{x}_{\mu\$}, \hat{y}_{\$}$) in figure 1 to the left, as illustrated in figure 2. The “keep” region is bigger for the high μ case, while the “sell” and “discontinue” regions are smaller.

Now consider the effect on the pair ($\hat{x}_{\mu\$}, \hat{y}_{\$}$) of increasing \$ with μ fixed. An increase in \$ shifts the $\hat{y}$ cutoff downward. The higher is permanent business quality, the greater the willingness to tolerate a low transient business quality before discontinuing the business. In addition to the effect on the $\hat{y}$ cutoff, a change in \$ also has an effect on the $\hat{x}$ cutoff. The direction of this effect depends upon μ as illustrated in figure 3. If μ is low, an increase in \$ shifts $\hat{x}_{\mu\$}$ to the right, i.e., $\hat{x}_{\mu_L\$_H} > \hat{x}_{\mu_L\$_L}$. If μ is high, an increase in \$ shifts $\hat{x}_{\mu\$}$ to the left, i.e., $\hat{x}_{\mu_H\$_H} < \hat{x}_{\mu_H\$_L}$. The basic intuition for the result is as follows.

The reason for the shape of figure 3 is that managers with a good match prefer a good business more than managers with a bad match, that is, we can show

(4.1) $Ev_{\mu_H\$_H} \; ! \; Ev_{\mu_H\$_L} > Ev_{\mu_L\$_H} \; ! \; Ev_{\mu_L\$_L}$,

where the expectation is taken with respect to the joint density of x and y. Why would a manager with a good match prefer a good business more than a manager with a bad match? To answer this, we ask another question: What are the benefits of a good business? If a manager sells the business at the beginning of the current period, a high \$ business brings a higher price. But a person with a good match and one with a bad match value this feature of a good business equally. If the individual ends up keeping the business in the current period, a high \$ business produces greater output. But again, a person with a good match and one with a bad match value this equally. However, a person with a good match places greater *future* value on having a good business today than does a person with a bad match. To see why this is the case consider point A in figure 3(a). At this x and y, the person with a low μ and a high \$ is indifferent between keeping and discontinuing the business. Since the individual is indifferent between keeping and discontinuing the business, he would be no worse off having a low \$ business, i.e., the additional benefit of a high \$ business is zero here. Now, if the individual had instead a high μ for the same x and y, he would strictly prefer keeping the business. Since he is keeping the business, the individual clearly prefers keeping a good business. This shows that for some x and y the incremental benefit of high \$ rather than low \$ is greater for high μ than for low μ. We can also show that it is never lower. This explains why inequality (4.1) holds.

An interpretation of this result is that even though there is no interaction between μ and \$ in the production of current output, μ and \$ are “complements” in a dynamic sense. Because of this dynamic complementarity, loosely speaking, a high \$ business is worth more to an existing owner who already has a high μ than it would be worth to a new owner since the new owner might draw a low μ. This explains why an owner with high μ and high \$ is so prone to keep rather than sell his business. Analogously, a high \$ business is worth less to an existing owner with low μ than it would be to a new owner since a new owner's μ can only be better than the existing owner's. Hence, an owner with a high \$ business but with low μ is eager to sell.

We can summarize our results by ranking the $\hat{x}$ cutoffs:

(4.2) $\hat{x}_{\mu_H \$_H} < \hat{x}_{\mu_H \$_L} < \hat{x}_{\mu_L \$_L} < \hat{x}_{\mu_L \$_H}.$

The first inequality is illustrated in figure 3(b), the second in figure 2, the third in figure 3(a). Using the inequalities in

(4.2), we can draw some conclusions about how the turnover probabilities $p^K_{\mu\$}$, $p^D_{\mu\$}$, and $p^S_{\mu\$}$ vary with μ and \$. Fixing \$, high μ means a higher probability of keeping, and a lower probability of selling and discontinuing, $p^K_{\mu_H\$} > p^K_{\mu_L\$}$, $p^S_{\mu_H\$} < p^S_{\mu_L\$}$, and $p^D_{\mu_H\$} < p^D_{\mu_L\$}$. Fixing μ, high \$ means a lower probability of discontinuance, $p^D_{\mu\$_H} < p^D_{\mu\$_L}$. However, the effect of increasing \$ on the probability of keeping or selling is more complicated. Holding μ fixed at μ_H, an increase in \$ raises the probability of keeping, i.e., $p^K_{\mu_H\$_H} > p^K_{\mu_H\$_L}$, but has an ambiguous effect on the probability of selling. Holding μ fixed at μ_L, an increase in \$ raises the probability of selling, $p^S_{\mu_L\$_H} > p^S_{\mu_L\$_L}$, but has an ambiguous effect on the probability of keeping.

Looking across all four different combinations of μ and \$, an owner with μ_H and $\$_H$ has the highest probability of keeping and the lowest probability of discontinuance. An owner with μ_L and $\$_H$ has the highest probability of selling. An owner with μ_L and $\$_L$ has the highest probability of discontinuance.

B. The Distribution of (μ,\$) *and the Calculation of Turnover Tables: First Special Case*

The determination of the fraction of businesses of type (μ,\$) in each cell in Table 1B, and hence, the various turnover probabilities, is analytically difficult. Hence, in this subsection, and the next, we examine special cases of the general model. The two special cases differ in their assumptions about μ_L. In the special case examined in this section, we set μ_L equal to its upper bound, that is, $\mu_L = \mu_H$. In this case there is no variation in permanent match quality. In the case of the next section, we set μ_L equal to its lower bound, that is, μ_L = !4 (more on the interpretation of this case below). This second case is of special interest since this parametric form is that chosen by the estimation procedures below.

Suppose then that $\mu_L = \mu_H$. Before discussing the business turnover patterns in this case, we first show that some of the analysis in Section A can be simplified when $\mu_L = \mu_H$. In particular, in the previous section we explained why $\hat{x}$ varied with $ as illustrated in figure 3. Part of the argument relied on the fact that the μ of the new owner could differ from the μ of the previous owner. When $\mu_L = \mu_H$, everyone has the same μ. For this special case we can show that $\hat{x}$ is independent of business quality $. This case is illustrated in figure 4. An increase in $ shifts $\hat{y}$ downward but leaves $\hat{x}$ unchanged. An increase in $ increases the probability of keeping and selling, and lowers the probability of discontinuance, i.e., $p^K_{\$_H} > p^K_{\$_L}$, $p^S_{\$_H} > p^S_{\$_L}$, and $p^D_{\$_H} < p^D_{\$_L}$ (where, note, we have dropped the μ subscript).

B.1 An Analysis of Managerial Tenure

Now consider some of the turnover patterns implied by this model. First, consider a cohort of *individuals* who start businesses in the same period. Suppose we keep track of the “surviving” members of this cohort, i.e., the individuals who keep their businesses period after period. These individuals correspond to those in the first column of Table 1B. Since individuals with high $

businesses are more likely to keep their businesses than those with low $ businesses, the fraction of surviving owners that have high $ will increase through time. This implies that the keep rate will increase over time in the cohort, while the discontinuance rate will decline. This pattern implied by the model matches that found in the first column of Table 1B.

Consider next a cohort of *individuals* who purchase businesses in the same period and, further, purchase businesses of the same age. Suppose we keep track of the surviving individuals in this cohort. These individuals correspond to those on the diagonals of Table 1B. For example, consider those who purchased businesses in 1982 that were less than two years old. These individuals correspond to those in the upper left-hand cell of the nonfounder group in Table 1B (age of business = 0–2, tenure of manager = 0–2). Among these individuals who stay with their business, both the age of the business and the tenure of the manager increase. Hence, if a new survey was taken next "period," these owners would be in the cell: age of business = 3–6, tenure of manager = 3–6. And so on. If we follow the surviving members of this cohort of individuals, then from the arguments in the paragraph above, the model implies that the keep rate increases as we move down the diagonal, while the discontinuance rate declines. This, too, is consistent with the pattern in the CBO. (Note, however, there are two "transitions" in which discontinuance rates increase; from 9 to 16 in one case, from 19 to 20 in another.)

B.2 Analysis of Business Age

Consider next a cohort of *firms* that are founded in a particular period. From figure 4, we see that high $ businesses are more likely to survive, that is, more likely to be kept or sold. This means the average business quality of the surviving members of the cohort increases with the age of the business cohort. Therefore, the probability of discontinuance decreases in the age of the business cohort.

Given these results, consider next examining businesses of different ages but whose managers have the same tenure. For example, consider what the model implies about discontinuance rates as we read down the column in Table 1B in which managers have 0–2 years of tenure. Since older businesses are of higher quality than younger ones, and since high quality businesses are more likely to be sold than low quality, then it follows that the average business quality increases in business age (with managerial tenure held fixed). This implies that discontinuance rates should fall as we read down the 0–2 year tenure column in Table 1B, as they do. This is true in the other columns as well, though there are two cases where this pattern does not hold.

B.3 Founder/Nonfounder Comparisons

We turn now to a comparison of firms that have been sold with firms that are still owned by their founders. Consider two businesses that have been kept by their founders up until date t. During period t, imagine one business is kept and the other is sold. Which business is more likely to be of high quality? That is, which is bigger, $pr(\$ = \$_H * p_0^H, a_t = S)$ or $pr(\$ = \$_H * p_0^H, a_t = K)$, where $pr(\$ = \$_H * p_0^H, a_t)$ is the probability $\$ = \$_H$ given the prior probability, p_0^H, that $\$ = \$_H$ (which is the same for both businesses since both have the same histories) and conditioned upon observing action a_t? The following lemma, proved in Appendix C, shows that the sold business is more likely to be high quality.

LEMMA. Assume the hazard function $g(y)/(1 ! G(y))$ is strictly increasing in y. Assume $\mu_L = \mu_H$. If p_0^H O (0,1), then $pr(\$ = \$_H * p_0^H, a_t = S) > pr(\$ = \$_H * p_0^H, a_t = K)$.

The regularity condition used in the lemma is frequently assumed in theoretical work and is satisfied by the normal distribution, among others. The intuition for the result can be seen by studying figure 4. The key step in the proof is to show that as we increase $\$$ and thus shift the $\hat{y}$ cutoff down, the “size” of the “Sell” region increases relatively more than the size of the “Keep” region. To see why this might be true, suppose for simplicity that y is uniformly distributed. In this case the $(\hat{x}, \hat{y})$ pair would lie in a “box.” The endpoints of the uniform distribution would determine the top and bottom and side boundaries of the box. In this case, as the $\hat{y}$ cutoff shifts down, the keep region runs into the lower endpoint of the distribution of y so relatively little is “added” to the keep region. The “Sell” region does not run into the lower endpoint so it gets relatively larger than the “Keep” region.

Because business sale is an indication of high quality, it is reasonable to expect that, holding business age fixed, that businesses that have been sold are of higher quality than those still owned by their founders. Straightforward calculations show this and hence prove the following corollary of the lemma.

Corollary. Under the same conditions as in the lemma, the probability of discontinuance is lower for a nonfounder firm than for a founder firm established the same year.

This implication of the model is at odds with the patterns in the CBO data. For example, for businesses that are 0–2 years of age, businesses that are owned by their founders have lower discontinuance rates than those that have been sold, 46 percent as compared to 59 percent. But note that because businesses owned by nonfounders are of higher average quality, the model implies that they have higher transfer rates than founder firms. This latter implication is consistent with the CBO patterns.

In summary, the model with $\mu_L = \mu_H$ is able to produce some of the turnover patterns found in the CBO data. However, the model "misses" some key features of the data. In particular, it implies that sold firms have lower discontinuance rates than founder firms of the same age. In the next special case, we add a match dimension to the model. This will act to increase the discontinuance rate of nonfounders relative to founders.

C. The Distribution of $(\mu,\$)$ *and the Calculation of Turnover Tables: Second Special Case*

Consider now the case where μ_L is “extremely” low. In particular, we want to assume a value for μ_L that insures that individuals drawing a bad permanent match will break the match after their first period in the business sector, that is, as soon as they can. But choosing such a value for μ_L takes some care. For example, if there are no restrictions on how large x can be, then to insure a bad match is broken after the first period we literally need to assume that $\mu_L = !4$. But if the return to a bad match takes this value, then the expected discounted returns to a new entrant are equal to !4. In order to bypass these technical problems, we make the following changes to the interpretation of the model. Individuals who enter the economy observe their permanent match in the first period but do not start receiving the return from this match until the next period. Hence, individuals learning they have a bad permanent match will break their match after the first period. Note that this change of interpretation does not influence any decisions in the economy. We will refer to this special case of the model as the case where “$\mu_L = !4$.”

As suggested, an alternative way to bypass these technical problems is to assume that there is some upper bound on the range of the x variable. If this is the case, we can make μ_L low enough so that it is never optimal for an owner with a bad permanent match to keep his business, whatever the level of x, y, or $\$$.

What business turnover patterns are implied by this special case? In order to answer this question, lets retrace the analysis for the previous special case. We will explore whether any of the results change here.

C.1 An analysis of tenure

The first turnover patterns that we sketched above were those in the first column of Table 1B. In order to sketch these patterns for this model, consider a cohort of individuals starting businesses in a particular period, say period t. A fraction 8_F of these individuals draw μ_H, a fraction $1 \; ! \; 8_F$ draw μ_L. At the beginning of period t + 1, no individual with match μ_L will decide to keep his business. Therefore, at the beginning of period t + 2, all surviving members of the cohort have high μ. Hence, average match quality increases as we move down the first column in Table 1B.

What happens to the distribution of business quality as we move down the first column of Table 1B? First, lets consider what happens to business quality from period t + 2 onwards. Since the cohort is comprised only of persons with good matches from this period onwards, and since $p^K_{\mu_H \$_H} > p^K_{\mu_H \$_L}$, the fraction of individuals that have $\$_H$ increases over time.

Now lets consider the change in business quality between periods t + 1 and t + 2. At the beginning of period t + 1, the fraction of good businesses equals $8_F >_{\mu_H} + (1 ! 8_F) >_{\mu_L}$. We will assume throughout this paper that $>_{\mu_H} \; \$ \; >_{\mu_L}$. This says that founders that draw a good match are at least as likely to draw a good business as are founders that draw a bad match. This seems plausible. If this is the case, then the fraction of founders in the cohort at the beginning of period t + 1 with $\$_H$ is no greater than $>_{\mu_H}$. In period t + 2 the fraction of individuals with good businesses will be at least $>_{\mu_H}$. Hence, business quality increases between t + 1 and t + 2.

The conclusion then is that both match and business quality increase as we move down the first column in Table 1B. Going back to the analysis in Section A on how the turnover probabilities varied with μ and $, it is easy to see that the model implies that probability of keeping strictly increases and that the probability of discontinuance decreases as the cohort ages. Falling discontinuance rates in the first column of Table 1B is consistent with the CBO pattern. Recall this was also an implication of the previous case where $\mu_L = \mu_H$.

In the previous section we next considered a cohort of individuals who purchase businesses in the same period and, further, purchase businesses of the same age. As mentioned above, these individuals correspond to those on the diagonals of Tables 1B. As in the previous section, the model here implies that discontinuance rates should decrease along the diagonal.

C.2 Founder/Nonfounder Comparisons

We turn again to a comparison of firms that have been sold with firms that are still owned by their founders. Consider businesses that have been kept by their founders up until date t. How do businesses that are kept during period t compare with those that are sold? We make two comparisons: first, we compare match quality for the managers and then business quality.

The owners who keep businesses have already undergone the process whereby bad matches are eliminated. The new owners of businesses purchased this period have not undergone this process. Hence, the average match quality μ for the founder firms is strictly greater than that for the nonfounder firms. If this were the only factor, then the discontinuance rate of the nonfounder businesses in the period after the acquisition would be greater than that of founder businesses. Recall that this implication is just the opposite of that in the special case above. Note as well that this implication means that this special case has a chance of matching the pattern found in Table 1B where nonfounders have higher discontinuance rates than founders.

The comparison of business quality between the businesses that are kept and those that are sold is not simple as the comparison of match quality. In general, the sold firms may have greater or lower average business quality $ compared to the kept firms. There are a number of factors at work.

The first factor explains why sold firms may be higher quality than kept firms. The point can be made with a simple example. Suppose that all firms have y = 0 (think of this as corresponding to a special case of the model where nearly all the probability mass is centered at y = 0). Then there are only two kinds of businesses on the market, $\$_L$ and $\$_H$ businesses (this contrasts with the general model where there is a continuum of firms on the market differentiated by y). For certain parameters of the model the following is the equilibrium. The price of a $\$_H$ business is zero. In each period there are individuals with $\$_H$ businesses that do not keep these businesses. Some of them discontinue their business, others sell their business (since the price is zero they are indifferent). All individuals with a $\$_L$ business who do not keep their business discontinue their businesses. In this example sale of a business is conclusive evidence that the business has high $\$$. Next, what can we infer about business quality when a business is kept? A low $\$$ business will be kept by its founder if he has a high permanent match μ and if the transient match x is high enough in the period. Therefore, all kept firms are not necessarily high $\$$ firms. Some are low $\$$ businesses that survive because of high match quality. Hence, sold firms have higher average business quality than kept firms in this example.

Having elucidated this principle, and having established in section B that without differences in μ sold firms have higher $\$$ than kept firms, one might be led to believe that there is a general result: namely that, everything else the same, a sold firm has higher average quality than a kept firm. But this is not always true. Recall that for the case of $\mu_L = \mu_H$ we obtained the result that the sold firms had higher $\$$ than the kept firms by showing in figure 4 that an increase in $\$$ led to an increases in the "sell" region that was relatively larger than the increase in the “keep” region. Now consider the case at hand of μ_L arbitrarily small and suppose the founders in the cohort have already undergone the “shakeout” period so that they all have high μ. The appropriate figure is figure 3(b). As we increase $, the keep region gets bigger. But now the sell region can actually decline. In such a case, sale of the firm is a signal of low quality. That is, for such a case, the posterior probability that a firm is $\$_H$ is lower for the sold firms than it is for the kept firms. In such a case sold firms would have higher discontinuance rates than founder firms because their lower average business quality would reinforce their lower average match quality.

5. Estimation of the Model

This section discusses our procedure for estimating the model parameters and presents the estimates. We delay discussion of the estimates until Section 6.

A. Description of the Estimation Procedure

Recall from Section 2 that a key feature of the CBO survey was that we could classify businesses into one of 27 cells defined by the age of the business, the tenure of the manager and the founder status of the owner as seen in Table 1A. Each of these 27 events can be further cross-classified by what happened to the business between 1982 and 1986, that is, whether it was kept, sold or discontinued.[5] These 81 (= 27 @ 3) cells are the focus of the

[5]Note that businesses which were sold in 1982 appear twice in the CBO universe, once when the original owner filed a tax return

analysis. Let the cells or events be indexed by k and let n_k denote the number of businesses in the CBO sample that are in cell k.

Roughly, the estimation procedure works as follows. For a given vector of model parameters, we use numerical methods to solve for the steady-state equilibrium of the model economy. This solution then is provisionally taken as the underlying universe of small businesses from which the CBO was drawn. In particular, we know the fraction of businesses in the universe that lie in each of the 81 cells. The fraction of businesses in each cell in the universe (that is, the model solution) can then be compared to the fraction of businesses in each cell in the CBO sample. The estimation procedure provides a way of choosing a vector of model parameters (that is, an underlying universe) so that the two fractions in each cell, that is, the fraction of businesses in the universe in a cell and the fraction of businesses in the CBO sample in that cell, are “close.” We now turn to a more formal description of the procedure.

Let 1 denote a vector of underlying parameters of the model economy where 1 = (*, e, parameters defining F(@) and G(@), w, (, μ_L, μ_H, $\$_L$, $\$_H$, 8_{NF}, 8_F, $>_{\mu_L}$, $>_{\mu H}$). For a given parameter vector 1,

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for the first part of the year and second when the new owner filed a tax return for the latter part. In constructing the model economy universe we therefore include individuals who were in the small business sector at the beginning of period 1982, or at the end of the period, or throughout the period. In sampling in this manner, businesses sold during period 1982 will appear twice in the analog universe, just as in the CBO universe.

we use numerical methods to calculate the steady-state competitive equilibrium. We next assume that the length of a period in the model economy is one year. We select an arbitrary period in the model to correspond to the year 1982. This solution then is provisionally taken as the underlying universe from which the CBO was drawn. Let $p_k(1)$ denote the fraction of all businesses in cell k in the model economy when the parameter vector equals 1. These fractions are easily calculated from the model solution.

When a business was sampled during the CBO survey, we think of it as being a random draw from the population which could result in one of the 81 mutually exclusive outcomes discussed above. Hence, the random vector $(n_1,n_2,...,...,n_{81})$ has a multinomial distribution. The probability of observing the CBO sample $(n_1,n_2,...,n_{81})$, given $(p_1(1),p_2(1),...,p_{81}(1))$, is therefore given by

$$(5.1) \qquad L(1) = \frac{n!}{n_1!...n_{81}!} \cdot \mathbf{p}_1(\Theta)^{n_1} \cdot \mathbf{p}_2(\Theta)^{n_2} \cdot \, ... \, \cdot \mathbf{p}_{81}(\Theta)^{n_{81}}.$$

Our estimation procedure is to find the parameter vector 1 which maximizes the (log of) equation (5.1), the likelihood function.[6]

[6]One difficulty in estimation is that there are cells which have zero probability in the model economy but for which there are observations in the CBO survey. Given our assumption that the period length is one year, there are no nonfounder firms in the model economy which were acquired in 1982 and established in 1982 (firms must be one period old before transfer can take place in the model). Yet there are 60 individuals in the CBO survey who claim to be nonfounders who acquired in 1982 a business established in 1982 (this is 20 percent of the 306 nonfounders who reported having acquired their business in 1982). We proceed by reallocating these observations in the cell which is the “nearest” neighbor, i.e., we shift the observations of nonfounders acquiring in 1982 businesses established in 1982 to the cells containing nonfounders acquiring in 1982, businesses established 1980–82.

B. The Model Parameters

We assume that transient match quality x_t and transient business quality y_t are both normally distributed with zero mean and variance F_x^2 and F_y^2, respectively. The parameters F_x, F_y, w, μ_L, μ_H, $\$_L$, $\$_H$, are all denoted in terms of units of the consumption good. Without loss of generality, we can normalize these units so that $F_x = 10$.

There are two other normalizations that are made. First, note that if one unit is added to the outside return w and to both μ_L and μ_H, then in each period the return to individuals is increased by a unit (independent of any decisions). Since there are no income effects in the model, these additions would not change any decisions. As an identifying assumption we therefore set w = 0. Second, note that if we add one unit to both μ_L and μ_H ,and subtract one unit from both $\$_L$ and $\$_H$, then the return in each period is unchanged, and so as above, these additions would not change any decisions. As an identifying assumption we therefore set $\$_L = 0$. In summary, regarding the parameters denoted in units of the consumption good, we make the identifying assumptions $F_x = 10$, w = 0, and $\$_L = 0$, and estimate F_y, μ_L, μ_H, and $\$_H$.

We chose not to estimate the discount factor and instead constrained * to equal 0.95. This is a plausible discount factor since the period length is one year. There is a sample analog to the parameter e, the fraction of new entrants who start businesses. The sample analog is the fraction of individuals in the CBO who entered in 1982 by starting their business. This fraction equals

0.874 and we directly set e = 0.874 rather than include this parameter in the maximum likelihood procedure. The growth rate parameter (does not have an exact sample analog so we did estimate this parameter.

The final set of parameters are the probabilities of drawing good matches and good businesses. Recall that the probability a nonfounder draws μ_H is 8_{NF} and that the probability a founder draws μ_H is 8_F. Given a founder draws μ_H, the probability he draws $\$_H$ is $>_{\mu_H}$ while if he draws μ_L, the probability of drawing a good business is $>_{\mu_L}$. We considered some alternative assumptions about these parameters. Our first assumption, which we call “Model 1,” is that the probability of drawing a good match is the same for nonfounders and founders, $8_{NF} = 8_F$, and that the probability a founder draws a good business is independent of the match that he draws, $>_{\mu_L} = >_{\mu_H}$. In the second specification, “Model 2,” we permit the probability that a founder draws a good business to depend upon the match that he draws, so that $>_{\mu_L}$ … $>_{\mu_H}$ is allowed. In “Model 3,” we further permit the probability of drawing a good match to depend on whether the business is being started or acquired from another owner, that is, we allow both 8_{NF} … 8_F and $>_{\mu_L}$ … $>_{\mu_H}$.

Table 2 presents the maximum likelihood estimates for the three models.[7] Standard errors of the estimates are presented as well.[8]

The estimates for the three specifications are qualitatively similar. In all three specifications, the standard deviation of the transient business shock y_t is estimated to be about 10 percent of the standard deviation of the transient match shock x_t which we

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[7]In this procedure we are wary of the fact that a local optimum is not necessarily a global optimum. In calculating the optimum we considered a wide range of starting points. We also plotted out the shape of the likelihood function for some key parameters. For example, we maximized the likelihood function for various fixed levels of the parameter 8 and plotted out this function of 8 to examine its shape. For model 1 there is a second (inferior) local optimum. We only found one local optimum for both models 2 and 3.

[8]We used the following “bootstrap” technique to estimate the standard errors: We took the parameter estimates and solved for the equilibrium distribution $p_k(2)$ across the 81 cells. We then drew 15,737 random draws from this distribution (the number of observations in the CBO survey) and then applied the estimation procedure to this simulated data set. We repeated this procedure 50 times and then calculated the distribution of the parameter estimates for these 50 simulated data sets. From this distribution of 50 realizations we calculated the standard errors. This much is standard. But note that for Models 2 and 3 our parameter estimate for μ_L is !4. The standard error is not a useful summary statistic of the distribution of the parameter estimated in this case, so we report other features of these distributions. In the case of Model 2, in 23 out of the 50 simulated data sets (46 percent) the estimate for μ_L was !4. In the remaining 54 percent of the data sets the estimate of μ_L ranged from a low of !29.8 to a high of !13.7. In terms of quintiles, 100 percent were below !13.7, 80 percent were below !18.0, 60 percent were below !21.9, and the 40 and 20 quintiles were both at !4. For Model 3, in 40 percent of the data sets the estimate for μ_L was !4 and for the remaining 60 percent the range was !59.9 to !13.8. The quintiles were !13.8, !18.9, !40.1, !4, !4. The key point here for both Models 2 and 3 is that although μ_L is not precisely estimated at !4, we have a high degree of confidence that it is a negative number with an extremely high absolute value compared with the other parameters of the model.

normalized at 10. In all three specifications, there is an extreme difference between μ_L and μ_H. In fact, in models 2 and 3, the likelihood is maximized by taking the parameter μ_L to its limit of !4. The data is choosing the polar case of the model that we discussed earlier. In this polar case owners with low μ exit the small business sector in the period immediately after they acquire the business; all owners that remain have high μ. In models 1 and 3, $\$_H$ is approximately equal to μ_H. In these models, given a high μ, having high $ doubles the permanent component of a firm's output. Model 2 is somewhat different in that $\$_H$ is more than twice μ_H. The growth rate (is between 1 and 2 percent for all three models. This is roughly consistent with the historical growth rate in the number of proprietorships.[9]

The probability of drawing a high μ is about 0.6 in all three specifications. In model 3, where 8_{NF} is permitted to differ from 8_F, the probability a nonfounder draws high μ is 0.65 which is somewhat larger than the probability that a founder draws high μ, 0.52. Turning to the probability that a founder draws high $, one can see that model 1 differs significantly from models 2 and 3. In model 1 this probability is 0.037. In the other two models this probability can depend upon the match quality drawn by the founder. In both models it is estimated that a founder drawing a

[9]The actual annual average growth rate in the number of proprietorships from 1957 to 1980 was 1.6 percent. The actual average annual growth rate from 1970 to 1980 was 3 percent. We stop at 1980 because the definition of the series changed in 1981. (Source: Statistics of Income Source Book on Sole Proprietorship Returns, 1957–84.)

bad match μ_L has a zero chance of drawing a good business $\$_H$. In contrast, a founder drawing a good match in models 2 and 3 has chances 0.18 and 0.14 of drawing a good business, respectively.

Consider next measures of goodness-of-fit for the various models. A conventional goodness-of-fit test is the chi-squared test. The model fails this test by a large margin. As discussed in Pakes (1986), and references cited therein, this problem occurs frequently in models designed to analyze proportions when the underlying sample size is large. The bottom of Table 2 presents some other measures of goodness-of-fit. Let $\tilde{p}_i = n_i/N$ denote the fraction of all observations in cell i in the CBO data. The sum of the absolute deviations between the empirical fractions $\tilde{p}_i$ and the predicted fractions $p_i(1)$ is 0.220 for model 1, 0.180 for model 2, and 0.187 for model 3. It is surprising that the figure for model 3 is larger than for model 2, since model 3 is a less restricted version of model 2. This illustrates that the maximum likelihood criterion is not perfectly correlated with other measures of goodness-of-fit. An alternative summary measure is provided by looking at the mean squared deviation between $\tilde{p}_i$ and $p_i(1)$ (MSE in Table 2) and comparing it to the variation in $\tilde{p}_i$ across the 81 cells (V in Table 2). The ratio MSE/V is presented in the last row of Table 2; it equals 4.3 percent, 2.8 percent, and 3.1 percent, respectively for models 1, 2, and 3. Again, model 2 fares best under this measure.

For the remainder of the paper we prefer to discuss a single model rather than all three. One way to possibly narrow the range of models is test the constraints on the probabilities (that is, the 8 and $>$ parameters) imposed in models 1 and 2. The log of the likelihood increases by 166 points when we relax the constraint that $>_{\mu_L} = >_{\mu_H}$ and by an additional 37 points when we further relax the constraint that $8_{NF} = 8_F$. The differences in the likelihood functions are sufficiently large that both constraints can be rejected in a likelihood ratio test by a large margin.

While this is true, we choose model 2 as the model to discuss in the rest of the paper. When we refer to the “model economy,” we shall mean the economy with parameters listed under model 2 in Table 2. Our reasons are as follows. Relaxing the constraint that moves us from model 1 to model 2 leads to a substantial improvement in the likelihood function and this improvement in fit is corroborated with the other measures of fit. This motivates our choice of model 2 over model 1. To explain our choice of model 2 over model 3, we first note that relaxing the constraint that takes us from model 2 to model 3 leads to a relatively small improvement in the likelihood function and there is actually a deterioration in the other measures of goodness-of-fit. Second, for the purposes of the next section where we study how the model economy works, we think the assumption that $8_{NF} = 8_F$ is attractive. Under this assumption founders are similar to nonfounders except for the fact that they are at different stages of the selection process. If 8_{NF} is different from 8_F, then founders and nonfounders are different for reasons that are outside the model.

C. Robustness of Estimates

This section discusses the robustness of the parameter estimates to alternative selections of the data. The data set used so far includes all nonminority male-owned businesses in the Characteristics of Business Owners sample. In this section we report the parameter estimates of the model when different subsets of the data are used in the estimation procedure. These estimates are presented in Table 3. For ease of comparison, the first column of Table 3 presents the estimates of the model for the entire data set (and so is analogous to the second column in Table 2). The data sets used in columns 2–6 are explained below.

The second column of Table 3 presents the model estimates when the data set is restricted to proprietorships. Proprietorships makes up about 90 percent of the CBO business population (the other 10 percent being partnerships and corporations). These businesses are all owned by a single individual. By looking at proprietorships we avoid the issue of how to treat multi-owned businesses.[10] Note that 8 in this table denotes the common probability that founders and nonfounders draw a good match, $8 / 8_{NF} = 8_{F}$. The

[10]The problem with multi-owned firms is that the different owners of the same firm may have different tenure and this doesn't happen in the model where all firms have a single owner. In the case of a multi-owned business our procedure was to randomly select one of the owners of the business and use his tenure. We considered some alternative procedures as well but the different treatments made little difference in the results in Table 2 because most businesses have a single -owner and in the multi-owner cases, the tenure of the owners are often the same because the group of owners often acquire the business together. These issues are further discussed in Holmes and Schmitz (1993).

estimates for the proprietors only data set are very similar to the estimates for the entire data set.

The next two columns in Table 3 correspond to data sets with “size” restrictions placed on the businesses. Many of the businesses in the CBO population are quite small. Many are part-time operations at which the owner works less than 10 hours a week. Some businesses have as little as $100 in receipts for all of 1982. While we think it is appropriate not to place size restrictions on the sample (after all, new businesses may start out small, and old businesses that are about to close may first undergo a reduction in size), it is worth asking how sensitive our results are to the inclusion of these smallest of businesses. The third column of Table 3 contains our estimates using the sample consisting of proprietors working 30 hours or more a week at the business. This hours restriction eliminates almost 40 percent of the proprietors. Nevertheless, the basic results are the same, in particular, the estimated model continues to display the polar case where a poor match is so bad that the owner leaves the business at the first opportunity.

Column four contains the estimates using the data set that excludes firms with less than $5,000 in receipts. The estimated return to a bad match is different than that in the first three columns: the estimate for μ_L is no longer at the limit point of minus infinity. Nevertheless, the results are qualitatively the same in that the difference between the returns to match qualities μ_H and μ_L of 9.79 ($0.43 - -8.36 = 9.79$) dwarfs the difference between the returns to business qualities $\$_H$ and $\$_L$ of 0.21 (0.21!0 = 0.21). While there is no longer a “shakeout” period where all bad matches are eliminated in the first period after acquisition, bad matches are eliminated rather quickly for these parameters. For example, the probability an owner drawing μ_L will keep his business three periods or more is less than 0.015.

The last issue we discuss is that related to industries. The businesses in the CBO population are in diverse industries. There are businesses from every Census two-digit industry except agriculture. Industries are likely to vary in the distribution of match and business quality for businesses within the industry. For example, it may be that business quality is of little importance in the taxi (transportation) industry. Whether a taxi driver is successful seems likely to be tied to whether the person is a good match to the unique aspects of the job.

The estimates in columns 1–4 use data sets that include businesses from all industries. There is nothing “wrong,” per se, with grouping industries in the estimation. For example, we can imagine that there are such businesses as taxi businesses in the model economy and that each taxi business has low business quality $\$_L$. We can imagine that there are other businesses, say restaurants, and that a certain fraction of these business have high business quality $\$_H$. The model economy is consistent with there being different industries that vary in their distribution over $\$_L$ and $\$_H$.

Still, it would be interesting to use the techniques of this paper to examine industry level data, to estimate the degree to which match quality and business quality vary across industries. Here we look at a more limited issue. In column 5 we present the estimates for the data set that only includes retail and service businesses. The "Retail Trade" sector and the "Services" sector together comprise more than one half of all the firms in the data set (i.e., SIC codes 5200-5999 and 7000-7999). These are the corner stores and barbershops that come to mind when one thinks of small businesses. Column 6 contains the estimates for the case of all firms *except* services and retail. The estimates for these two mutually exclusive sets of industries are remarkably similar to each other and to the estimates from the combined data set in column 1.

6. Discussion of Estimated Model Economy

In this section we examine the estimated model economy. We begin by presenting the business turnover rates in the estimated model economy, comparing them to those in the CBO survey.

A. A Comparison of the Model and CBO Business Turnover Rates

Table 4 presents business turnover rates. The top panel of the table gives turnover rates from the CBO survey; this panel reproduces the information that was presented in Tables 1B and 1C above. The bottom panel presents turnover rates for the estimated model economy (again, the model economy associated with Model 2 in Table 2).

As a way to compare the turnover rates in the top and bottom panels, recall the discussion of the CBO survey in Section 2. There we highlighted a few points about the CBO turnover rates. The first point was that, examining businesses of the same age, the probability that the business fails is initially decreasing in the tenure of the manager. At some point the discontinuance rate begins to increase. As can be seen in the bottom panel of Table 4, this pattern is true of the model economy for nonfounders. Below we will discuss why the model produces this pattern. Note, however, that discontinuance rates for founders do not begin to increase in tenure after some period.

The second point was that, examining businesses whose managers have the same tenure at their business, the probability that the business fails is typically decreasing in the age of the business. This pattern is true of the model economy as well.

The last points we mentioned concerned comparisons of turnover rates for founder and nonfounder businesses of the same age. We mentioned that, examining businesses of the same age, businesses owned by nonfounders with tenure of 0–2 years have higher discontinuance rates than businesses owned by their founders (except for the very oldest businesses, those of 23+ years), while the opposite was true for transfer rates. These patterns are true of the model economy as well, though the magnitudes in the CBO differ from those in the model.

It should be kept in mind that the procedure we used to “fit” the data tries to match the age and tenure distribution of businesses in the CBO survey as well as their turnover behavior. Table 5 compares the age distribution of firms in the CBO sample and the model economy. It also tabulates the percent of firms that are nonfounder firms by the age of the firm. The distributions in the CBO sample and the model economy are similar, particularly the fraction of the business population that is nonfounder firms.

B. Turnover Probabilities and the Distribution of (μ,$)

In this section we provide some intuition for what is driving the turnover patterns in the model economy that were presented in Table 4. To do this we first describe the probabilities of turnover given (μ,$), then describe how the distribution of (μ,$) changes over time.

The turnover probabilities for the estimated model economy, that is, the probability of keeping, selling and discontinuing are given in Table 6, along with other selected variables. First, consider the probability of keeping. If a manager draws a bad match then the probability that the business is kept is zero. If the business manager draws a good match, the probability that the business is kept is high, 0.904 in the case of a bad business, 0.953 in the case of a good business.

Consider next the probability of sale. If the manager draws a bad match then the probability of sale is higher than if a good match was drawn. For managers with bad matches, the probability that a business is sold is much higher if the business is a good one (0.690 for good businesses as compared to 0.064 for bad businesses). Recall that there was no general result regarding how the probability of sale varied with β given $\mu = \mu_H$. For the parameter values of the estimated model economy, the probability of sale is decisively higher for managers with good businesses (0.03 for good businesses as compared to 0.004 for bad businesses).

We now discuss how the distribution of match and business qualities evolves over time. First, consider a cohort of new businesses. We are interested in describing how the distribution of match and business qualities evolves over time among the businesses in this cohort that are kept by their founders. This distribution is displayed in Table 7A. After the businesses are initially started, when the businesses are age one, all the businesses are still owned by their original founders. A fraction $(1-\lambda) = 0.439$ of these owners drew a bad match; all of these owners therefore drew a bad business as well (recall $\xi_{\mu_L} = 0$ in the estimated model parameters in Table 2). Hence, the first two numbers in the first row of Table 7A are 0.439 and 0.000. Next, a fraction 0.562 of the managers drew good matches; among these owners, a fraction 0.176 drew good businesses. Hence, a fraction 0.099 ($\approx 0.562 \cdot 0.176$) of the managers have good matches and good businesses; the remaining portion, 0.462, have a good match but a bad business. Hence, the last two numbers in the first row are 0.462 and 0.099.

All founders that draw bad matches either discontinue or sell their businesses after the first period, hence the first two numbers in the second row are 0.000 and 0.000. Among founders that have a good match, those with a good business are more likely to keep the business than those with a bad one (the probabilities of keeping are 0.953 and 0.904, respectively). Hence, in moving from age one to two, the fraction of good businesses in the cohort continues to increase, from 0.099 to 0.184. The share of good businesses in the cohort continues to increase over time. However, since the difference between 0.904 and 0.953 is "small," the selection process works slowly. Even by age of 20 only 0.367 percent of the remaining founders have a good business. This table supports the following characterization of founders firms: Most founder firms have low business quality. Those founder businesses that survive do so because their managers have high match quality.

The story is different for firms that have been sold at least once (i.e., nonfounder firms). Table 7b presents the quality distribution for nonfounder firms by the age of the business.[11] Note that in this table we are not controlling for the tenure of the manager. For example, among businesses that are age five, there are managers with 1, 2, ..., 4 years of tenure.

We first discuss the case of businesses that are five years old or greater. As compared to founder businesses of the same age, a relatively large fraction of the nonfounder businesses are

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[11]The nonfounder table does not have an entry for businesses of age one because at the beginning of a period all nonfounder businesses are necessarily two years old or greater.

good businesses. For example, of nonfounder businesses of age 10, a fraction 0.593 (= 0.539+0.054) are good businesses; the analogous fraction for founder firms is only 0.255. Good businesses are relatively more likely to be sold than are bad businesses in the model. Hence, if a business has been sold, this is an indication of quality in the model.

An exception to this last point arises for the case of very young businesses. To see this, consider a cohort of new businesses. At age one, a fraction 0.439 of the new business owners have a bad match and a bad businesses. While any given owner with both a bad match and a bad business has a low probability of selling the business (only 0.064), because there are so many such owners, a large fraction of the businesses that are sold immediately after startup are bad businesses. Hence, among businesses that are age 2, a larger fraction of nonfounder businesses are of bad quality than are founder businesses, 0.908 (= 0.398+0.510) as compared with 0.816. Early sale, then, is an indication here of poor business quality.

We are now in a position to describe the intuition for some of the turnover patterns implied by the model. The last effect above explains why businesses that are started and then sold right away have high discontinuance rates. In the model economy businesses of age 0–2 owned by nonfounders have higher discontinuance rates than those still owned by founders (54 percent compared to 51 percent). This effect also accounts for why the discontinuance of nonfounders begins to increase in tenure after a certain point holding the age of the business fixed. When we fix the age of the business and increase the tenure of the nonfounder owning the business, we decrease the age of the business at the time of acquisition. Sale of the business when the business is recently established is a negative signal about business quality in the model economy. In the model economy for businesses of age 13–22 owned by nonfounders, discontinuance rates start at 24 percent, fall to 11 percent, and then increase to 19 percent.

Consider next a comparison of the first two columns of discontinuance rates for the model economy, that is, compare discontinuance rates for founder businesses and recently acquired nonfounder businesses (0–2 years of tenure) of the same business age. For businesses that are age 3 or greater, nonfounders have higher average business quality than their counterpart founder firms of the same age. On the other hand, the selection process eliminating bad matches has not been completed yet for the non-founder firms while the process is complete at this point for founder firms. The disadvantage the nonfounder firms have in lower average match quality more than offsets their advantage in higher business quality so that the nonfounders have higher discontinuance rates than founder firms of the same age. For the youngest firms (0–2 years of age), the selection process eliminating bad matches is incomplete for both founder and the nonfounder firms. Hence, average match quality is basically the same for both groups. But in this case the nonfounder firms have lower business quality and this accounts for why they have higher discontinuance rates than their founder counterparts.

Finally, it is instructive to briefly compare model 1 and model 2. In model 1, founders drawing a bad match are as likely to start a good business as founders drawing a good match (i.e., $>_{\mu_L} = >_{\mu_H}$). Because of this, individuals drawing bad matches and leaving their businesses include owners of both good and bad businesses. Hence, even for the youngest firms, sold firms have higher quality than firms that are kept. The result is that in model 1 businesses of age 0–2 that have been sold have lower discontinuance rates than their founder counterparts. This is in contrast to the pattern in the actual CBO data. Model 1 also does not generate the pattern that the discontinuance rate for nonfounders is U-shaped in tenure for businesses of a given age. Model 2 is able to match these patterns in the CBO data by taking $>_L$ to zero.

Appendix A:

The Supply of Businesses

In this appendix we derive the "supply" of businesses. It will be useful to first derive the number of businesses in the steady-state equilibrium. Let $n_{\mu\$,t}$ denote the number of businesses in existence at the beginning of time t with match quality μ and business quality \$. The number of businesses in period t + 1 is a simple function of the number of businesses in period t and the actions of individuals as described by the policy function in figure 1. The number of such businesses that are good businesses ($\$ = \$_H$) and whose owners are a good match ($\mu = \mu_H$) with the business are

(A1) $$n_{\mu_H\beta_H,t+1} = \lambda_F \xi_{\mu_H} \cdot \mathbf{e} \cdot M_t + \lambda_{NF} \cdot \left(\mathbf{p}^S_{\mu_L\beta_H} \cdot n_{\mu_L\beta_H,t} + \mathbf{p}^S_{\mu_H\beta_H} \cdot n_{\mu_H\beta_H,t}\right).$$

The first term consists of those businesses that were newly started in period t. The total number of new firms created was $e @ M_t$; a fraction $8_F>_{\mu_H}$ had a good match μ_H and good business quality $\$_H$. The second term consists of the businesses that were purchased in time t. It equals the probability 8_{NF} that an individual draws a good match times the total number of firms of quality $\$_H$ that were sold in period t. The latter is obtained by summing, over both possible match qualities, the number of $\$_H$ quality businesses multiplied by the probability of business sale for this type of business. The formula for the other three μ and \$ combinations are similarly defined.

In steady-state equilibrium, the number of established businesses in each period of type $\mu\beta$ grows at the rate γ (the rate of new entry into the economy). Let n_t be the (4×1) matrix consisting of the number of businesses of each type $\mu\beta$. In steady-state equilibrium,

$$\text{(A2)}\quad n_{t+1} = n_t \cdot (1+\gamma).$$

Substituting (A1) (and the analogs of (A1) for the other three μ and β combinations) into (A2) yields four linear equations in four unknowns. For $\gamma \geq 0$ we can show that for each price vector (b_L,b_H) that there exists a unique solution $n_t(b_L,b_H)$ to these four equations.

We can now calculate the total number of businesses available for sale in period t as a function of the prices (b_L,b_H). We call this "supply" in period t. It equals

$$\text{(A3)}\quad \text{Supply}_t(b_L,b_H) = \sum_{\mu\beta} p^S_{\mu\beta}(b_L,b_H) \cdot n_{\mu\beta,t}(b_L,b_H).$$

Appendix B

The purpose of this appendix is to present a proof for the inequalities stated in equation (4.2) in the text, that is, $\hat{x}_{\mu_H\$_H} < \hat{x}_{\mu_H\$_L} < \hat{x}_{\mu_L\$_L} < \hat{x}_{\mu_L\$_H}$.

Before presenting the proof, we state and prove a rather lengthy lemma. To simplify notation we drop the μ and $ in subscripts. In this appendix the first subscript denotes match quality and the second denotes business quality. For example, $\hat{x}_{HL} = \hat{x}_{\mu_H\$_L}$.

LEMMA. Assume $0 < 8_{NF} < 1$ and $\mu_L < \mu_H$. Assume b_L and b_H satisfy:

(B1) $b_H = b_L + \$_H \,!\, \$_L + (1!8_{NF})*[Ev_{LH}!Ev_{LL}] + 8_{NF}*[Ev_{HH}!Ev_{HL}]$.

Then $Ev_{HH} \,!\, Ev_{HL} > Ev_{LH} \,!\, Ev_{LL}$.

The condition (B1) on b_L and b_H imposed by the lemma is condition (ii) in the definition of equilibrium that individuals buying businesses be indifferent between $\$_L$ and $\$_H$. The lemma states that if this condition holds then the value function for an individual has a certain property. This property is given by condition (4.1) in Section 4.

In order to prove this lemma, it will be useful to first prove a similar lemma for an individual that faces a decision problem that lasts a finite number of periods, say T periods, rather than the infinite horizon studied in the text. The lemma for the finite horizon will take the same form: given a certain condition on prices, the value function has certain properties. Let us briefly set up the finite horizon decision problem before stating the finite period version of the lemma.

So consider an individual that faces the same choices as an individual in the text except that the person's horizon lasts only T periods. At the T'th period in that person's life, the person must either sell or discontinue the business (if the person is still in the business). Given a sequence of prices for businesses (that may depend on time), we define $v^{a}_{\mu\beta,t}(x,y)$ to be the maximized discounted return to the individual from picking action a, $a \in \{K,S,D\}$, at time $t \in \{0,1,...,T\}$ and let $v_{\mu\beta,t}(x,y)$ be the maximum value of these three choices. These are the value functions for the finite horizon problem.

The condition that will be assumed for prices is as follows. Let the price of a bad business be a constant equal to b_L for all t. We define the sequence of prices for a good business recursively. In this construction, without loss of generality, we set $w = 0$. Let $b_{H,T} = b_L$. Now $v^{S}_{\mu\beta,T}(x,y) = b_\beta + y$ and $v^{D}_{\mu\beta,T}(x,y) = 0$. The individual cannot keep the business at $t = T$ so $v_{\mu\beta,T} = \max\{v^{S}_{\mu\beta,T}(x,y), v^{D}_{\mu\beta,T}\}$. For $t < T$, define these objects recursively by

(B2) $b_{H,t} = b_L + \beta_H - \beta_L + (1-\lambda_{NF})*[Ev_{LH,t+1} - Ev_{LL,t+1}] + \lambda_{NF}*[Ev_{HH,t+1} - Ev_{HL,t+1}]$,

(B3) $v^{K}_{\mu\beta,t}(x,y) = \mu + \beta + x + y + Ev_{\mu\beta,t+1}$,

(B4) $v^{S}_{\mu\beta,t}(x,y) = b_{\beta,t} + y$,

(B5) $v^{D}_{\mu\beta,t}(x,y) = 0$,

(B5′) $v_{\mu\beta,t} = \max\{v^{K}_{\mu\beta,t}(x,y), v^{S}_{\mu\beta,t}(x,y), v^{D}_{\mu\beta,t}(x,y)\}$.

We are now in a position to state the finite horizon version of the lemma.

LEMMA. (Finite Horizon). Assume $0 < 8_{NF} < 1$ and $\mu_L < \mu_H$. Assume the horizon is T periods. Assume b_L is constant and that $b_{H,t}$ satisfies (B2). Then $Ev_{HH,t} ! Ev_{HL,t} > Ev_{LH,t} ! Ev_{LL,t}$, $t < T$.

PROOF. The first step of the proof is to show that the value functions satisfy a weak inequality, that is, (B6) below. The second step is to show that they satisfy the strong inequality, that is, (B14) below.

<u>Step 1</u>.

Turning to the first step then, we want to show that

(B6) $Ev_{HH,t} ! Ev_{HL,t} \$ Ev_{LH,t} ! Ev_{LL,t}, \quad t \# T.$

Since $b_{H,T} = b_L$, the LHS and RHS of (B6) are both zero so (B6) holds for $t = T$. So we now suppose (B6) is true for $t + 1$ and show it is true for t. In order to do this, it is sufficient to show that (B7) holds at each point (x,y), that is,

(B7) $v_{HH,t}(x,y) ! v_{HL,t}(x,y) \$ v_{LH,t}(x,y) ! v_{LL,t}(x,y).$

Let $a_{\mu\$,t}(x,y)$ be the optimal action given μ, \$, t, x, and y and let $\hat{x}_{\mu\$,t}$ and $\hat{y}_{\$,t}$ be the corresponding cutoffs. The point $\hat{x}_{\mu\$,t}$ solves $v^{K}_{\mu\$,t}(x,y) = v^{S}_{\mu\$,t}(x,y)$, or

(B8) $\hat{x}_{\mu\$,t} = b_{\$,t} ! \mu ! \$! Ev_{\mu\$,t+1}.$

It is immediate from (B8) that $\hat{x}_{H\$,t} < \hat{x}_{L\$,t}$ for either \$. Using (B8) we have

$$\text{(B9)}\quad \hat{x}_{HH,t} ! \hat{x}_{HL,t} = b_{H,t} ! b_L ! (\$_H!\$_L) ! [Ev_{HH,t+1}!Ev_{HL,t+1}].$$

But now note that the definition of $b_{H,t}$ in (B2) and the fact that (B6) holds by assumption for t + 1 then implies that (B9) is nonpositive, i.e., $\hat{x}_{HH,t} \# \hat{x}_{HL,t}$. An analogous argument shows $\hat{x}_{LH,t}$ $\$ \hat{x}_{LL,t}$. (Note that it is precisely these inequalities that it is our ultimate objective to prove hold for the infinite horizon case.) We now show that these inequalities imply that (B7) holds at each point (x,y).

There are two cases:

Case 1. $x \# \hat{x}_{HH,t}$

At such an x, $a_{HH,t}(x,y) \ldots K$. We have shown that $\hat{x}_{HH,t} \# \hat{x}_{\mu\$,t}$ for all μ and $. Hence $a_{\mu\$,t}(x,y) \ldots K$ for all μ and $. Since no type is keeping (where we refer to a (μ,$) pair as a type), payoffs at this (x,y) are independent of μ. This implies condition (B7) holds with equality.

Case 2. $x > \hat{x}_{HH,t}$

If $a_{HH,t}(x,y) = D$ at this point (x,y) all the other types also discontinue so that the LHS and RHS of (B7) are both zero so that (B7) holds. So now assume that y is high enough so that $a_{HH,t}(x,y)$ = K. We consider three subcases.

Subcase (i) $a_{HL,t}(x,y) = K$. This implies

$$\text{(B10)}\qquad v_{HH,t}(x,y) ! v_{HL,t}(x,y) = \$_H ! \$_L + *[Ev_{HH,t+1}!Ev_{HL,t+1}].$$

Suppose $a_{LH,t}(x,y) = K$. Then, since $v_{LL,t}(x,y) \# v^K_{LL,t}(x,y)$,

$$\text{(B11)}\qquad v_{LH,t}(x,y) ! v_{LL,t}(x,y) \# \$_H ! \$_L + *[Ev_{LH,t+1}!Ev_{LL,t+1}].$$

Since (B6) holds for t + 1, (B10) and (B11) imply that (B7) holds.

Suppose $a_{LH,t}(x,y) = S$. Then since $v_{LL,t}(x,y) \# v^{S}_{LL,t}(x,y)$,

(B12) $\quad v_{LH,t}(x,y) ! v_{LL,t}(x,y) \# b_{H,t} ! b_{L}$.

But then the definition of $b_{H,t}$ in (B2) and the fact that (B6) holds for t + 1 together imply that (B7) holds.

Then final possibility is $a_{LH,t}(x,y) = D$. In this case the RHS of (B7) is zero. So the inequality holds.

Subcase (ii) $a_{HL,t}(x,y) = S$.

Since $\hat{x}_{LL,t} > \hat{x}_{HL,t}$, since $\hat{x}_{LH,t} \$ \hat{x}_{LL,t}$, and since $\hat{y}_{L,t} \$ \hat{y}_{H,t}$, $a_{LL,t}(x,y) = S$ and $a_{LH,t}(x,y) = S$. Since $v_{HH,t}(x,y) \$ v^{S}_{HH,t}(x,y)$, to prove inequality (B7) holds it is sufficient to prove

(B13) $\quad v^{S}_{HH,t}(x,y) ! v^{S}_{HL,t}(x,y) \$ v^{S}_{LH,t}(x,y) ! v^{S}_{LL,t}(x,y)$,

which holds since both sides equal $b_{H,t} ! b_{L}$.

Subcase (iii) $a_{HL,t}(x,y) = D$.

In this case $a_{LL,t}(x,y) = D$ so $v_{HL,t}(x,y) = V_{LL,t}(x,y)$. So (B7) holds if $v_{HH,t}(x,y) \$ v_{LH,t}(x,y)$ which is immediate.

We have now completed each case and each subcase. Therefore condition (B7) holds at each point (x,y). This implies (B6) holds.

<u>Step 2.</u>

We now show a strict inequality holds for each t < T, i.e.,

(B14) $\quad Ev_{HH,t} ! Ev_{HL,t} > Ev_{LH,t} ! Ev_{LL,t}, \quad t < T$.

Since the weak inequality in (B7) holds at each point (x,y), it is sufficient to show that a strict inequality holds for a set of (x,y) that is not measure zero. We showed above that for t < T $\hat{x}_{HH,t}$ # $\hat{x}_{HL,t}$ < $\hat{x}_{LL,t}$ # $\hat{x}_{LH,t}$. Note also that $\hat{y}_{H,t} < \hat{y}_L$. For x just greater than $\hat{x}_{HH,t}$ and y just greater than $\hat{y}_{H,t}$, all types choose D except HH. For x and y in this region the strict inequality holds for (B7). This proves that (B14) holds. □

This completes the proof for the version of the lemma where the individual faces a finite horizon. Taking the objects v $_{\mu\$,t}$(x,y) and $b_{H,t}$ from this lemma, and using standard dynamic programming arguments shows that these objects converge (as t goes to minus infinity) to their infinite horizon analogs that are stated in the original lemma.

With the original lemma for the infinite horizon problem in hand, we are now in a position to state the main proposition.

PROPOSITION. Assume that 0 < 8_{NF} < 1, $\mu_L < \mu_H$, and $\$_L < \$_H$. Assume that b_L and b_H satisfy (B1) above. Then

(B15) $\hat{x}_{HH} < \hat{x}_{HL} < \hat{x}_{LL} < \hat{x}_{LH}$.

PROOF. Recall that $\hat{x}_{\mu\$}$ solves $v^K_{\mu\$}(x,y) = v^S_{\mu\$}(x,y)$. Using the definitions for $v^K_{\mu\$}(x,y)$ and $v^S_{\mu\$}(x,y)$ in the text and solving for $\hat{x}_{\mu\$}$ yields

(B16) $$\hat{x}_{\mu\$} = b_\$ + \frac{w}{1-\delta} \ ! \ \$ \ ! \ \mu \ ! \ *Ev_{\mu\$}.$$

It follows immediately that $\hat{x}_{HL} < \hat{x}_{LL}$. We now show that $\hat{x}_{HH} < \hat{x}_{HL}$. We can write

(B17) $$\hat{x}_{HH} \ ! \ \hat{x}_{HL} = [b_H!\$_H!*Ev_{HH}] \ ! \ [b_L!\$_L!*Ev_{HL}].$$

This is negative if and only if

(B18) $$b_H \,!\, b_L < \$_H \,!\, \$_L + *(Ev_{HH}!Ev_{HL}].$$

But from (B1),

(B19) $$b_H \,!\, b_L = \$_H \,!\, \$_L + (1!8_{NF})*[Ev_{LH}!Ev_{LL}] + 8_{NF}*[Ev_{HH}!Ev_{HL}].$$

From the lemma we know that $Ev_{HH} \,!\, Ev_{HL} > Ev_{LH} \,!\, Ev_{LL}$. This fact along with $8_{NF} < 1$ and equation (B19) imply that (B18) holds which proves that $\hat{x}_{HH} < \hat{x}_{HL}$. A parallel argument proves that $\hat{x}_{LL} < \hat{x}_{LH}$.

□

Appendix C

PROPOSITION. Assume the hazard function $g(y)/(1 ! G(y))$ is strictly increasing in y. Assume $\mu_L = \mu_H$. If p_0^H O $(0,1)$, then $pr(\$ = \$_H * p_0^H, a_t = S) > pr(\$ = \$_H * p_0^H, a_t = K)$.

PROOF. Since μ is constant, the cutoff $\hat{x}_{\mu\$}$ will depend only on $\$$. Let $\hat{x}_L$ denote the cutoff for $\$_L$ and $\hat{x}_H$ the cutoff for $\$_H$. We first show that $\hat{x}_L = \hat{x}_H$. To see this, recall that $\hat{x}_\$$ is the point where $v_\$^K(x,y) = v_\$^S(x,y)$. From equations (3.2) and (3.3) in the text, this equality yields $\$ + x + y + *Ev_\$ = b_\$ + y$. Canceling y from both sides yields $\hat{x}_\$ = b_\$! \$! *Ev_\$$. But then equation (3.6) from the text implies that $\hat{x}_L = \hat{x}_H$. Henceforth denote this common cutoff as $\hat{x}$.

Using Bayes rule to calculate $pr(\$ = \$_H * p_0^H, a_t)$, we need to show

$$(C1)\quad \frac{\mathbf{p}_H^S \cdot \mathbf{p}_0^H}{\mathbf{p}_H^S \cdot \mathbf{p}_0^H + \mathbf{p}_L^S \cdot (1-\mathbf{p}_0^H)} > \frac{\mathbf{p}_H^K \cdot \mathbf{p}_0^H}{\mathbf{p}_H^K \cdot \mathbf{p}_0^H + \mathbf{p}_L^K \cdot (1-\mathbf{p}_0^H)}$$

where $p_\$^a$ denotes the probability of action a given $\$$. But this holds if and only if $p_L^S/p_H^S < p_L^K/p_H^K$, or equivalently, if and only if the ratio $p_\$^S/p_\$^K$ is higher for $\$_H$ than for $\$_L$. This ratio equals

$$(C2)\quad \frac{\mathbf{p}_\beta^S}{\mathbf{p}_\beta^K} = \frac{F(\hat{x} \cdot [1 - G(\hat{y}_\beta)]}{\int_{\hat{x}}^{\infty} f(x) \cdot [1 - G(\hat{y}_\beta + \hat{x} - x)] \cdot dx}.$$

Straightforward calculations reveal that (C2) is strictly increasing in $\hat{y}$ if the hazard rate condition on $G(\cdot)$ holds. This completes the proof since $\hat{y}_{\$_H} < \hat{y}_{\$_L}$.

References

Bureau of the Census. 1987. *1982 Characteristics of business owners,* August.

Dunne, Timothy; Roberts, Mark; and Samuelson, Larry. 1989. The growth and failure of U.S. manufacturing plants. *Quarterly Journal of Economics* (November): 495–515.

Eckstein, Zvi, and Wolpin, Kenneth. 1989. The specification and estimation of dynamic stochastic discrete choice models: A survey. *Journal of Human Resources* (Fall): 562–98.

Evans, David S. 1987. Tests of alternative theories of firm growth. *Journal of Political Economy* 95: 657–74.

Greenwald, Bruce C. N. 1979. *Adverse selection in the labor market.* Garland Publishing, New York.

Holmes, Thomas, and Schmitz, James. 1990. A Theory of entrepreneurship and its application to the study of business transfer. *Journal of Political Economy,* April.

___________. 1993. Managerial tenure, business age, and small business turnover. Manuscript.

Hopenhayn, Hugo. 1992. Entry, exit and firm dynamics in long run equilibrium. *Econometrica* (September): 1127–50.

Jovanovic, Boyan. 1979. Job-matching and the theory of turnover. *Journal of Political Economy* 87.

___________. 1982a. Selection and evolution of industry. *Econometrica* 50 (May): 649–70.

___________. 1982b. Truthful disclosure of information. *Bell Journal of Economics,* 36–44.

Jovanovic, Boyan, and Moffitt, Robert. 1990. An estimate of a sectoral model of labor mobility. *Journal of Political Economy* 98 (August): 827–52.

Pakes, Ariel. 1986. Patents as options: Some estimates of the value of holding European patent stocks. *Econometrica* 54 (July): 755–84.

Pakes, Ariel, and Ericson, Richard. 1988. Empirical implications of alternative models of firm dynamics. SSRI WP 8803. University of Wisconsin, January.

Rao, C. Radharkrishna. 1965. *Linear statistical inference and its applications.* Wiley, New York.

Table 1

Crosstabulations

1982 Characteristics of Business Owners Survey

Nonminority Males

A. Cell Counts by Age of Business, Tenure of Manager, and Founder/Nonfounder Status

		Nonfounders					
		Tenure of Manager (years)					
Age of Business (years)	Founders	0	1–2	3–6	7–12	13–22	23+
0	2,147	60					
1–2	2,909	56	108				
3–6	2,967	40	117	116			
7–12	2,043	29	77	98	73		
13–22	1,515	31	70	106	106	75	
23 and over	1,463	93	208	291	292	344	303

B. Percent Discontinued

		Nonfounders				
		Tenure of Manager (years)				
Age of Business (years)	Founders	0–2	3–6	7–12	13–22	23+
0–2	46	59				
3–6	26	38	33			
7–12	20	25	17	26		
13–22	22	25	19	9	19	
23 and over	26	20	13	10	16	20

C. Percent Sold

		Nonfounders	
		Tenure of Manager (years)	
Age of Business (years)	Founders	0–2	3 or more

0–2	3	7	
3–6	3	15	8
7–12	3	15	14
13–22	4	16	9
23 and over	4	15	12

Table 2

Parameter Estimates
(estimated standard errors in parentheses)

Parameter	Model 1 $8_{NF} = 8_F$ $>_{\mu_L} = >_{\mu_H}$	Model 2 $8_{NF} = 8_F$ $>_{\mu_L}$ … $>_{\mu_H}$	Model 3 8_{NF} … 8_F $>_{\mu_L}$ … $>_{\mu_H}$
F_y	0.93 (0.08)	1.61 (0.11)	1.00 (0.11)
μ_L	!10.99 (2.66)	!4 *	!4 *
μ_H	0.30 (0.01)	0.23 (0.02)	0.31 (0.02)
$\$_H$	0.29 (0.02)	0.47 (0.02)	0.28 (0.03)
(	0.012 (0.002)	0.020 (0.001)	0.018 (0.001)
8_{NF}	0.59 (0.01)	0.56 (0.01)	0.65 (0.02)
8_F	0.59 (0.01)	0.56 (0.01)	0.52 (0.05)
$>_{\mu_H}$	0.037 (0.005)	0.18 (0.01)	0.14 (0.01)
$>_{\mu_L}$	0.037 (0.005)	0.000 (0.003)	0.00 (0.004)

Summary Statistics			
-Log(likelihood)	49,867	49,704	49,677
SAD (sum of absolute deviations = $3_{i=1}^{81}*p_i(1)$! $\tilde{p}_i*$)	0.220	0.180	0.187
MSE (mean squared errors = 1/81 $3_{i=1}^{81}*p_i(1)$! $\tilde{p}_i*^2$)	2.9×10!5	1.9×10!5	2.1×10!5
V[p;data] = 1/81$3_{i=1}^{81}*\tilde{p}_i$!1/81$*^2$	68.6×10!5	68.6×10!5	68.6×10-!5
MSE/V[p;data]	0.043	0.028	0.031

*See footnote 8 in the text for a discussion of the distribution of this estimate.

Table 3

Parameter Estimates for Alternative Subsets of the Data

	1 All Firms	2 Propri-etors	3 Proprietors Working Full Time	4 Receipts Above $5000	5 Services and Retail	6 All Other Indus-tries
F_y	1.61	1.25	0.82	0.95	1.59	1.78
μ_L	!4	!4	!4	!8.36	!4	!4
μ_H	0.23	0.26	0.37	0.43	0.18	0.27
$\$_H$	0.47	0.40	0.23	0.21	0.52	0.46
(	0.020	0.016	0.013	0.021	0.017	0.023
8	0.56	0.53	0.61	0.69	0.53	0.60
$>_{\mu_H}$	0.18	0.14	0.15	0.16	0.20	0.15
$>_{\mu_L}$	0.00	0.00	0.00	0.00	0.00	0.00
Number of Observations	15,737	13,972	8,929	10,611	8,124	7,613

Table 4

Comparison of Turnover Rates in the CBO Survey and the Model Economy

CBO Data

Age of Business (years)	Discontinuance Rates: Founders	Discontinuance Rates: Nonfounders, Tenure of Manager (years): 0–2	3–6	7–12	13–22	23+	Sale Rates: Founder	Sale Rates: Nonfounders, Tenure: 0–2	3+
0–2	46	59					3	7	
3–6	26	38	33				3	15	8
7–12	20	25	17	26			3	15	14
13–22	22	25	19	9	19		4	16	9
23 and over	26	20	13	10	16	20	4	15	12

Model Economy

Age of Business (years)	Discontinuance Rates: Founders	Discontinuance Rates: Nonfounders, Tenure of Manager (years): 0–2	3–6	7–12	13–22	23+	Sale Rates: Founder	Sale Rates: Nonfounders, Tenure: 0–2	3+
0–2	51	54					2	5	
3–6	27	30	27				3	14	3
7–12	26	27	16	24			3	20	5
13–22	24	24	11	13	19		4	22	7
23 and over	19	22	9	9	9	10	6	24	9

Table 5

Comparison of Distribution of

Businesses in
CBO Survey and Model Economy

Age Distribution (Percent in each age category)

Age of Business (years)	CBO	Mod-el
0–2	34	31
3–6	21	20
7–12	15	18
13–22	12	15
23 and over	19	16

Fraction of Business Population that are
Nonfounder Businesses by Age of Business

Age of Business (years)	CBO	Model
0–2	4	3
3–6	8	7
7–12	12	11
13–22	20	20
23 and over	51	52

Table 6

Equilibrium Levels of Selected Variables in the Model

	Bad Match ($\mu = \mu_L$)		Good Match ($\mu = \mu_H$)	
	Bad Business ($\$ = \$_L$)	Good Business ($\$ = \$_H$)	Bad Busi- ness ($\$ = -\$_L$)	Good Business ($\$ = \$_H$)
$b_{\$}$	!2.4	0.8	!2.4	0.8
$\hat{y}$	2.4	!0.8	2.4	!0.8
$\hat{x}$	4	4	!15.7	!17.1
p^K	0.000	0.000	0.904	0.953
p^S	0.064	0.690	0.004	0.030
p^D	0.936	0.310	0.092	0.017

Table 7a

Distribution of Qualities Among Founder Businesses
By Age of Business

Age of Business	Bad Match ($\mu = \mu_L$)		Good Match ($\mu = \mu_H$)	
	Bad Business ($\$ = \$_L$)	Good Business ($\$ = \$_H$)	Bad Business ($\$ = -\$_L$)	Good Business ($\$ = \$_H$)
1	0.439	0.000	0.462	0.099
2	0.000	0.000	0.816	0.184
3	0.000	0.000	0.808	0.192
5	0.000	0.000	0.791	0.209
10	0.000	0.000	0.745	0.255
20	0.000	0.000	0.633	0.367

Table 7b

Distribution of Qualities Among Nonfounder Businesses

By Age of Business

Age of Business	Bad Match ($\mu = \mu_L$)		Good Match ($\mu = \mu_H$)	
	Bad Business ($\$ = \$_L$)	Good Business ($\$ = \$_H$)	Bad Business ($\$ = -\$_L$)	Good Business ($\$ = \$_H$)
2	0.398	0.040	0.510	0.051
3	0.047	0.073	0.718	0.163
5	0.024	0.069	0.598	0.309
10	0.013	0.054	0.395	0.539
20	0.004	0.040	0.190	0.766

Figure 1

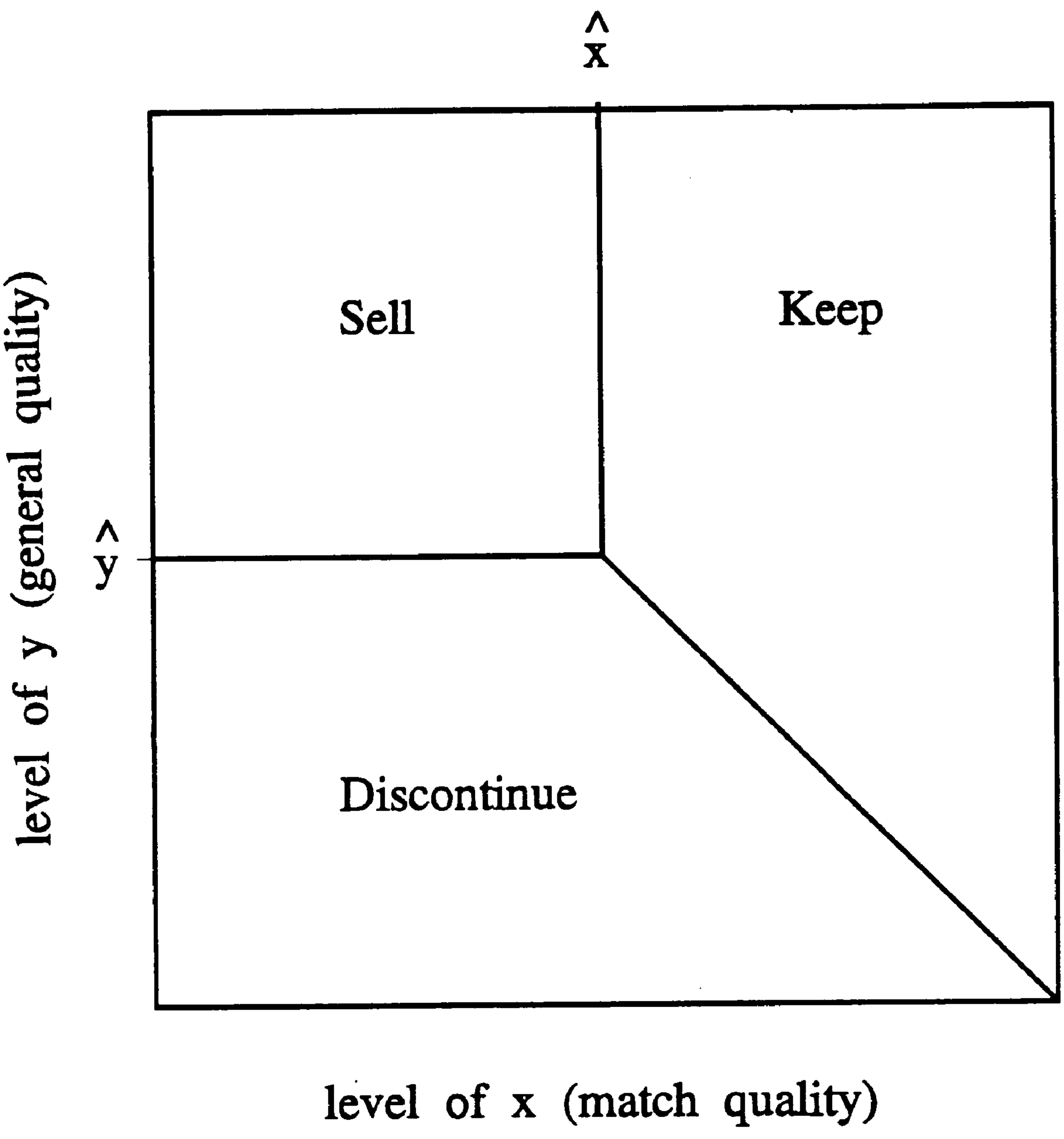
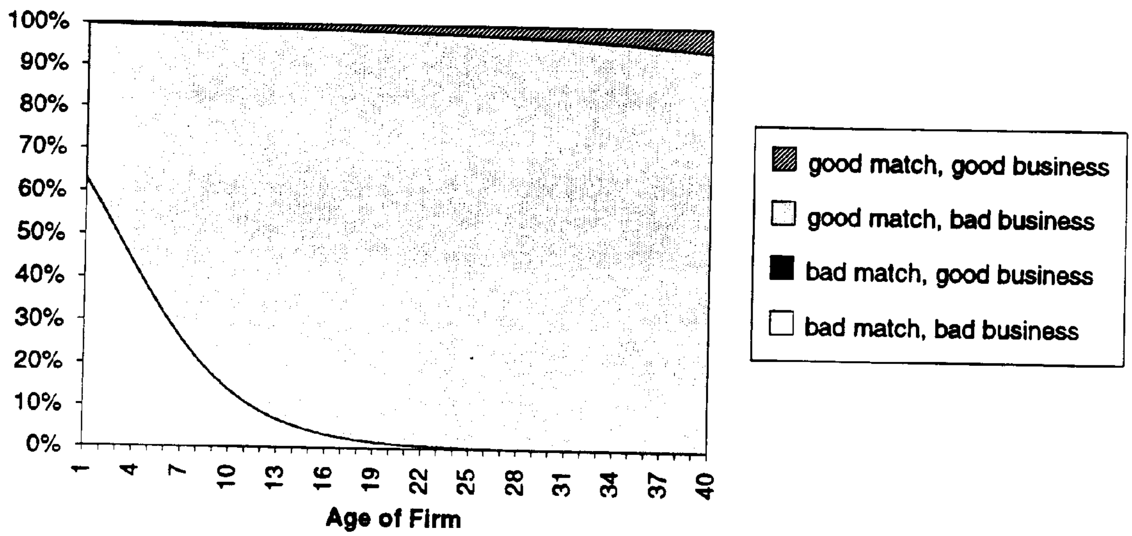

$\hat{x}$
Sell
Keep
$\hat{y}$
Discontinue
level of y (general quality)
level of x (match quality)

Figure 2

Part A
Distribution of Founders



Part B
Distribution of Nonfounders