Logarithmic Conformal Field Theories of Disordered Dirac Fermions

Miraculous Joseph Bhaseen
Christ Church

Department of Physics, Theoretical Physics
University of Oxford

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"The human heart likes a little disorder in its geometry..."
Louis de Bernieres, Captain Corelli's Mandolin.

Abstract

In this thesis we investigate different theoretical approaches used in the treatment of low-dimensional disordered systems at criticality. We focus on a specific model of two-dimensional Dirac fermions interacting with a random non-Abelian vector potential. This model was originally introduced by Nersesyan, Tsvelik and Wenger to describe the effects of non-magnetic impurities on the low-energy properties of d-wave superconductors. We exploit the fact that this model is amenable to three independent non-perturbative approaches. These are based on the commonly employed replica and supersymmetry techniques, together with a third model specific analysis valid in the limit of strong disorder. These approaches are implemented by means of the powerful methods of non-Abelian bosonization. We investigate the emergent Wess–Zumino–Novikov–Witten (WZNW) models and their associated logarithmic conformal field theories (LCFTs). Illuminating simplifications are observed to occur in the supersymmetric approach and we are able to reconcile troublesome inconsistencies previously encountered in the other approaches. In contradistinction to earlier treatments, we demonstrate that this *ménage à trois* of theoretical approaches yields identical results for the disorder averaged four-point correlation functions of the local density of states (LDoS). Moreover, this marriage of approaches reveals an underlying structure inherited from the non-unitary minimal model with central charge $c = -2$. This theory describes the geometrical critical behaviour of dense polymers and has served as an instructive example of a LCFT. We highlight the important rôle played by the $\mathbb{Z}_N$ twist operators of the polymer theory. Further considerations of the internal consistency of the various approaches leads to natural relations for the mutual cancellation of logarithms between different LCFTs.

Contents

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Preface

This preface is intended as a brief guide to the motivation, the contents, and the omissions, of this thesis. We hope that this may provide some orientation for the reader.

In this thesis we use an exactly solvable model of disorder to compare different theoretical approaches used to investigate the critical behaviour of disordered (possibly electronic) systems. A prominent example of a disordered critical point, and a source of many challenging problems, is that governing the plateau transitions in the integer quantum Hall effect (IQHE) — see for example [1]. Despite the numerous field theoretic approaches to this problem based on either Pruisken's non-linear sigma model [2–4] or the Chalker–Coddington network model [5] and its superspin chain descendants [6–10], the nature of the critical point remains elusive and inaccessible by perturbation theory [11, 12]. Other recent studies of disordered critical points have included Dirac fermions subjected to various random potentials [13–16], the random XY model [17], the plateau transitions occurring in the spin quantum Hall effect and its relation to critical percolation [18–20], and the Nishimori line in the random bond Ising model [21]. In order to gain insight into such problems it is imperative to fully exploit, and indeed develop, the methods available for treating disordered systems, and to obtain as many non-perturbative results as possible. In low-dimensional systems we may be aided in this pursuit by the availability of powerful techniques such as conformal field theory [22–24] and the Bethe ansatz [25, 26].

We focus on a fermionic model originally introduced by Nersesyan, Tsvelik and Wenger to describe the effects of non-magnetic impurities on the low-energy properties of two-dimensional d-wave superconductors [27–29]. Prominent examples of such d-wave superconductors include the high temperature superconducting cuprates [30, 31] and understanding the nature and stability of d-wave quasi-particle excitations is a subject of ongoing intensive research [32–39]. The quasi-particle spectrum of a pure spin-singlet $d_{x^2-y^2}$ superconductor exhibits four isolated 'nodes' in the vicinity of which reside low-energy gapless excitations with a formally relativistic conical dispersion relation. Non-magnetic impurities introduce scattering processes within and between these nodes. In the presence of nodal back-scattering, and in the absence of nearest neighbour skew-scattering [16], this problem may be mapped onto a relativistic field theory of massless Dirac fermions interacting with a random non-Abelian gauge field [27–29]. The task of disorder averaging in this theory has, along with other related problems, been the subject of an important series of treatments, refinements and elaborations [27, 28, 40–48]. In this thesis we shall build on the extensive foundations laid in these previous studies.

In addition to some considerable advances, the subject of disordered d-wave superconductors has also aroused controversy in the literature — for a clear and recent review see reference [16]. In rather broad and oversimplified terms, one may identify two distinct scenarios which add fuel to these (and indeed many other) debates:

- A given model (as described by its action, say) may not capture all the essential physics or symmetries of the physical problem. The model may have an inherently limited (or indeed vanishing) domain of validity. Naturally, different models may give conflicting results and it is important to distinguish between them, both experimentally and theoretically.

- The theoretical analysis of a given model, quite aside from its experimental relevance, may be (and often is) rather involved and subtle. It may be plagued by delicate limits for instance, or may involve unfamiliar ideas or emerging mathematics. The method of solution may have an inherently limited (or indeed vanishing) domain of validity.

Although the issues of 'model' versus 'method' are distinct, it is important to bear in mind that any meaningful discrimination between candidate models assumes that our methods of solution are trustworthy. In this respect, the model of two-dimensional Dirac fermions interacting with a non-Abelian gauge field has the virtue of being amenable to three non-perturbative approaches. These are based on the commonly used (yet notoriously delicate) replica [49] and supersymmetry methods [50, 51], together with a third method valid in the limit of strong disorder [46]. In this thesis we combine and extend the results of Bernard and LeClair [52] and Maassarani and Serban [53] in order to demonstrate that considerable simplifications occur in the supersymmetric formulation [54–56]. Most notably, an entire conformal block decouples in the approach and we are able to make extensive use of the complete elliptic integrals. In the light of these simplifications we are able to reconcile a number of troublesome inconsistencies in previous works [40, 48]. In contradistinction to these earlier treatments we are able to demonstrate that this *ménage à trois* of theoretical approaches yields identical results for the four-point correlation functions of the local density of states (LDoS) [55]. It is worth emphasizing that whilst the reliability of the replica approach outside of perturbation theory has been frequently and legitimately questioned — see for example [50, 51, 57, 58] — we use it here (in a quite straightforward manner) to reproduce the results of the (mathematically more rigorous) supersymmetric approach. Although the implementation of the replica method remains a delicate issue in general, and in many cases one must adopt various (replica) symmetry breaking schemes — see references [59, 60] for recent examples in random matrix theory — it is noteworthy that the disordered non-Abelian Dirac fermion problem renders itself here to a textbook (replica) treatment, albeit with the benefit of hindsight. Whilst our immediate goal has not been to discriminate between the various models of disordered d-wave superconductivity, we hope that our renewed confidence in the available non-perturbative methods may ultimately assist in this process.

In addition to this marriage of approaches, the analysis of the non-Abelian Dirac fermion problem reveals intimate connections to the non-unitary minimal model with central charge $c = -2$ [55, 61]. This theory describes the geometrical critical behaviour of dense polymers [62, 63] and has served as an instructive example of a LCFT [64, 65]. In particular, we highlight the important rôle played by the $\mathbb{Z}_N$ twist operators of the polymer theory [55, 56, 61]. Further considerations of the internal consistency of the various approaches leads to natural relations for the mutual cancellation of logarithms between different LCFTs [54]. These relations are based on a corollary to Legendre's relation in the theory of elliptic integrals.

The structure of this thesis is as follows. In chapter 1 we provide a brief introduction to the main ideas and methods employed in this thesis. We have endeavored to develop a natural flow: continuous phase transitions $\rightarrow$ CFT $\rightarrow$ disordered Dirac fermions $\rightarrow$ replicas and supersymmetry $\rightarrow$ non-Abelian bosonization $\rightarrow$ WZNW $\rightarrow$ LCFT, in the hope that this will assist the reader. In chapter 2 we study the non-Abelian Dirac fermion problem by three different methods. In particular, the presentation of approaches (replicas, strong disorder, supersymmetry) is in the reverse order to which they were originally studied by the author; it is felt that this may ease the reader through this work. The appendices attached to this chapter are an important and integral part of this thesis and are relegated solely for clarity of presentation. In chapter 3 we study the theoretical approaches to the Dirac fermion problem in a little more detail. In particular, we provide evidence for the mutual cancellation of logarithms between different logarithmic conformal field theories. In chapter 4 we present our conclusions. We end this thesis with a short technical appendix on hypergeometric series.

Publications

We gather here a list of the single author and joint publications which form the basis of this thesis [54–56].

M. J. Bhaseen
Legendre's Relation and the Quantum Equivalence $osp(4|4)_1 = osp(2|2)_{-2} \oplus su(2)_0$
Nucl. Phys. **B604** (2001) 537-550 [`cond-mat/0011229`]

Using explicit results for the four-point correlation functions of the Wess–Zumino–Novikov–Witten (WZNW) model we discuss the conformal embedding $\mathrm{osp}(4|4)_1 = \mathrm{osp}(2|2)_{-2} \oplus \mathrm{su}(2)_0$. This embedding has emerged in Bernard and LeClair's recent paper [52]. Given that the $\mathrm{osp}(4|4)_1$ WZNW model is a free theory with power law correlation functions, whereas the $\mathrm{su}(2)_0$ and $\mathrm{osp}(2|2)_{-2}$ models are CFTs with logarithmic correlation functions, one immediately wonders whether or not it is possible to combine these logarithms and obtain simple power laws. Indeed, this very concern has been raised in a revised version of [52]. In this paper we demonstrate how one may recover the free field behaviour from a braiding of the solutions of the $\mathrm{su}(2)_0$ and $\mathrm{osp}(2|2)_{-2}$ Knizhnik–Zamolodchikov equations. We do this by implementing a procedure analogous to the conformal bootstrap programme [22]. Our ability to recover such simple behaviour relies on a remarkable identity in the theory of elliptic integrals known as Legendre's relation.

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M. J. Bhaseen, J.-S. Caux, I. I. Kogan and A. M. Tsvelik
Disordered Dirac Fermions: the Marriage of Three Different Approaches
Accepted for publication in Nuclear Physics B [`cond-mat/0012240`]

We compare the critical multipoint correlation functions for two-dimensional (massless) Dirac fermions in the presence of a random $\mathrm{su}(N)$ (non-Abelian) gauge potential, obtained by three different methods. We critically reexamine previous results obtained using the replica approach and in the limit of infinite disorder strength and compare them to new results (presented here) obtained using the supersymmetric approach to the $N = 2$ case. We demonstrate that this *ménage à trois* of different approaches leads to identical results. Remarkable relations between apparently different conformal field theories (CFTs) are thereby obtained. We further establish a connection between the random Dirac fermion problem and the $c = -2$ theory of dense polymers. The presence of the $c = -2$ theory may be seen in all three different treatments of the disorder.

M. J. Bhaseen
Random Dirac Fermions: The $su(N)$ Gauge Potential and $\mathbb{Z}_N$ Twists
`cond-mat/0012420`

We obtain the disorder averaged (critical) four-point correlation functions for N species of (two-dimensional Euclidean) Dirac fermions subject to a (Gaussian) random su(N) gauge field. The replica approach and the strong disorder approach yield identical results, as one might expect in the light of recent developments in the $N = 2$ case [55]. We establish a connection with 'dressed' $\mathbb{Z}_N$ twist fields in the $c = -2$ logarithmic conformal field theory (LCFT), thereby extending the recent connection to 'dressed' $\mathbb{Z}_2$ twists [55, 61]. We draw attention to the fact that 'dressed' $\mathbb{Z}_2$ twist operators have previously appeared in the description of half flux-quantum quasiholes in the bulk excitations of the Haldane–Rezayi Quantum Hall state [66].

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We note that some of the results of this latter publication [56] have been subsequently incorporated into the previous joint publication [55]. In addition, an early draft of the publication [55] focusing on the $su(2)$ case, appears in *New Theoretical Approaches to Strongly Correlated Systems* [67]. We note further, that we have not included the following joint publication [12] in this thesis:

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M. J. Bhaseen, I. I. Kogan, O. A. Soloviev, N. Taniguchi and A. Tsvelik
Towards a Field Theory of the Plateau Transitions in the Integer Quantum Hall Effect
Nucl. Phys. **B580** (2000) 688-720 [`cond-mat/9912060`]

We suggest a procedure for calculating correlation functions of the local densities of states (DoS) at the plateau transitions in the Integer Quantum Hall effect (IQHE). We argue that their correlation functions are appropriately described in terms of the SL$(2,\mathbb{C})$/SU(2) WZNW model (at the usual Kač–Moody point and with the level $6 \leq k \leq 8$). In this model we have identified the operators corresponding to the local DoS, and derived the partial differential equation determining their correlation functions. The OPEs for powers of the local DoS obtained from this equation are in agreement with available results.

Conventions

- Dirac Fermions: We use Greek subscripts and superscripts on Dirac fermions to denote 'colour' indices pertaining to the non-Abelian gauge potential. We use the Latin script to denote replica and other indices.

- Complex coordinates: In terms of two-dimensional Euclidean space coordinates x_0 and x_1 we denote $z = x^0 + ix^1$ (holomorphic) and $\bar{z} = x^0 - ix^1$ (antiholomorphic). The coordinate derivatives take the form $\partial = \frac{1}{2}(\partial_0 - i\partial_1)$ and $\bar{\partial} = \frac{1}{2}(\partial_0 + i\partial_1)$.

- Anharmonic Ratio: We adopt the definition of the anharmonic ratio used by Knizhnik and Zamolodchikov [68] and by Maassarani and Serban [53]:
$$z = \frac{z_{12}z_{34}}{z_{23}z_{41}}, \quad 1 - z = \frac{z_{13}z_{42}}{z_{23}z_{41}},$$
where $z_{ij} \equiv z_i - z_j$. In particular, upon interchanging the coordinates z_2 and z_3, the anharmonic ratio z as defined above is transformed into $1 - z$.

- Conformal Dimensions: We shall use the symbol h to denote conformal dimensions. In the case of spinless fields (which are the only ones we deal with in this thesis) the full scaling dimension is $\Delta = 2h$.

- Complete Elliptic Integrals: We shall make frequent use of the complete elliptic integrals. Our conventions differ from those of many standard tables [69, 70], but are more suited to their present field theoretic application. We shall use the somewhat more convenient choice of z in place of the modulus k^2. This may lead to cosmetic changes in the relations satisfied by these functions.

- Logarithms: Although we shall study aspects of logarithmic conformal field theories, the reader will find very few explicit logarithms. They are there however, and are often encoded in the analytic structure of the hypergeometric functions; a function $F(z)$ which converges inside the unit disc surrounding $z = 0$, may develop logarithmic behaviour in the vicinity of $z = 1$. We shall also employ the 'tilde-hat' notation $\tilde{F}(z) \equiv F(1 - z)$ and $\hat{F}(z) \equiv F(1/z)$.

- Q-matrix: In our study of the random non-Abelian Dirac fermion problem we shall introduce the Q-matrix. In all cases, the Q-matrix will be a primary field of the appropriate WZNW model.

- Kač–Moody algebra: We shall denote the extended symmetry algebra of the WZNW models by $\widehat{\mathcal{A}}_k$, where $\mathcal{A}$ is the underlying Lie algebra, and k is the level or central extension.

Chapter 1

Introduction

The experimental and theoretical study of continuous phase transitions (critical phenomena) occurring in a wide variety of physical systems continues to generate a wealth of interesting physics and mathematics — see for example [71–76]. Well known examples of continuous phase transitions include the (zero magnetic field) paramagnetic to ferromagnetic transition occurring at the Curie point in iron, the order–disorder transition in β-brass, the superconducting transition in the absence of an applied magnetic field[1], and the He I to He II superfluid transition occurring in liquid ^{4}He. In contrast to the rather abrupt, first-order transitions such as the freezing of water into ice, continuous phase transitions are not associated with the production of a latent heat. Indeed, the epithet continuous stems from the fact that the first derivatives of the free energy are continuous in these transitions, with the higher derivatives being either discontinuous or divergent.

The signatures of continuous phase transitions include the power law divergence of response functions such as the specific heat capacity, the power law divergence of a correlation length (loosely defined here as the distance over which fluctuations are statistically correlated) and the power law decay of their correlation functions — see for example [74]. These dimensionless powers, or *critical exponents*, are an important characteristic of the physical system under consideration, and provide a challenge for the theorist to calculate. Moreover, distinct physical systems may share the same set of critical exponents, and fall into so-called *universality classes*. A detailed understanding of these phenomena came with the advent of the renormalization group in the late 1960s, contributions for which K. G. Wilson was awarded the 1982 Nobel Prize in Physics. An important ingredient in the renormalization group approach to critical phenomena is the

[1]In type I superconductors a non-zero magnetic field renders a first-order transition rather than a continuous one. In type II superconductors the phase transition is continuous even in the presence of a non-zero applied field.

notion that critical systems are *scale invariant*. That is to say, the divergence of their correlation length renders them insensitive to the global rescaling of their microscopic lattice spacing — for further details see for example [74, 75].

An insightful step forward came from the suggestion of Polyakov in 1970, to elevate the notion of scale invariance in critical systems to the larger symmetry of conformal invariance [77]. In this short paper, the consequences of (global) conformal invariance on the structure of the multipoint correlation functions (in arbitrary dimension) was firmly established. Over a decade passed before the seminal papers of Belavin, Polyakov and Zamolodchikov studied the implications of conformal invariance in *two-dimensional* critical systems, and set forth the discipline of conformal field theory [22,78]. In principle, the ideas of conformal symmetry are relevant in all dimensions, but as these authors demonstrated, conformal invariance plays a crucial rôle in the description of continuous phase transitions occurring in two dimensions. As we discuss in §1.1, the conformal transformations form a group, and in two dimensions they coincide with the analytic transformations of the complex plane. In particular, as the titles of references [22, 78] suggest, there are an *infinite* number of such transformations in two dimensions, in stark contrast to higher dimensions. This greatly enlarged (local conformal) symmetry allows for the exact determination of multipoint correlation functions and the extraction of critical exponents. As in any branch of physics where symmetries are important, the tools of group theory and algebras are invaluable. The generators of conformal transformations satisfy the so-called Virasoro algebra, and its representation theory is central to the study of two-dimensional critical phenomena. In particular, the so-called *minimal models* contain a *finite* number of representations of the Virasoro algebra and may be identified with the critical points of two-dimensional lattice models such as the Ising and 3-state Potts models. The basis states of the representation correspond to physical degrees of freedom such as the Ising spin [22, 78] — see also §1.4.

In this thesis we shall use the powerful machinery of conformal field theory to study a fermionic model originally introduced by Nerseysan, Tsvelik and Wenger to describe the effects of non-magnetic impurities on the low-energy properties of two-dimensional d-wave superconductors [27–29]. The applicability of these techniques stems not only from the reduction in spatial dimension, but also from the fact that the low-energy quasi-particle spectrum of a clean d-wave superconductor exhibits four isolated 'nodes'— see figure 2.1. In the vicinity of these nodes reside low-energy gapless excitations with a formally relativistic dispersion relation; the Fermi velocity plays the rôle of the speed of light. In particular, the presence of gapless excitations in this system is a signal of scale invariance, since an energy gap, ε, between the ground state and the first excited state constitutes an inverse length scale, $\lambda = hc/\varepsilon$. This motivates a description of the pure d-wave superconductor in terms of four independent 'species' of massless Dirac fermions. The presence of 'disorder' in the form of non-magnetic impurities, may lead to scattering processes within and between the nodes, and to a coupling between the nodal fermions. This led the authors of references [27–29] to introduce a model of massless Dirac fermions interacting with a random non-Abelian gauge field. The task of disorder averaging in this theory has, along with other related problems, been the subject of an important series of treatments, refinements and elaborations [27, 28, 40–48]. In this thesis we shall build on the extensive foundations laid in these previous studies and study this model in three independent ways. In this pursuit we shall combine the 'traditional' methods for treating disordered systems such as the replica and supersymmetry approaches, with the powerful non-perturbative methods of non-Abelian bosonization, to be described below. In particular, we shall focus on calculating the disorder-averaged four-point correlation functions of the local density of states (LDoS). The motivation for studying these multipoint correlation functions stems from the following fact: the two- and three-point functions are fixed, up to normalization, by the finite global conformal invariance *alone* [77] — see §1.2. That is to say, the functional form of the two- and three-point functions depends only on the scaling dimensions of the fields under consideration. However, the four-point functions are determined by the infinite-parameter local conformal algebra *and* any extended symmetries that may, and as we shall see, will be present. They are an important non-perturbative probe of different conformal field theories, and hence of different theoretical approaches to the non-Abelian Dirac fermion problem. As we shall see, the disorder-averaged four-point functions of the non-Abelian Dirac fermion problem turn out to be quite illuminating.

In quite general terms, the correlation functions of a set of fields $\psi_i(\mathbf{x}_i)$, in the presence of a fixed realization of disorder $\mathcal{A}(\mathbf{x})$, may be represented as a Feynman path integral:

$$\langle \psi_1(\mathbf{x}_1)\psi_2(\mathbf{x}_2)\ldots\rangle = \frac{\int [\mathcal{D}\psi]\ \psi_1(\mathbf{x}_1)\psi_2(\mathbf{x}_2)\ldots e^{-S[\psi,\mathcal{A}]}}{Z[\mathcal{A}]},$$

where $S[\psi, \mathcal{A}]$ is the action, and the partition function is given by,

$$Z[\mathcal{A}] = \int [\mathcal{D}\psi]\ e^{-S[\psi,\mathcal{A}]}.$$

We wish to calculate the disorder average of these correlation functions:

$$\overline{\langle \psi_1(\mathbf{x}_1)\psi_2(\mathbf{x}_2)\ldots\rangle} \equiv \int [\mathcal{D}\mathcal{A}]\, \langle \psi_1(\mathbf{x}_1)\psi_2(\mathbf{x}_2)\ldots\rangle \mathcal{P}[\mathcal{A}],$$

where $\mathcal{P}[\mathcal{A}]$ is the probability distribution functional of the disorder field $\mathcal{A}(\mathbf{x})$. Unfortunately, this task is complicated by the presence of the disorder dependent partition function in the definition of the original correlation functions; there is disorder dependence in the numerator (arising from the action) *and* the denominator (arising from the partition function). We shall consider here two frequently employed approaches to this problem — see also §2 of reference [46]. The first is the so-called replica approach which was introduced by Edwards and Anderson in the context of spin glasses [49]. The basic idea is to multiply this correlation function by $Z^n[\mathcal{A}]$, where n is some integer greater than unity. This removes the troublesome denominator, and we consider instead:

$$\langle \psi_1^{(1)}(\mathbf{x}_1)\psi_2^{(1)}(\mathbf{x}_2)\ldots\rangle_n \equiv \int \prod_{r=1}^{n} \left[\mathcal{D}\psi^{(r)}\right] \ \psi_1^{(1)}(\mathbf{x}_1)\psi_2^{(1)}(\mathbf{x}_2)\ldots \exp\left\{-\sum_{r=1}^{n} S[\psi^{(r)},\mathcal{A}]\right\},$$

where the so-called replica indices take the integer values $r = 1, 2, \ldots, n$. In particular, one may perform the disorder average more straightforwardly in this replicated theory. It is the assertion of the replica 'trick', that one may subsequently send $n \to 0$, and in this limit recover the disorder averaged correlation functions of the original problem. Although the rigorous validity of this procedure is far from obvious, we shall demonstrate in this thesis, that the non-Abelian Dirac fermion problem renders itself to a text-book replica treatment.

The second approach is the so-called supersymmetry method which was introduced by Efetov in the study of disordered metals [50]. The method has since been applied to a broad range of topics including mesoscopic physics, random matrix theory, and quantum chaos [51]. It is based on the well known reciprocal relationship between bosonic (c-number) and fermionic (Grassmann) Gaussian integrals — see for example [76]. In particular, in theories which are Gaussian at fixed disorder, the inverse partition function (which is a functional determinant) may be represented as a path integral over fields, $\phi(\mathbf{x})$ say, with the opposite statistics. This yields the following 'supersymmetric' representation for the original correlation functions:

$$\langle \psi_1(\mathbf{x}_1)\psi_2(\mathbf{x}_2)\ldots\rangle = \int [\mathcal{D}\psi]\,[\mathcal{D}\phi] \ \psi_1(\mathbf{x}_1)\psi_2(\mathbf{x}_2)\ldots\, e^{-S[\psi,\mathcal{A}]-S[\phi,\mathcal{A}]}.$$

Once again, the disorder average becomes easier to evaluate. In particular, the supersymmetry method rigorously avoids the delicate replica limit. It does however, have the drawback that one is forced to work with both bosons and fermions, and (unlike the replica trick) it is limited to systems which are Gaussian at fixed disorder. As we shall see, in the non-Abelian Dirac fermion problem, these are not limiting factors.

Having decided on the method of treating the disorder, e.g. replicas or supersymmetry, it is important to bear in mind that there are two distinct path integrals to perform. The first is the 'disorder average', which is the path integral over the disorder field $\mathcal{A}(\mathbf{x})$ weighted by the probability distribution functional $\mathcal{P}[\mathcal{A}]$. The second, is the 'field theory average' over the fields such as $\psi(\mathbf{x})$ and $\phi(\mathbf{x})$ which define the correlation function we are interested in. In particular, evaluating the disorder average will lead (in general) to a coupling between the different replicas, or between the bosons and fermions, as appropriate. This leads one to evaluate the field theory average against an 'effective action' and to consider a class of *perturbed* conformal field theories — see for example [46, 52, 79, 80]. In the non-Abelian Dirac fermion problem, these perturbations will only couple to a *subsector* of the replicated or supersymmetrized Dirac Hamiltonians. In order to isolate the appropriate correlation functions, it is convenient to adopt a representation where these subsectors are manifest. As we shall see more fully in chapter 2, the powerful methods of non-Abelian bosonization will enable us to 'break up' the replicated and supersymmetrized Dirac Hamiltonians, and aid us in this pursuit.

The modern approach to (Abelian) bosonization was developed independently by Coleman [81], Mandalstam [82], Mattis [83], and Luther (in collaboration with Peschel and Emery) [84, 85] — a collection of reprints of many of the 'classic' papers relating to bosonization (including the aforementioned) is to be found in the book [86]. The most basic premise in all of these works is the (non-local) equivalence between the theory of a massless Dirac fermion in 1+1 dimensions, and the theory of a massless scalar field (free boson) — hence the term bosonization.[2] In an important series of works by Polyakov and Wiegmann [88–91], and Witten [92], these considerations were extended to theories of Dirac fermions with non-Abelian symmetries. In particular, we shall make profitable use of the equivalence between a theory of massless non-Abelian Dirac fermions and the so-called (bosonic) Wess–Zumino–Novikov–Witten (WZNW) models [92,93]. In addition to conformal invariance, the WZNW models possess further symmetries whose conserved currents generate an extended conformal algebra. This is the so-called Kač–Moody algebra, and it is sometimes referred to as a current algebra or an affine Lie algebra — see for example [94]. In particular, the Kač–Moody constraints on correlation functions are encoded in the so-called Knizhnik–Zamolodchikov equations [68]; we shall solve such equations many times in this thesis. Moreover, we shall employ supersymmetric extensions of the rules of non-Abelian bosonization and encounter WZNW models on supergroups [52, 53].

In the course of our study of the non-Abelian Dirac fermion problem we shall be confronted with a special class of CFTs, known as the logarithmic conformal field theories

[2] In fact, these authors managed to demonstrate that the Bose-Fermi equivalence can be extended to interacting theories in 1+1 dimensions: the massive fermionic Thirring model is equivalent to the bosonic sine-Gordon model. In condensed matter physics one frequently encounters problems of interacting electrons, and this correspondence yielded the solutions to some previously intractable problems. An exposition of this technique in condensed matter physics may be found in the book [87].

(LCFTs). This class of CFTs is of growing importance in the study of disordered critical points. Unlike ordinary CFTs, the correlation functions in logarithmic CFTs are not always simple power laws, but may also involve logarithms. In the WZNW models, this is reflected in the (Frobenius) series solution of the Knizhnik–Zamolodchikov equations: the roots of the indicial equation differ by integers. The LCFTs thus have a dual status: they are on the one hand limiting cases of ordinary CFTs (in which the indicial roots differ by integers) and on the other hand (due to the presence of logarithms) they are distinct and novel theories in their own right — see §1.6 for more details and references.

In the remainder of this introduction we discuss some of these subjects in a little more detail in order to highlight some of the main ideas.

§1.1 Conformal Invariance

In this section we provide a short introduction to conformal invariance in d-dimensions. In particular, we shall discuss the important and rather special case of two dimensions. There are many excellent books [23, 24, 87, 94, 95] and review articles [96, 97] on the subject, and we refer the reader to the literature for further details. A good starting place for the uninitiated is chapter 11 of reference [75].

Under a general coordinate transformation $x \to x'$ the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ will undergo a change and it is natural to view such coordinate transformations as acting upon the metric tensor:

$$g'_{\mu\nu}(\mathbf{x}') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\, g_{\alpha\beta}(\mathbf{x}), \tag{1.1}$$

in which we adopt the standard Einstein summation convention over repeated indices. By definition, conformal transformations are the subgroup of all possible coordinate transformations which leave the metric tensor invariant up to a (local) scale factor:

$$g'_{\mu\nu}(\mathbf{x}') = \Lambda(\mathbf{x})g_{\mu\nu}(\mathbf{x}). \tag{1.2}$$

A conformal transformation is thus locally equivalent to a translation a rotation and a dilatation (or scale transformation). It is readily seen that such coordinate transformations preserve the angle $\boldsymbol{a}.\boldsymbol{b}/\sqrt{\boldsymbol{a}^2\boldsymbol{b}^2}$ (where $\boldsymbol{a}\cdot\boldsymbol{b} = g_{\mu\nu}a^\mu b^\nu$) at which two arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ cross; the epithet *conformal* literally means *preserving form.* The reader familiar with complex variable theory may well recognize this to be a property of coordinate transformations generated by analytic functions. Indeed, the subjects are intimately related and we shall return to this connection below.

Under an infinitesimal coordinate transformation $x'^\mu = x^\mu + \epsilon^\mu$, the metric tensor transforms according to equation (1.1) and yields:

$$g'_{\mu\nu}(\mathbf{x}') = g_{\mu\nu}(\mathbf{x}) - (\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu). \tag{1.3}$$

The condition (1.2) that such an infinitesimal transformation be conformal becomes:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu - \frac{2}{d}(\partial_\alpha \epsilon^\alpha) g_{\mu\nu} = 0, \tag{1.4}$$

where the factor of $2(\partial_\alpha \epsilon^\alpha)/d$ is fixed by contraction with $g^{\mu\nu}$. In dimensions $d > 2$, the solutions to equation (1.4) are strongly constrained. In this case, one may demonstrate that ϵ is at most quadratic in the coordinates x — we refer the reader to chapter 4 of reference [23] for a proof of this. The zeroth order solutions correspond to infinitesimal global translations $\epsilon^\mu = a^\mu$, whilst the linear solutions correspond to infinitesimal rotations $\epsilon^\mu = \omega^\mu{}_\nu \, x^\nu$ (where ω is antisymmetric) and scale transformations $\epsilon^\mu = \lambda \, x^\mu$. The quadratic solutions are the so-called *special conformal transformations* and may be shown to have the form $\epsilon^\mu = b^\mu \mathbf{x}^2 - 2x^\mu(\mathbf{b}.\mathbf{x})$. It is readily seen that these special transformations may be expressed as an inversion, followed by a translation, followed by an inversion: $x'^\mu/x'^2 = x^\mu/x^2 + b^\mu$. Thus, the conformal group in $d > 2$ is exhausted by the operations of translation, rotation, scale transformation and inversion. The generators of these infinitesimal conformal transformations form a Lie algebra. With a Euclidean metric, this Lie algebra is isomorphic to $so(d+1,1)$ and is spanned by a *finite* number $(d+1)(d+2)/2$ of generators.

However, in two dimensions the conformal constraint equations (1.4) admit an *infinite* parameter family of solutions. Adopting the Euclidean metric $g_{\mu\nu} = \delta_{\mu\nu}$, the equations (1.4) coincide with the Cauchy–Riemann conditions in the theory of analytic functions:

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2, \quad \partial_1 \epsilon_2 = -\partial_2 \epsilon_1. \tag{1.5}$$

It is natural to introduce the complex variables $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$ and the coordinate derivatives $\partial_z = (\partial_1 - i\partial_2)/2$, $\partial_{\bar{z}} = (\partial_1 + i\partial_2)/2$ obeying $\partial_z z = 1$ and $\partial_{\bar{z}} z = 0$. Defining $\epsilon = \epsilon_1 + i\epsilon_2$ and $\bar{\epsilon} = \epsilon_1 - i\epsilon_2$, the equations (1.5) may be written

$$\partial_{\bar{z}} \epsilon = 0, \quad \partial_z \bar{\epsilon} = 0. \tag{1.6}$$

That is to say, ϵ is a function of z only and is termed *holomorphic*, whilst $\bar{\epsilon}$ is a function of $\bar{z}$ only and is termed *antiholomorphic*. The conformal transformations thus coincide with the analytic transformations:

$$z \to z + \epsilon(z), \quad \bar{z} \to z + \bar{\epsilon}(\bar{z}). \tag{1.7}$$

An infinite number of Laurent expansion coefficients ε_n are needed to specify the space of all analytic transformations and it is conventional to define:

$$z \to z + \sum_{n=-\infty}^{\infty} \varepsilon_n z^{n+1}, \quad \bar{z} \to \bar{z} + \sum_{n=-\infty}^{\infty} \bar{\varepsilon}_n \bar{z}^{n+1}. \tag{1.8}$$

The corresponding generators read:

$$l_n = -z^{n+1}\partial_z, \quad \bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}}. \tag{1.9}$$

It is readily seen that these generators satisfy the so-called *local* conformal algebra:

$$[l_n, l_m] = (n-m)l_{n+m}, \quad [\bar{l}_n, \bar{l}_m] = (n-m)\bar{l}_{n+m}, \quad [l_n, \bar{l}_m] = 0. \tag{1.10}$$

The epithet *local* stems from the fact that not *all* the generators are globally well defined on the Riemann sphere. As may be seen from their definitions (1.9), only the generators l_{-1}, l_0, l_{+1} of the so-called *global* conformal algebra are free of singularities at $z = 0$ and $z = \infty$. In spite of this, it is the *infinite* local conformal algebra (or more accurately the Virasoro algebra) which is at the heart of conformal field theory in two dimensions. The group of global conformal transformations correspond to the so-called *projective transformations* of the complex plane:

$$z' = \frac{az+b}{cz+d} \quad \text{where} \quad ad - bc = 1, \tag{1.11}$$

and $a, b, c, d \in \mathbb{C}$. In §1.2 we shall indicate how invariance under the global projective conformal transformations constrains the possible form of correlation functions, and in §1.4 we shall see how the local Virasoro algebra may constrain these even further.

§1.2 Correlation Functions

An important class of fields in conformal field theory are the so-called *quasi-primary* fields which transform under *global* conformal transformations in the following 'tensorial' manner:

$$\mathcal{O}'(z', \bar{z}') = \left(\frac{dz'}{dz}\right)^{-h} \left(\frac{d\bar{z}'}{d\bar{z}}\right)^{-\bar{h}} \mathcal{O}(z, \bar{z}), \tag{1.12}$$

where h ($\bar{h}$) is the so-called holomorphic (antiholomorphic) *conformal dimension* of the field $\mathcal{O}(z, \bar{z})$. A field which transforms according to (1.12) under the full *local* conformal algebra is termed *primary*. In particular, the local density of states in the non-Abelian Dirac fermion problem will be a primary field. Global conformal invariance under the projective transformations (1.11) restricts the two-point correlation functions of quasi-primary fields $\mathcal{O}_i(1) \equiv \mathcal{O}_i(z_i, \bar{z}_i)$ to be of the form

$$\langle \mathcal{O}_1(1)\mathcal{O}_2(2)\rangle = \frac{C_{12}}{z_{12}^{2h}\,\bar{z}_{12}^{2\bar{h}}}, \tag{1.13}$$

if $h_1 = h_2 = h$ and $\bar{h}_1 = \bar{h}_2 = \bar{h}$, and is zero otherwise; $z_{ij} \equiv z_i - z_j$ and with suitably normalized fields one may take $C_{12} = 1$. The three-point functions are similarly constrained to be of the form:

$$\langle \mathcal{O}_1(1)\mathcal{O}_2(2)\mathcal{O}_3(3)\rangle = \frac{C_{123}}{\left(\prod_{i<j}^3 z_{ij}^{h_i+h_j-h_k}\right)\left(\prod_{i<j}^3 \bar{z}_{ij}^{\bar{h}_i+\bar{h}_j-\bar{h}_k}\right)}. \tag{1.14}$$

The four-point functions on the other hand, are *not* completely determined by *global* conformal invariance alone, but are of the form

$$\langle \mathcal{O}_1(1)\mathcal{O}_2(2)\mathcal{O}_3(3)\mathcal{O}_4(4)\rangle = \frac{\mathcal{F}(z,\bar{z})}{\left(\prod_{i<j}^4 z_{ij}^{h_i+h_j-h/3}\right)\left(\prod_{i<j}^4 \bar{z}_{ij}^{\bar{h}_i+\bar{h}_j-\bar{h}/3}\right)}, \tag{1.15}$$

where $h = \sum_{i=1}^4 h_i$ and $\bar{h} = \sum_{i=1}^4 \bar{h}_i$ and $\mathcal{F}(z,\bar{z})$ is *thus far* an arbitrary function of the so-called *anharmonic ratio*:

$$z = \frac{z_{12}z_{34}}{z_{23}z_{41}}, \quad \bar{z} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{23}\bar{z}_{41}}. \tag{1.16}$$

The residual dependence on the anharmonic ratio may be understood as follows. The global conformal transformations (1.11) involve three independent complex parameters, and one may always find a transformation to map three distinct points z_2, z_3, z_4 to three pre-assigned points $0, 1, \infty$, say. The generic four-point function retains its dependence on the remaining point $z_1 = z$.

It is worth emphasizing that equations (1.13), (1.14) and (1.15) are consequences of *global* conformal invariance *alone*, and may thus be generalized to higher dimensions. Although global conformal invariance tells us nothing about the structure constants C_{123}, and the dependence of the four-point correlation function on the anharmonic ratio, the requirement of *local* conformal invariance will constrain these dramatically. It is the infinite dimensional group of local conformal transformations (analytic transformations of the complex plane) — and their current algebra extensions — that enables exact solutions of two-dimensional conformal field theories. In general the *holomorphic* dependence on the anharmonic ratio is given as the solution of an appropriate differential equation — see for example §1.4. The full correlation function with its holomorphic and antiholomorphic dependence is obtained by combining these so-called *conformal blocks*[3] so as to respect the stringent constraints of *single-valuedness* and *crossing symmetry.* The constraint of single-valuedness simply asserts that our physical correlation functions should be single-valued functions over the entire complex plane. Crossing symmetry (or duality as it is sometimes referred to) is the requirement that correlation functions should be covariant (or invariant up to Fermionic minus signs) under the interchange of identical fields. Implementing these constraints is commonly referred to as the *conformal bootstrap* procedure. One of our major goals (and hopefully achievements) in this thesis will be the computation and comparison of these non-trivial four-point functions in the non-Abelian Dirac fermion problem obtained by three different methods.

§1.3 Virasoro Algebra

In conformal field theory, and indeed throughout this thesis, an important rôle is played by the energy-momentum tensor, $T_{\mu\nu}$. This may be defined (up to normalization) from the variation of the action under an arbitrary change in the coordinates, $x^\mu \to x^\mu + \epsilon^\mu$:

$$\delta S = -\int d^d x \, T_{\mu\nu} \, \partial^\mu \epsilon^\nu. \tag{1.17}$$

Translational invariance of the action requires $T_{\mu\nu}$ to be conserved, $\partial^\mu T_{\mu\nu} = 0$. Under infinitesimal conformal transformations it follows from equation (1.4), and the fact that $T_{\mu\nu}$ may be made symmetric in rotationally invariant theories, that

$$\delta S = -\frac{1}{d} \int d^d x \, T^\mu{}_\mu \, \partial_\alpha \epsilon^\alpha. \tag{1.18}$$

Thus, tracelessness of the energy–momentum tensor implies the invariance of the action under conformal transformations. In two dimensions, the conservation, symmetry and trace conditions imply that the energy–momentum tensor may be factorized into holomorphic and antiholomorphic components:[4]

$$T(z) \equiv T_{zz}(z), \quad \bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}(\bar{z}). \tag{1.19}$$

The fields $\epsilon(z)T(z)$ and $\bar{\epsilon}(\bar{z})\bar{T}(\bar{z})$ are conserved Noether currents, and (as in any field theory) their conserved charges generate the corresponding symmetry. As we shall see, these fields generate conformal transformations in our field theory [22]. In order to

[3] In the case of WZNW models these are sometimes referred to as *chiral blocks*.

[4] The vanishing trace implies $0 = T^\mu{}_\mu = g^{\mu\nu} T_{\nu\mu} = 2T_{z\bar{z}} + 2T_{\bar{z}z}$, and thus $T_{z,\bar{z}} = T_{\bar{z},z} = 0$ by symmetry. The conservation law $g^{\mu\alpha}\partial_\alpha T_{\mu\nu} = 0$ reduces to $\partial_{\bar{z}} T_{zz} = 0$, and $\partial_z T_{\bar{z}\bar{z}} = 0$ for $\nu = z, \bar{z}$ respectively.

define a conserved charge, we must define 'time' on our Euclidean complex plane. It is conventional to define this as increasing radially outwards from the origin, and the ensuing technical manipulations are often referred to as *radial quantization* — see chapter 6 of the book [23] for more details. In the framework of radial quantization, the conserved charge takes the form of a contour integral:

$$Q = \frac{1}{2\pi i}\oint dz\, \epsilon(z) T(z) \ldots \tag{1.20}$$

The ellipsis in equation (1.20) are used to highlight the fact that Q is an *operator*, and as such must act on some field *before* the contour integral is evaluated. The action of the charge on a given field is thus encoded in the singularities of the energy-momentum tensor with that field; the singularities determine the value of the contour integral by the residue theorem. In particular, the transformation law (1.12) of a quasi-primary field translates into this singularity (or Operator Product Expansion) language as

$$T(z)\mathcal{O}(w,\bar{w}) \sim \frac{h}{(z-w)^2}\,\mathcal{O}(w,\bar{w}) + \frac{1}{z-w}\,\partial_w \mathcal{O}(w,\bar{w}) \tag{1.21}$$

where the symbol $\sim$ means equality modulo terms regular as $w \to z$. Moreover, it turns out that the energy-momentum tensor is not (in general) a primary field, and its OPE with itself may be put in the form:

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{1.22}$$

where c is model dependent parameter known as the *central charge*, and where the (previously unspecified) normalization of $T(z)$ has been chosen to yield $h = 2$. In particular, defining our conserved charges (1.20) in the following manner

$$L_n = \frac{1}{2\pi i}\oint dz\, z^{n+1}\, T(z) \ldots \quad n \in \mathbb{Z} \tag{1.23}$$

one may demonstrate (using the techniques of radial quantization) that these operators obey the so-called Virasoro algebra:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}. \tag{1.24}$$

The Virasoro algebra is the field-theoretic version of the conformal algebra (1.10), and it differs only by the additional central charge term. In some sense, the central charge counts the massless degrees of freedom in a given theory [98] — see also [99, 100] for physical interpretations of the central charge. In this thesis (and for disordered critical models in general) we shall study theories with $c = 0$. As the reader will see, they are by no means trivial or devoid of degrees of freedom; this stems from the fact that they are associated with *non-unitary* representations of the Virasoro algebra.

The study of the Virasoro algebra in conformal field theory is not unlike the study of the $su(2)$ algebra in the quantum mechanical theory of angular momentum. In particular the eigenstates of the Virasoro operator L_0 carry the 'quantum number' h, the conformal dimension. The Virasoro generator L_0 plays a rôle analogous to J_z, with h corresponding to m. The Virasoro operators L_- (L_+) act like (an infinite set of) raising (lowering) operators, in that they increase (decrease) the value h. The *primary states* are (lowest-weight) eigenstates of the operator L_0 with eigenvalue h:

$$L_0|h\rangle = h|h\rangle, \tag{1.25}$$

and are annihilated by all the Virasoro lowering operators:

$$L_n|h\rangle = 0, n \geq 1. \tag{1.26}$$

The primary *states* are the Hilbert space representatives of the primary *fields* we discussed earlier. The states obtained by applying the raising operators L_{-n} in all possible ways to the primary states are *all* eigenstates of L_0 and are termed *descendant states*; a primary state and its descendants form a *conformal family*. That is to say, to each primary state in a CFT there is a conformal family which is in fact a representation of the Virasoro algebra.

In CFT, an important rôle is played by the so-called *singular vectors* (or null states) of the Virasoro algebra. A singular vector $|s\rangle$ is a descendant in a conformal family which is annihilated by all the Virasoro lowering operators: $L_n|s\rangle = 0, n \geq 1$. In particular, the singular vectors *decouple* from the Hilbert space, and are orthogonal to all the other members of the conformal family. This parallels the situation in quantum mechanics, where the Hilbert space of states truncates for integral and half-integral values of the angular momentum. The primary states which give rise to singular vectors are termed *degenerate states* [22]. As we shall see in §1.4, the existence of singular vectors will lead to differential equations for the correlation functions involving degenerate fields.

§1.4 Minimal Models

An important class of conformal field theories are the so-called *minimal models*. The Hilbert space description of these theories contains a finite number of representations of the Virasoro algebra. That is to say, they are characterized by a finite number of primary fields. The condition of unitarity imposes constraints on the possible central charges and the conformal dimensions that may be present.[5] The *unitary* minimal models have central charge $0 < c < 1$ given by

$$c = 1 - \frac{6}{m(m+1)}, \quad m = 3, 4, 5, \ldots \tag{1.27}$$

The allowed conformal dimensions of the *degenerate* fields, $\phi_{r,s}$, are given by the so-called Kač formula:

$$h_{r,s} = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}, \quad 1 \leq s \leq r \leq m-1 \tag{1.28}$$

In particular, the critical point of the two-dimensional Ising model is described by the minimal model with $m = 3$. The central charge as given by equation (1.27) coincides with that of a single Majorana fermion $c = 1/2$ as one might expect from Onsager's exact solution. The fields $\phi_{1,1}$, $\phi_{2,2}$ and $\phi_{2,1}$ with conformal dimensions $h_{1,1} = 0$, $h_{2,2} = 1/16$ and $h_{2,1} = 1/2$ are interpreted as the identity operator, the continuum Ising spin (σ_i) and the energy operator $(\sigma_i\sigma_j)$ respectively. Moreover, the tricritical Ising model corresponds to $m = 4$, and the three-state Potts model corresponds to $m = 5$.

Let us suppose for a moment that we are interested in the critical correlation functions of the Ising energy operator $\phi_{2,1}$. In the Hilbert space description, the state $|\phi_{2,1}\rangle$ with conformal dimension $h_{2,1}$ has a singular vector at level 2:

$$|s\rangle = \left[\mathcal{C}L_{-1}^2 - L_{-2}\right]|\phi_{1,2}\rangle \tag{1.29}$$

where $\mathcal{C} = 3/2(2h_{2,1}+1)$. That is to say, the specific combination of (descendant) states (1.29) obtained by the action of 'two-step' lowering operators on the state $|\phi_{2,1}\rangle$, is annihilated by all the Virasoro lowering operators and completely *decouples* from the Hilbert space. In particular, any correlation function involving this descendant state (or more accurately the corresponding descendant field) must be zero. As is discussed more fully in §7.3 of reference [23], equation (1.29) translates into the following partial differential constraint on the correlation functions:

$$\left\{\mathcal{C}\frac{\partial^2}{\partial z_1^2} - \sum_{i=2}^{n}\left[\frac{1}{z-z_i}\frac{\partial}{\partial z_i} + \frac{h_i}{(z-z_i)^2}\right]\right\}\langle\phi_{2,1}(z_1)\mathcal{O}_2(z_2)\cdots\mathcal{O}_n(z_n)\rangle = 0, \tag{1.30}$$

where the h_i are the conformal dimensions of the fields $\mathcal{O}_i(z_i)$. In the case of the four-point correlation functions, one may use global conformal invariance to reduce this *partial*

[5] In the language of Hilbert spaces, the condition of unitarity is the requirement that we are free of states with negative norm. The explicit evaluation of a given norm (or inner product) $\langle h|L_{l_n}\cdots L_{l_1}L_{-l_n}\cdots L_{-l_1}|h\rangle$ depends on the value of h, and through the Virasoro commutation relations, on the value of c.

differential equation into an *ordinary* differential equation. Specifically, one may first substitute into equation (1.30), the generic holomorphic dependence of the four-point function as given by equation (1.15):

$$\langle \phi_{2,1}(z_1) \cdots \mathcal{O}_4(z_4) \rangle = \mathcal{F}(z) \prod_{i<j}^{4} z_{ij}^{-h_i - h_j + h/3}. \tag{1.31}$$

By means of the projective conformal transformations (1.11) one may then map the points z_2, z_3, z_4 to $0, 1, \infty$, respectively, and thus from the definition (1.16) identify z_1 with z. This yields an ordinary differential equation for $\mathcal{F}(z)$. The solutions to this second-order differential equation $\mathcal{F}^{(a)}(z)$ $(a = 1, 2)$ — the so-called *conformal blocks* — yield the hitherto unknown dependence of the non-chiral four-point function on the anharmonic ratio:

$$\mathcal{F}(z, \bar{z}) = \sum_{a,b=1}^{2} X_{ab} \mathcal{F}^{(a)}(z) \mathcal{F}^{(b)}(\bar{z}). \tag{1.32}$$

The coefficients X_{ab} appearing in equation (1.32) may be determined from the requirements of single-valuedness and crossing symmetry of the correlation function. In principle, the four-point correlation functions of all the degenerate fields $\phi_{r,s}$ may be determined in a similar way from a differential equation of order rs. Integral representations for these have been elucidated by Dotsenko and Fateev [101, 102].

In recent years some interesting statistical models have been identified with *non-unitary* minimal models. In particular the $c = -2$ theory $(m = 1)$ describes the geometrical critical behaviour of dense polymers [62]. This theory has turned out to be very illuminating and we shall return to it in §1.6 and throughout this thesis.

§1.5 Wess–Zumino–Novikov–Witten model

Another important class of CFTs are the Wess–Zumino–Novikov–Witten (WZNW) models [68, 92, 103, 104]. We shall meet them many times in the course of this thesis, and we refer the reader to the books [23, 87, 94, 95] for a more detailed introduction. In this section we provide a brief glossary of the main ideas and constructions. We begin by considering the action:

$$S[g] = \frac{1}{4\lambda^2} \int d^2\xi \, \mathrm{Tr}'(\partial^\mu g^{-1} \partial_\mu g) + k\Gamma[g], \tag{1.33}$$

where (the WZNW term) Γ reads

$$\Gamma[g] = -\frac{i}{24\pi}\int d^3x\, \epsilon^{\mu\nu\rho}\, \mathrm{Tr}'(g^{-1}\partial_\mu g\, g^{-1}\partial_\nu g\, g^{-1}\partial_\rho g), \tag{1.34}$$

and $g(\xi)$ is a matrix field taking values on the group manifold G associated with the Lie algebra g. The integration region in the WZNW term is over the three-dimensional ball, whose boundary is the compactification of our original two-dimensional Euclidean space (ξ^1, ξ^2). In order for this term to be well defined, k should be an integer. The classical equation of motion follows from the requirement that the action be stationary $(0 = \delta S \equiv S[g+\delta g] - S[g])$ under the replacement $g \to g+\delta g \equiv g(1+g^{-1}\delta g)$. Substituting the latter form of the variation into the so-called Polyakov–Wiegmann identity [89]:

$$S[ab] = S[a] + S[b] + \int d^2\xi\, \omega^{\mu\nu}\, \mathrm{Tr}'(a^{-1}\partial_\mu a\, b\partial_\nu b^{-1}); \quad \omega^{\mu\nu} = \frac{\delta^{\mu\nu}}{2\lambda^2} - \frac{ik\epsilon^{\mu\nu}}{8\pi}, \tag{1.35}$$

and keeping terms of order δg,

$$0 = \delta S = -\int d^2\xi\, \omega^{\mu\nu}\, \mathrm{Tr}'(g^{-1}\partial_\mu g\, \partial_\nu(g^{-1}\delta g)) \Rightarrow \partial_\nu(\omega^{\mu\nu} g^{-1}\partial_\mu g) = 0. \tag{1.36}$$

In particular, for $\lambda^2 = 4\pi/k$, the field equation becomes

$$\partial(g^{-1}\bar\partial g) = 0 \Longleftrightarrow \bar\partial(\partial g g^{-1}) = 0. \tag{1.37}$$

That is to say, for this special relationship between the parameters λ and k, the model (1.33) possesses conserved holomorphic and antiholomorphic currents

$$J(z) = \sum_a J^a(z)t^a = -k\partial g g^{-1}, \tag{1.38}$$

$$\bar J(\bar z) = \sum_a \bar J^a(\bar z)\bar t^a = k g^{-1}\bar\partial g. \tag{1.39}$$

where the generators t^a of the Lie algebra g obey $[t^a, t^b] = f^{ab}{}_c t^c$ (repeated summation implied) and are normalized according to $\mathrm{Tr}'(t^a t^b) = 2\delta_{a,b}$. Moreover, for this relationship the model is also conformally invariant: the one-loop β-function vanishes [92], and more generally, the (Sugawara) energy momentum tensor yields the Virasoro algebra. We shall denote the action $S[g]$ with $\lambda^2 = 4\pi/k$ (*the* WZNW model) by $W_k[g]$:

$$W_k[g] = \frac{k}{16\pi}\int d^2\xi\, \mathrm{Tr}'(\partial^\mu g^{-1}\partial_\mu g) + k\Gamma[g]. \tag{1.40}$$

The parameter k is the so-called level of the WZNW model, and we shall often refer to the action (1.40) as the $\widehat{g}_k$ WZNW model. The conserved currents (1.37) arise from the invariance of the action (1.40) under the semi-local transformation

$$g(z,\bar{z}) \to \Omega(z) g(z,\bar{z}) \bar{\Omega}^{-1}(\bar{z}), \tag{1.41}$$

where $\Omega(z)$ and $\bar{\Omega}(\bar{z})$ are arbitrary elements of G. In addition to the presence of conserved holomorphic and antiholomorphic currents, it is readily verified that the solutions of the classical field equations (1.37) factorize: $g(z,\bar{z}) = f(z)\bar{f}(\bar{z})$. An important example of this, and of non-Abelian bosonization, is the equivalence between the fundamental bosonic field $g_{\alpha\bar{\alpha}}$ of the $\widehat{u}(N)_1$ WZNW model, and a bilinear of free massless Fermi fields [68, 92]:

$$M g_{\alpha\bar{\alpha}}(z,\bar{z}) = L_\alpha(z) R^\dagger_{\bar{\alpha}}(\bar{z}), \tag{1.42}$$

where the mass scale M is dependent upon the UV regularization scheme. In particular, the currents (1.38) may be shown to obey the operator product expansion:

$$J^a(z) J^b(w) \sim \frac{k\,\delta^{ab}}{(z-w)^2} + \frac{f^{ab}{}_c\, J^c(w)}{(z-w)}. \tag{1.43}$$

In analogy with equations (1.20) and (1.23) one may introduce the generators

$$J^a_n = \frac{1}{2\pi i} \oint dz\, z^n J^a(z) \ldots \tag{1.44}$$

The operator product expansions (1.43) translate through the methods of radial quantization into the following commutation relations:

$$[J^a_n, J^b_m] = f^{ab}{}_c\, J^c_{n+m} + kn\delta_{ab}\delta_{n+m,0}. \tag{1.45}$$

This is referred to as a Kač–Moody algebra (or affine Lie algebra or current algebra) at level k, and is often denoted $\widehat{g}_k$ — see for example [94, 105]. The energy–momentum tensor of the WZNW model is a bilinear in the conserved currents. This is known in the literature as the Sugawara construction:

$$T(z) = \frac{1}{2(k+g^\vee)} \sum_a (J^a J^a)(z) \tag{1.46}$$

where $g^\vee$ is the so-called dual Coxeter number of the algebra, and the normalization is fixed by the requirement that $T(z)$ has conformal dimension $h = 2$ [68, 106]. It may be shown that this energy-momentum tensor gives rise to a Virasoro algebra with central charge given by:

$$c = \frac{k \dim g}{k + g^\vee}. \tag{1.47}$$

Moreover, the multipoint correlation functions of the WZNW primary fields satisfy the so-called Knizhnik–Zamolodchikov equations [68]:

$$\left[\partial_{z_i} - \frac{1}{k+g^\vee} \sum_{j \neq i} \frac{t_i^a t_j^a}{z_i - z_j} \right] \langle \phi_1(z_1) \ldots \phi_n(z_n) \rangle = 0. \tag{1.48}$$

In this thesis we shall also be faced with WZNW models based on Lie superalgebras — see for example [11, 53, 107]. A generic Lie superalgebra is spanned by both bosonic and fermionic generators obeying

$$[t^a, t^b\} = f^{ab}{}_c \, t^c, \tag{1.49}$$

where the *Lie superbracket* or *supercommutator* $[,\}$ represents an anticommutator if both generators are fermionic and a commutator otherwise — see for example [108]. Although, the representation theory of superalgebras may be more involved than ordinary Lie algebras, in many applications of the rather general formulas presented above, one need only make minor modifications. These include replacing trace by supertrace [51], commutators by supercommutators, and the dimension of the algebra by the superdimension — the difference between the number of bosonic and fermionic generators.

§1.6 Logarithmic Conformal Field Theory

In this section we shall briefly discuss the topic of logarithmic conformal field theory (LCFT) originally introduced by Saleur [62] and Gurarie [64]. Our interest in these theories stems from their increasing relevance in the study of disordered critical points [12, 20, 40, 43–45, 53, 79, 109, 110]. The LCFTs have been extensively studied in both statistical physics and string theory [63, 65, 111–142]. Aside from their relevance in critical disordered systems, they have emerged in problems of gravitational dressing [111,112], D-branes [143–148], Seiberg–Witten models [149,150], magnetohydrodynamics [151–155] and the Haldane–Rezayi Quantum Hall state [66]. In this section we shall focus on a much studied, and indeed highly relevant, example of a LCFT — the $c = -2$ minimal model.

As we discussed in §1.4, the *unitary* minimal models have central charge given by equation (1.27) with $m \geq 3$. In particular, one may consider the *non-unitary* theory obtained by setting $m = 1$ in this expression; this leads to a model with central charge $c = -2$. This non-unitary theory plays an important rôle in the work of Saleur on dense polymers [62], and also in this present work. In particular, for this value of m (or equivalently the central charge) one may consider the field $\phi_{1,2}$ with conformal dimension $h_{1,2} = -1/8$ as given (formally) by equation (1.28). Although, the reader may worry about operators with negative conformal dimensions, we encourage them to indulge a little; in the non-Abelian Dirac fermion problem these operators will be 'dressed' by other fields and will ultimately acquire a *positive* dimension. In the spirit of §1.4 one may endeavor to calculate the four-point correlation functions of these fields; indeed, this procedure was carried out in the seminal works [62, 64]. We shall adopt the established notation of labeling the $c = -2$ field $\phi_{1,2}$, as μ. [64, 129]. The four-point correlation function of the field μ are found to be given by [62, 64, 129]

$$\langle \mu(1)\mu(2)\mu(3)\mu(4)\rangle = |z_{14}z_{23}z(1-z)|^{1/4} \sum_{a,b=1}^{2} X_{ab}\, F^{(a)}(z)F^{(b)}(\bar{z}), \tag{1.50}$$

where the conformal blocks $F^{(a)}(z)$ are solutions of the hypergeometric equation

$$z(1-z)G''(z) + [\gamma - (\alpha+\beta+1)z]G'(z) - \alpha\beta G(z) = 0, \tag{1.51}$$

with $\alpha = 1/2$, $\beta = 1/2$ and $\gamma = 1$, and where the coefficients X_{ab} are determined by the requirements of single-valuedness and crossing symmetry on the correlator (1.50). In the cases where γ is *not* an integer, one may write the two independent solutions of the hypergeometric equation (1.51) in the form:

$$G^{(1)}(z) = {}_2F_1[\alpha, \beta; \gamma; z], \quad G^{(2)} = z^{1-\gamma}\, {}_2F_1[\alpha-\gamma+1, \beta-\gamma+1; 2-\gamma; z], \tag{1.52}$$

where the hypergeometric series is defined by

$${}_2F_1[\alpha, \beta; \gamma; z] = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!} z^n; \quad (\alpha)_n \equiv \alpha(\alpha+1)\cdots(\alpha+n-1), \tag{1.53}$$

and is convergent in the domain $|z| < 1$; outside of this domain one must analytically continue this series — see appendix A. In the case at hand, with $\gamma = 1$, the solutions $G^{(1)}(z)$ and $G^{(2)}(z)$ are observed to be *degenerate*; they are both equal to ${}_2F_1[\frac{1}{2}, \frac{1}{2}; 1; z]$. We remark for the future benefit of the reader, that this hypergeometric series is in fact a complete elliptic integral of the first kind $2K(z)/\pi$ — see equation (A.3). In this case, the established method of Frobenius (see for example [156]) indicates that the second independent solution is a *logarithm* times the first solution plus a another regular series:

$$G^{(2)}(z) = G^{(1)}(z)\ln z + H(z), \tag{1.54}$$

where (in this example) $H(z)$ is the rather unpleasant series 9.153(2) of reference [69]:

$$H(z) = 2\sum_{m=0}^{\infty} z^m \frac{(1/2)_m(1/2)_m}{(m!)^2}\left[\psi(k+1/2) - \psi(k+1) - \psi(1/2) + \psi(1)\right], \qquad (1.55)$$

and $\psi(x) = d\ln\Gamma(x)/dx$. The form of equations (1.54) and (1.55) are somewhat cumbersome and one may perform the following 'trick' in order to make them more palatable. The hypergeometric equation (1.51) with $\alpha = 1/2, \beta = 1/2, \gamma = 1$, is invariant under the transformation $z \to 1-z$, and it follows that ${}_2F_1[\frac{1}{2},\frac{1}{2};1;1-z]$ is also a solution. Moreover, this solution is regular in the unit disc surrounding $z = 1$, but under analytic continuation develops a logarithmic singularity in the vicinity of $z = 0$ — see equation (A.15). Since there cannot be more than two independent solutions to a second-order differential equation, it must follow that (the analytic continuation of) ${}_2F_1[\frac{1}{2},\frac{1}{2};1;1-z]$ is a suitable substitute to (1.54). That is to say one may finally write the four-point correlation function of the fields μ in the form:

$$\langle\mu(1)\mu(2)\mu(3)\mu(4)\rangle = |z_{14}z_{23}z(1-z)|^{1/4}\left[K(z)K(1-\bar{z}) + K(1-z)K(\bar{z})\right], \qquad (1.56)$$

where we have embraced the compact notation afforded by the complete elliptic integrals (A.3), and have enforced the requirements of single-valuedness. The form of equation (1.56) is not only convenient for compactness, but also for symmetry; under the interchange of coordinates z_2 and z_3, the anharmonic ratios z and $1-z$ are also interchanged, and the correlation function is manifestly crossing symmetric. We remark, that whilst equation (1.56) has been obtained by solving the appropriate differential equation, the same result could also be obtained by taking an appropriate $c \to -2$ limit of the non-logarithmic formulae given in [101, 102]. We further invite the reader to compare the result (1.56) to the computations (2.19) and (2.41) in the non-Abelian Dirac fermion problem; the $c = -2$ theory will emerge again...

Chapter 2

Random Non-Abelian Dirac Fermions

§2.1 Introduction

An important driving force in the study of disordered critical points, is the quest to find a complete theoretical description of the localization–delocalization transitions which occur between the plateaus in the integer quantum Hall effect (IQHE) — see for example [1]. These transitions arise from a delicate interplay between quenched disorder in the sample, which tends to localize the two-dimensional electrons, and the strong magnetic field, which tends to delocalize them. On rather general grounds, it is natural to assume that these second-order phase transitions are governed by a two-dimensional conformal field theory, whose explicit identity would yield predictions for critical exponents and correlation functions. Despite numerous field theoretic approaches to find this conformal field theory, based on either Pruisken's non-linear sigma model [2–4] or the Chalker–Coddington network model [5] and its superspin chain descendants [6–10], the nature of the critical point remains elusive and inaccessible by perturbation theory [11, 12].

An early alternative to these approaches was suggested by Ludwig, Fisher, Shankar and Grinstein who observed that a two-dimensional Dirac fermion exhibits an analogue of the IQHE transition as the fermion mass is tuned through zero [13]. In particular, these transitions occur in the absence of quenched disorder, and one may systematically include and study the effects of its presence. As was discussed by these authors, disorder may be included in the form of a random mass, a random scalar potential, and a random (Abelian) vector potential for the Dirac fermion. Indeed, it was later shown that the Chalker–Coddington network model maps on to such a model in the continuum limit [157]. Unfortunately, the generic IQHE transition (with all three types of disorder) has, thus far, also proven inaccessible in this representation. However, these authors set the stage for studies of Dirac fermions interacting with random potentials [17,46,53,158–167].

In this chapter we shall study a non-Abelian variant of the model of reference [13] which was introduced by Nersesyan, Tsvelik and Wenger in the context of disordered d-wave superconductivity [27,28]. As we shall see, this model comprises of N_c 'colours' of two-dimensional massless Dirac fermions interacting with a random non-Abelian gauge field, and has proved to be extremely instructive example of a disordered critical point. Indeed this model has also arisen in studies of non-Hermitian quantum mechanics [14] and the spin quantum Hall effect [52, 168]. In particular this model is amenable to a variety of theoretical approaches [46]. Amongst its non-trivial features are the presence of logarithmic operators [40], and the multifractal scaling of moments of the local density of states [41, 43–45, 48].

The structure of this chapter is as follows: in §2.2 we provide a brief outline of the emergence of the non-Abelian Dirac fermion problem in the context of disordered d-wave superconductivity. In §2.3 we define the problem in more general terms. In §2.4–§2.6 we focus our attention on the four-point correlation functions of the local density of states (LDoS) as encoded in the so-called Q-matrix. Each of these sections is devoted to a different non-perturbative field theoretical approach to the disordered Dirac fermion problem: §2.4 deals with the replica approach, §2.5 deals with the so-called strong disorder approach and §2.6 deals with the supersymmetric approach. Within each of these sections we provide an outline of the theoretical approach and the form of the appropriate Q-matrix, together with a discussion of the relevant conformal dimensions and four-point correlation functions. In order to increase the transparency of these sections we have relegated many of the important (but arguably involved) technical details on the solution of the relevant WZNW models into a rather substantial appendix. The interested reader will find gathered in this appendix the solutions to the Knizhnik–Zamolodchikov equations which arise in this work. In §2.7 we continue our study of the supersymmetric approach and provide a remarkably simple free field representation of the disorder averaged theory comprising of a two-component symplectic fermion and a pair of free bosons (one compact and the other non-compact). The two-component symplectic fermion with central charge $c = -2$ arises in the theoretical description of dense polymers and we demonstrate that the twist operators of the latter theory play a fundamental rôle in the non-Abelian Dirac fermion problem. In §2.8 we discuss the convergence of approaches to the disordered non-Abelian Dirac fermion problem. Finally we present concluding remarks and technical appendices. In particular, appendix 2.A is devoted to the $\widehat{su}(N)_k$ WZNW model and is subdivided into our considerations of the $\widehat{u}(0)_k$ WZNW model — arising in the replica approach — and the $\widehat{su}(N)_{-2N}$ WZNW model — arising in the strong disorder approach. Appendix 2.B is devoted to the $\widehat{osp}(2|2)_k$ WZNW model which arises in the supersymmetric approach.

§2.2 Disordered D-Wave Superconductivity

Following [27, 28] we consider the Hamiltonian describing the low-energy degrees of freedom in a two-dimensional d-wave superconductor:

$$H_0 = \sum_{\mathbf{k},\sigma}[\epsilon(\mathbf{k}) - \mu]c^\dagger_{\mathbf{k},\sigma}c_{\mathbf{k},\sigma} + \sum_{\mathbf{k}}[\Delta(\mathbf{k})c^\dagger_{\mathbf{k}\uparrow}c^\dagger_{-\mathbf{k}\downarrow} + \text{h.c.}], \tag{2.1}$$

where $\epsilon(\mathbf{k})$ is a tight-binding spectrum respecting the point symmetry of the underlying lattice, $\Delta(\mathbf{k})$ is a pairing amplitude with the $\mathrm{d}_{\mathrm{x}^2-\mathrm{y}^2}$ symmetry, and μ is the chemical potential — see for example [169]. Without loss of generality one may take the tight-binding spectrum to be of the form

$$\epsilon(\mathbf{k}) = -2t\left[\cos(k_x a) + \cos(k_y a)\right], \tag{2.2}$$

and the d-wave order parameter to be given by

$$\Delta(\mathbf{k}) = \Delta\left[\cos(k_x a) - \cos(k_y a)\right]. \tag{2.3}$$

In a realistic d-wave superconductor $\Delta \ll t$. The quasi-particle energy spectrum of such a superconductor is given by

$$E(\mathbf{k}) = \pm\sqrt{(\epsilon(\mathbf{k}) - \mu)^2 + \Delta(\mathbf{k})^2} \tag{2.4}$$

In particular, the quasi-particle energy spectrum $E(k)$ vanishes at four points or 'nodes'. Setting $\mu = 0$ and $a = 1$ for simplicity, these are given by $(k_x, k_y) = \left(\pm\frac{\pi}{2}, \pm\frac{\pi}{2}\right)$ — see figure 2.1. As may be seen quite clearly from this figure, in the vicinity of these nodes reside low-energy gapless excitations with a formally relativistic spectrum. In the notation of references [27, 28] the nodes are labeled $\mathbf{1}, \bar{\mathbf{1}}, \mathbf{2}, \bar{\mathbf{2}}$:

$$\mathbf{1} = (\tfrac{\pi}{2}, \tfrac{\pi}{2}), \quad \bar{\mathbf{1}} = (-\tfrac{\pi}{2}, -\tfrac{\pi}{2}), \quad \mathbf{2} = (-\tfrac{\pi}{2}, \tfrac{\pi}{2}), \quad \bar{\mathbf{2}} = (\tfrac{\pi}{2}, -\tfrac{\pi}{2}). \tag{2.5}$$

In the strict absence of nearest neighbour scattering processes $(\mathbf{1}, \bar{\mathbf{1}}) \leftrightarrow (\mathbf{2}, \bar{\mathbf{2}})$ [16], one may treat the pairs of nodes $(\mathbf{1}, \bar{\mathbf{1}})$ and $(\mathbf{2}, \bar{\mathbf{2}})$ independently [27, 28]; for definiteness we shall choose the $(\mathbf{1}, \bar{\mathbf{1}})$ pair. In this limit, the authors of references [27, 28] were able to map this problem (after coordinate rescaling and field redefinition) on to a model

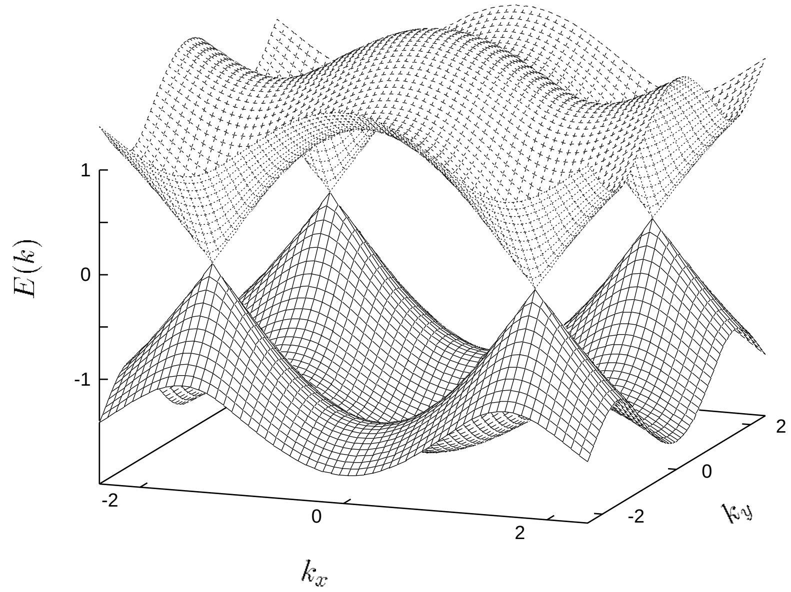

Figure 2.1: The quasi-particle spectrum of a clean d-wave superconductor showing the existence of four Dirac nodes.

of $(2+1)$–dimensional massless Dirac fermions interacting with a random non-Abelian gauge field. In particular, by focusing on static disorder one may consider the Fourier components of the fermionic fields with different frequencies separately. This renders the theory two-dimensional. In this thesis we shall focus on the zero frequency critical theory:

$$S = \int d^2\xi \sum_{\alpha,\beta} \bar{\psi}_\alpha \gamma^\mu (\delta^{\alpha\beta}\partial_\mu - iA_\mu^{\alpha\beta})\psi_\beta. \tag{2.6}$$

The 'colour' indices $\alpha, \beta \in (\mathbf{1}, \bar{\mathbf{1}})$ are the node labels, and the field ψ_α represents a two-component Dirac fermion at node α. The gauge field describes scattering processes within and between the nodes $\mathbf{1}$ and $\bar{\mathbf{1}}$, and resides in the algebra $su(2)$: $A_\mu = A_\mu^a \tau_a^{\alpha\beta}$. The probability distribution of the gauge field is taken to be Gaussian. In this chapter we shall consider the $su(N_c)$ generalization of the model (2.6) and our (primary) goal will be to calculate and compare the disorder averaged multipoint correlation functions of the LDoS obtained by three different non-perturbative approaches.

§2.3 Non-Abelian Dirac Fermions

We consider N_c colours of two-dimensional massless Dirac fermions minimally coupled to a non-Abelian gauge field $A_\mu \in su(N_c)$:

$$S = \int d^2\xi \sum_{\alpha,\beta=1}^{N_c} \bar{\psi}^\alpha \not{D}^{\alpha\beta} \psi^\beta, \tag{2.7}$$

where the gauge covariant derivative takes the form

$$\not{D}^{\alpha\beta} = \gamma^\mu(\delta^{\alpha\beta}\partial_\mu - iA_\mu^{\alpha\beta}). \tag{2.8}$$

The γ^μ matrices form a two-dimensional representation of the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ with Euclidean metric $g^{\mu\nu} = \mathrm{diag}(1,1)$, and the gauge field $A_\mu^{\alpha\beta} = A_\mu^a \tau_a^{\alpha\beta}$ may be expanded in terms of the generators τ_a of $su(N_c)$. Physical quantities are obtained by disorder averaging products of Green's functions. We use the distribution functional

$$P[A_\mu] \propto e^{-S[A_\mu]}, \quad S[A_\mu] = \frac{1}{g_A} \int d^2\xi \,\mathrm{Tr}\, A_\mu(\xi) A_\mu(\xi) \tag{2.9}$$

representing the usual choice of δ-correlated Gaussian white noise for the random vector potential. The local density of states (LDoS) of the Dirac fermion problem is given by [27–29]

$$\rho(\mathbf{r}) = \sum_{\alpha=1}^{N_c} \left[L_\alpha^\dagger R_\alpha + R_\alpha^\dagger L_\alpha \right]. \tag{2.10}$$

In §2.4–§2.6 we shall investigate the disorder averaged multipoint correlation functions of the LDoS. In particular, we shall focus our attention on the long distance properties (obtained for example by integrating out the massive modes) encoded in the so-called Q-fields:

$$Q \sim \sum_{\alpha=1}^{N_c} L_\alpha^\dagger R_\alpha, \quad Q^\dagger \sim \sum_{\alpha=1}^{N_c} R_\alpha^\dagger L_\alpha \tag{2.11}$$

We shall study the critical correlation functions of the Q-fields by three different approaches; in each case our Q-fields will be represented by Q-matrices governed by appropriate WZNW models.

§2.4 Replica Approach

In order to average over disorder we introduce N_f flavours (or replicas) of the massless Dirac fermions appearing in equation (2.7):

$$S^{\mathrm{Replica}} = \int d^2\xi \sum_{i=1}^{N_f} \sum_{\alpha,\beta=1}^{N_c} \bar{\psi}^{\alpha,i} \not{D}^{\alpha\beta} \psi^{\beta,i} \tag{2.12}$$

The (replicated) action (2.12) enjoys a global $SU(N_f) \times SU(N_c) \times U(1)$ symmetry and may be recast using Witten's non-Abelian bosonization rules [92] — for reviews of the bosonization approach see for example [87, 170]. It is well known that the free Dirac action (in the absence any gauge fields) with the flavour-colour symmetry described above may be represented as the sum of three critical Wess–Zumino–Novikov–Witten (WZNW) models of the form

$$W_k[g] = \frac{k}{16\pi} \int d^2\xi \,\mathrm{Tr}'(\partial^\mu g^{-1} \partial_\mu g) + k\Gamma[g] \tag{2.13}$$

where (the WZNW term) Γ reads

$$\Gamma[g] = \frac{i}{24\pi} \int d^3x \, \epsilon^{\mu\nu\rho} \,\mathrm{Tr}'(g^{-1}\partial_\mu g \, g^{-1}\partial_\nu g \, g^{-1}\partial_\rho g). \tag{2.14}$$

The celebrated equivalence between the free (flavoured-coloured) Dirac fermion theory and the bosonic WZNW models may be written in the symbolic form

$$\text{Free Dirac Fermions} = \widehat{su}(N_c)_{N_f} \times \widehat{su}(N_f)_{N_c} \times \widehat{u}(1) \tag{2.15}$$

where it is understood that the *chiral* blocks of the free Dirac theory are obtained as products of the *chiral* blocks of the WZNW models; here $\widehat{g}_k$ denotes the group manifold g and the so-called level k of the WZNW action (2.13). That this is indeed an equality between *chiral* blocks is seen most clearly in the work of Fuchs [171], and is discussed more fully in chapter 3. Of paramount importance in the application of (2.15) to our disordered problem is that the disorder — the su(N_c) gauge potential — couples *only* to (currents from) the $\widehat{su}(N_c)_{N_f}$ sector. Averaging over disorder (equivalently integrating over the su(N_c) gauge potential) generates a (quadratic) interaction between su(N_c) currents only. This interaction scales to the strong coupling regime where a mass gap $M \sim \exp[-2\pi/N_c g_A]$ is dynamically generated in the $\widehat{su}(N_c)_{N_f}$ sector of the theory. As follows from the decomposition (2.15) the massless degrees of freedom are SU(N_c) singlets and are described by the remaining WZNW models $\widehat{su}(N_f)_{N_c} \times u(1) \sim \widehat{u}(N_f)_{N_c}$.

Upon integrating out the massive (colour) degrees of freedom, the fermion bilinears are expressed in terms of the so-called Q-matrix

$$Q_{r\bar{r}} \sim \frac{1}{M}\sum_{\alpha=1}^{N_c} L^{\dagger}_{\alpha,r} R_{\alpha,\bar{r}} \tag{2.16}$$

whose indices reside in the flavour (replica) space; this is simply a replicated version of the Q-field introduced to describe the LDoS in equations (2.10) and (2.11). The Q-matrix assumes values in the group $\mathrm{U}(N_f)$ and is governed by a (critical) effective action of the WZNW form (2.13) with level $k = N_c$. The WZNW model thus plays an analogous rôle to the sigma model in the conventional theory of localization [50] — it is an effective action for the slow degrees of freedom, once the fast degrees of freedom have been integrated out.

§2.4.1 Conformal Dimensions

In the replica approach, the conformal dimension of the Q-matrix is that of a primary field of the $(N_f \to 0)$ $\widehat{u}(N_f)_{N_c}$ WZNW model transforming in the fundamental representation — see equation (2.80) of appendix 2.A:

$$h_Q = \frac{1}{2N_c^2} \tag{2.17}$$

As we shall see in §2.5.1 and §2.6.1, the conformal dimension of the Q-matrix (LDoS) is given by (2.17) in all three treatments of the disorder. This ensures the equality of their two-point and three-point correlation functions. It is well known in CFT however, that the four-point correlation functions are *not* determined solely by scaling dimensions, but in general have a non-trivial dependence on the so-called anharmonic ratios (2.67) [77]; in CFTs of the WZNW form, this dependence is obtained by solving the appropriate Knizhnik–Zamolodchikov equations [68]. In order to compare different theoretical approaches to the random Dirac fermion problem it is thus *essential* to study the four-point correlation functions of the Q-matrix.

§2.4.2 Correlation Functions of the Q-field

The correlation functions of the Q-matrix are those pertaining to the $\widehat{u}(N_f)_{N_c}$ WZNW model in the replica $(N_f \to 0)$ limit, and are thus obtained by solving the $\widehat{u}(N_f)_{N_c}$ Knizhnik–Zamolodchikov equations [68]. This fact was recognised in the early replica treatment of the random Dirac fermion problem [27,28]. The approach was subsequently refined to accommodate the logarithmic solutions to the Knizhnik–Zamolodchikov equations which appear in the replica limit [40]. Despite the advances made in these works they suffer (in places) from a naïve implementation of the replica trick; an extraneous trace over replica indices was performed in a number of instances. In order to address these issues, and to demonstrate the convergence of approaches to the random Dirac fermion problem, we rederive a number of results and adopt a uniform notation throughout.

The four-point correlation function of the Q-matrix admits the $U(N_f) \times U(N_f)$ invariant decomposition [40, 68] (see appendix 2.A.1):

$$\langle Q_{r_1\bar{r}_1}(1)Q^\dagger_{r_2\bar{r}_2}(2)Q^\dagger_{r_3\bar{r}_3}(3)Q_{r_4\bar{r}_4}(4)\rangle = |z_{14}z_{23}|^{-2/N_c^2} \sum_{i,j=1}^{2} I_i^r \bar{I}_j^{\bar{r}} F_{ij}(z,\bar{z}) \tag{2.18}$$

where r denotes the ordered sequence of flavour (replica) indices r_1, r_2, r_3, r_4, and where the invariant tensors I_1 and I_2 are defined as $I_1^r = \delta^{r_1,r_2}\delta^{r_3,r_4}$ and $I_2^r = \delta^{r_1,r_3}\delta^{r_2,r_4}$, together with similar equations for $\bar{I}_1$ and $\bar{I}_2$. The anharmonic ratio z is defined as $z = z_{12}z_{34}/z_{14}z_{32}$ and similarly for $\bar{z}$. The functions $F_{ij}(z,\bar{z})$ are single-valued combinations of the solutions to the $\widehat{u}(N_f)_{N_c}$ Knizhnik–Zamolodchikov equations, which in the replica limit ($N_f \to 0$) are given by equations (2.87) with $k = N_c$. For example, setting all replica indices equal to 1 yields

$$\langle Q_{1\bar{1}}(1)Q^\dagger_{1\bar{1}}(2)Q^\dagger_{1\bar{1}}(3)Q_{1\bar{1}}(4)\rangle \sim |\Upsilon|^2 \left[K_{N_c}(z)K_{N_c}(1-\bar{z}) + K_{N_c}(1-z)K_{N_c}(\bar{z})\right] \tag{2.19}$$

where $\Upsilon = [z_{14}z_{23}\, z(1-z)]^{-1/N_c^2}$ and $K_{N_c}(z)$ and $E_{N_c}(z)$ are natural generalizations of the complete elliptic integrals — see equation (2.84). In addition one may consider 'mixed' correlation functions of the form:

$$\langle Q_{1\bar{1}}(1)Q^\dagger_{1\bar{1}}(2)Q^\dagger_{2\bar{2}}(3)Q_{2\bar{2}}(4)\rangle \sim |\Upsilon|^2 \left[E_{N_c}(z)(K_{N_c}(1-\bar{z}) - E_{N_c}(1-\bar{z})) + c.c.\right] \tag{2.20}$$

where $c.c$ stands for complex conjugation — replacement of z by $\bar{z}$. Correlation functions calculated with respect to this effective (replicated) action correspond to correlation functions averaged with respect to the initial action with quenched disorder [46, 80]:

$$\overline{\langle Q(1)Q^\dagger(2)Q^\dagger(3)Q(4)\rangle_A} = \langle Q_{r_i\bar{r}_i}(1)Q^\dagger_{r_i\bar{r}_i}(2)Q^\dagger_{r_i\bar{r}_i}(3)Q_{r_i\bar{r}_i}(4)\rangle_{\text{rep}}, \tag{2.21a}$$

$$\overline{\langle Q(1)Q^\dagger(2)\rangle_A\langle Q^\dagger(3)Q(4)\rangle_A} = \langle Q_{r_i\bar{r}_i}(1)Q^\dagger_{r_i\bar{r}_i}(2)Q^\dagger_{r_j\bar{r}_j}(3)Q_{r_j\bar{r}_j}(4)\rangle_{\text{rep}}, \tag{2.21b}$$

where $r_i \neq r_j$ in equation (2.21b). The left hand side of (2.21a) represents the four-point correlation function of the Q-field (calculated with respect to the original action in the presence of quenched disorder) averaged over disorder (denoted by an overline); the left hand side of (2.21b) represents the product of two two-point correlation function of the Q-field (again calculated with respect to the original action in the presence of quenched disorder) averaged over disorder.

We note that as a direct consequence of the rather simple replica index structure of (2.18) the results (2.21a) and (2.21b) are independent of which replicas are actually considered; this is a manifestation of replica symmetry — see for example §3.3 of reference [172]. The correlation functions of the LDoS may be obtained from these results by means of the decomposition (2.10) together with crossing symmetry. We emphasize that one does *not* perform a trace over replica indices in order to extract the LDoS; the traces over replica indices appearing in equation (60) of reference [28] and equation (5) of reference [40] are erroneous. We further draw attention to the simplicity and manifest crossing symmetry of the results (2.19) and (2.20) as compared with those obtained in reference [40]

§2.5 Strong Disorder Approach

As was first discussed in [41, 46], and subsequently developed in [44], the random Dirac fermion problem is amenable to a direct treatment in the limit of infinite disorder strength ($g_A \to \infty$) without invoking either replicas or supersymmetry. We note that in this limit the probability measure (2.9) is absent. We summarize here only the important details. We separate the action (2.7) into its chiral components

$$S = \int d^2\xi \sum_{\alpha=1}^{N_c} \left[L_\alpha^\dagger(\bar\partial + i\bar A)L_\alpha + R_\alpha^\dagger(\partial + iA)R_\alpha \right] \tag{2.22}$$

and parameterize the gauge fields in terms of group elements $g(\xi) \in SU^{\mathbb{C}}(N) \sim SL(N;\mathbb{C})$ residing in the complex extension of $SU(N)$

$$A = i\partial g g^{-1}, \quad \bar A = i\bar\partial \bar g \bar g^{-1}. \tag{2.23}$$

This enables one to perform the chiral gauge transformations

$$L \to \bar g \mathcal{L}, \quad R \to g\mathcal{R}, \tag{2.24}$$

so as to render a theory of free fermions $(\mathcal{L}, \mathcal{R})$ decoupled from the gauge fields. An important feature of this procedure is that the Jacobian of the transformations (2.24) is proportional to the partition function of the original action (2.7) at fixed disorder, $Z[A_\mu]$ — this cancels the normalizing partition function in (fixed disorder) correlation functions and removes the need to invoke replicas or supersymmetry in order to perform disorder averaging. The Jacobian associated with the change of variables (2.23) is well known to involve the WZNW action (2.13) on the (non-compact) manifold $h = g^\dagger g \in SU^{\mathbb{C}}(N)/SU(N)$ at level $k = -2N_c$ [41, 46, 88, 173]:

$$\mathcal{D}A = \mathcal{D}g \, \exp\left(2N_c W_{-2N_c}[g^\dagger g]\right) \tag{2.25}$$

In reference [44] the Wakimoto free-field representation of the $SU^{\mathbb{C}}(N)/SU(N)$ WZNW model at $k = -2N$ was constructed, and it was shown to generate the $\widehat{su}(N)_{-2N}$ Kač–Moody algebra; in other words, the correlation functions of the h-fields may be obtained from the solution of the $\widehat{su}(N)_{-2N}$ Knizhnik–Zamolodchikov equations — see appendix 2.A.2. In this approach the Q-fields of the random Dirac fermion problem with N_c colours are expressed as primary fields of the $\widehat{su}(N_c)_{-2N_c}$ WZNW model 'dressed' by free fermions:

$$Q = \sum_{\alpha,\bar{\alpha}=1}^{N_c} \mathcal{L}_\alpha^\dagger \, h_{\alpha\bar{\alpha}} \, \mathcal{R}_{\bar{\alpha}}, \quad Q^\dagger = \sum_{\alpha,\bar{\alpha}=1}^{N_c} \mathcal{R}_{\bar{\alpha}}^\dagger \, h_{\bar{\alpha},\alpha}^\dagger \, \mathcal{L}_\alpha, \tag{2.26}$$

We shall study the conformal dimensions and correlation functions of these fields in the subsections below.

§2.5.1 Conformal Dimensions

It is readily seen from the strong disorder decomposition (2.26) that the conformal dimension of the Q-field is that of free Dirac fermion ($h = 1/2$) and an $\widehat{su}(N_c)_{-2N_c}$ primary field — see equation (2.68) and set $N = N_c$ and $k = -2N_c$:

$$h_Q = \frac{1}{2} + \frac{N_c^2 - 1}{2N_c(N_c - 2N_c)} = \frac{1}{2N_c^2} \tag{2.27}$$

This agrees with the replica result (2.17) and ensures the equality of the two-point and three-point correlation functions in both approaches. The coincidence of higher correlation functions will be discussed below.

§2.5.2 Correlation Functions of the Q-field

The correlation functions of the Q-field are thus obtained by a fermionic 'dressing' of the correlation functions of the $\widehat{su}(N_c)_{-2N_c}$ WZNW model. We note that whilst the chiral solutions to the $\widehat{su}(N_c)_{-2N_c}$ WZNW model given in [44] are correct, their normalization constants are erroneous — they do not satisfy the conformal bootstrap. We provide the correct conformal blocks in equations (2.90) and their expression in terms of generalized elliptic integrals in equations (2.91). Moreover, in the light of the new results obtained in the replica approach, we shall find it convenient to generalize the decomposition (2.26) slightly so as to accommodate disorder averages of *products* of quenched correlation functions such as those appearing in equation (2.21b). To this end we introduce as many *additional* fermionic species (denoted by an index p) as quenched correlation functions we wish to disorder average. We emphasize that these additional indices are *not* required to *perform* the disorder averaging (as would be true of replicas) but simply encode which combinations of quenched correlation functions are to be averaged over disorder — in this approach we do *not* perform an $N_p \to 0$ limit. The Q-matrix acquires a pseudo-replica index structure (labeled by the index p) analogous to that appearing in equation (2.16):

$$Q_{p,\bar{p}} = \sum_{\alpha,\bar{\alpha}=1}^{N_c} \mathcal{L}^{\dagger}_{\alpha,p}\, h_{\alpha,\bar{\alpha}}\, \mathcal{R}_{\bar{\alpha},\bar{p}}, \quad Q^{\dagger}_{p,\bar{p}} = \sum_{\alpha,\bar{\alpha}=1}^{N_c} \mathcal{R}^{\dagger}_{\bar{\alpha},\bar{p}}\, h^{\dagger}_{\bar{\alpha},\alpha}\, \mathcal{L}_{\alpha,p}, \tag{2.28}$$

The correlation functions of the fields (2.28) may be written in the form

$$\langle Q_{p_1\bar{p}_1}(1) Q^{\dagger}_{p_2\bar{p}_2}(2) Q^{\dagger}_{p_3\bar{p}_3}(3) Q_{p_4\bar{p}_4}(4)\rangle = |z_{14}z_{23}|^{-4h} \sum_{a,b=1}^{2} C_{ab}\, \mathcal{G}^{(a)}_{p}(z_i)\, \mathcal{G}^{(b)}_{\bar{p}}(\bar{z}_i) \tag{2.29}$$

where $h = (1-N_c^2)/2N_c^2$ is the conformal dimension of the $\widehat{su}(N_c)_{-2N_c}$ primary field $h_{\alpha,\bar{\alpha}}$ as given by equation (2.68). $\mathcal{G}^{(a)}_p(z_i)$ is the (holomorphic) product of the free Dirac and $\widehat{su}(N_c)_{-2N_c}$ correlation functions traced over the colour indices:

$$\mathcal{G}^{(a)}_p(z_i) = \sum_{\alpha=1}^{N_c} \left(\frac{I_1^{\alpha} I_1^{p}}{z_{12}z_{34}} - \frac{I_2^{\alpha} I_2^{p}}{z_{13}z_{24}} \right) \times \left(I_1^{\alpha} F_1^{(a)}(z) + I_2^{\alpha} F_2^{(a)}(z) \right) \tag{2.30}$$

The coefficients C_{ab} appearing in equation (2.29) have the values $C_{12} = C_{21} = 1$, $C_{11} = C_{22} = 0$ to ensure single-valuedness and their off-diagonal form reflects the logarithmic nature of the underlying $\widehat{su}(N_c)_{-2N_c}$ LCFT. We use the symbol p here to denote the ordered sequence of pseudo-replica indices p_1, p_2, p_3, p_4 and the invariant tensors I_1 and I_2 are defined as $I_1^p = \delta^{p_1,p_2}\delta^{p_3,p_4}$, $I_2^p = \delta^{p_1,p_3}\delta^{p_2,p_4}$; analogous expressions hold for α. Performing the trace over colour indices appearing in (2.30) by means of the identities

$$\sum_{\alpha=1}^{N_c} I_1^{\alpha} I_1^{\alpha} = N_c^2 \quad \sum_{\alpha=1}^{N_c} I_1^{\alpha} I_2^{\alpha} = N_c \tag{2.31}$$

and utilizing the relations (2.94) satisfied by the $\widehat{su}(N_c)_{-2N_c}$ chiral blocks one obtains:

$$\mathcal{G}_p^{(1)}(z_i) = \frac{c'}{z_{14}z_{32}}\,\Lambda\{I_1^p E_{N_c}(z) + I_2^p\left[K_{N_c}(z) - E_{N_c}(z)\right]\} \tag{2.32a}$$

$$\mathcal{G}_p^{(2)}(z_i) = \frac{2c'N_c}{z_{14}z_{32}}\,\Lambda\left\{I_1^p\left[\tilde{E}_{N_c}(z) - \tilde{K}_{N_c}(z)\right] - I_2^p\tilde{E}_{N_c}(z)\right\} \tag{2.32b}$$

where $c' = cN(N^2-1)$ is a normalization constant, $\Lambda = [z(1-z)]^{-1/N_c^2}$, and where we have adopted the notation that $\tilde{f}(z) \equiv f(1-z)$ for an arbitrary function $f(z)$. In particular by inserting the explicit results (2.32) into the decomposition (2.29) and collecting the coefficients of $I_i^p\bar{I}_j^{\bar{p}}$ one may recast (2.29) so as to read:

$$\langle Q_{p_1\bar{p}_1}(1)Q^\dagger_{p_2\bar{p}_2}(2)Q^\dagger_{p_3\bar{p}_3}(3)Q_{p_4\bar{p}_4}(4)\rangle = -\frac{2c'^2}{X_{12}}|z_{14}z_{23}|^{-2/N_c^2}\sum_{i,j=1}^{2} I_i^p\bar{I}_j^{\bar{p}}F_{ij}(z,\bar{z}) \tag{2.33}$$

where the $F_{ij}(z,\bar{z})$ are (both fortunately and remarkably) single-valued combinations of the solutions to the $\widehat{u}(0)_{N_c}$ Knizhnik–Zamolodchikov equations as given by equations (2.87) with $k = N_c$. That is to say (up to an irrelevant normalization) we have recovered the replica result (2.18) in which our pseudo-replica indices play the rôle of replica indices. In particular the results (2.19) and (2.20) together with their interpretations (2.21) follow straightforwardly.

§2.6 Supersymmetric Approach

In the supersymmetric approach to disordered fermionic systems one introduces bosonic copies of the original Grassmann fields [50, 51]. The partition function of the resulting supersymmetric theory is equal to unity due to the inverse relationship between ordinary c-number Gaussian functional integrals and their Grassmann counterparts. The absence of a disorder dependent partition function normalizing quenched correlation functions drastically simplifies their disorder averaging. The supersymmetric approach to the random Dirac fermion problem has been outlined by Bernard and LeClair [52] who, following the general principles of the supersymmetric approach, have introduced bosonic copies of the Grassmann fields coupled to the same gauge potential. The resulting action is given by (2.12) where the summation over replicas is replaced by a summation over fermionic (Grassmann) and bosonic (c-number) fields:

$$S^{\mathrm{SUSY}} = \int d^2\xi \sum_{i=1}^{2}\sum_{\alpha,\beta=1}^{N_c} \bar{\psi}^{\alpha,i}\not{D}^{\alpha\beta}\psi^{\beta,i} \tag{2.34}$$

The symmetry of the free supersymmetrized Dirac action (in the absence of any gauge fields) is $OSp(2N_c|2N_c)$ [46] and it may be recast as the $\widehat{osp}(2N_c|2N_c)_1$ WZNW model. In particular, the random $su(N_c)$ gauge potential couples only to currents J^a (and $\bar{J}^a$) residing in the $\widehat{su}(N_c)_0$ subalgebra of the complete $\widehat{osp}(2N_c|2N_c)_1$ Kač–Moody algebra:

$$S^{\mathrm{SUSY}} = \widehat{osp}(2N_c|2N_c)_1 + \int d^2\xi\, (J^a A^a + \bar{J}^a \bar{A}^a) \tag{2.35}$$

In the special case $N_c = 2$, Bernard and LeClair have demonstrated that the Sugawara energy momentum tensor for $\widehat{osp}(2N_c|2N_c)_1$ may be decomposed into the sum of two commuting pieces pertaining to different symmetries [52]:

$$T_{\mathrm{osp}(4|4)_1} = T_{\mathrm{osp}(2|2)_{-2}} + T_{\mathrm{su}(2)_0} \tag{2.36}$$

Exploiting the decomposition (2.36) one may rewrite equation (2.35) in the *decoupled* form

$$S^{\mathrm{SUSY}} = \widehat{osp}(2|2)_{-2} + \widehat{su}(2)_0 + \int d^2\xi\, (J^a A^a + \bar{J}^a \bar{A}^a) \tag{2.37}$$

We note that the rôle of the decomposition (2.36) in this approach closely mirrors that of the decomposition (2.15) employed in the replica approach — both allow us to decouple the effects of the gauge potential disorder. Indeed, the latter two terms appearing in equation (2.37) are precisely those which appear in the replica approach as $N_f \to 0$. As was rigorously proven in references [27, 28], and utilized to our advantage in §2.4, this $su(2)$ sector becomes massive for the simple Gaussian distribution of the gauge fields given in equation (2.9); a perturbative renormalization group argument was given in reference [52] where the one loop beta function was calculated with the result that the coupling constant g_A flows to strong coupling.[1] That is to say, the low-energy effective theory governing the Q-field is the $\widehat{osp}(2|2)_{-2}$ WZNW model. Fortunately, the $\widehat{osp}(2|2)_k$ WZNW model has been discussed quite extensively in the work of Maassarani and Serban [53]. In particular the model undergoes a dramatic simplification at $k = -2$ [54] and we provide a rather extensive discussion of this model in Appendix 2.B. In this approach the Q-field may be represented as

$$Q = Q^{1,\bar{1}}, \quad Q^\dagger = Q^{4,\bar{4}} \tag{2.38}$$

[1] Although a formal proof of the gap generation exists only for the case when the disorder distribution is Gaussian, it appears reasonable to assume that the $su(2)$ sector remains massive for a broader choice of disorder distributions. At scales smaller than the gap, the effective action is given by the critical $\widehat{osp}(2|2)_{-2}$ WZNW model. The critical point is stable with respect to variations of the disorder: all such variations affect only the massive modes and hence do not generate relevant perturbations of the critical action.

where the $Q^{\alpha,\bar{\alpha}}$ are primary fields transforming in the $[0,1/2]$ representation of $osp(2|2)$ — see Appendix 2.B.

§2.6.1 Conformal Dimensions

In the supersymmetric approach to the random $su(2)$ Dirac fermion problem, the conformal dimension of the Q-field coincides with that of a primary field transforming in the fundamental $[0,1/2]$ representation of $\widehat{osp}(2|2)_{-2}$ — see equation (2.101) and set $k=-2$:

$$h_Q = \frac{1}{4-2(-2)} = \frac{1}{8} \tag{2.39}$$

This is in agreement with the replica result (2.17) and the strong disorder result (2.27) when $N_c = 2$.

§2.6.2 Correlation Functions of the Q-field

The four-point correlation function of the Q-matrix admits the $osp(2|2) \times osp(2|2)$ invariant decomposition — see equations (2.98), (2.99) and (2.102):

$$\langle Q^{\alpha_1,\bar{\alpha}_1}(1) Q^{\alpha_2,\bar{\alpha}_2}(2) Q^{\alpha_3,\bar{\alpha}_3}(3) Q^{\alpha_4,\bar{\alpha}_4}(4)\rangle = |z_{14}z_{23}|^{-1/2} \sum_{i,j=1}^{3} I_i^{\alpha} \bar{I}_j^{\bar{\alpha}} F_{ij}(z,\bar{z}) \tag{2.40}$$

where α denotes the ordered sequence of indices $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, which label the basis states of the four-dimensional representation of $osp(2|2)$, and where the invariant tensors I_1, I_2 and I_3 are defined in equations (2.103) together with similar equations for $\bar{I}_1$, $\bar{I}_2$ and $\bar{I}_3$. The anharmonic ratio z is defined as $z = z_{12}z_{34}/z_{14}z_{32}$ and similarly for $\bar{z}$. The functions $F_{ij}(z,\bar{z})$ are single-valued combinations of the solutions to the $\widehat{osp}(2|2)_{-2}$ Knizhnik–Zamolodchikov equations, which are summarized by equations (2.124) — (2.128). For example, focusing on the correlation function pertaining to the decomposition (2.38):

$$\langle Q^{1\bar{1}}(1) Q^{4\bar{4}}(2) Q^{4\bar{4}}(3) Q^{1\bar{1}}(4)\rangle \sim |\Upsilon|^2 \left[K(z)K(1-\bar{z}) + K(1-z)K(\bar{z})\right]. \tag{2.41}$$

where here $\Upsilon = [z_{14}z_{23}\, z(1-z)]^{-1/4}$ and $K(z)$ and $E(z)$ are the complete elliptic integrals. This is in agreement with both the replica result (2.19) and the strong disorder result (2.33) when the number of colours $N_c = 2$.

§2.7 Dense Polymers and Twist Operators

It turns out that the supersymmetric description provides an extremely economical and straightforward approach to the disordered Dirac fermion problem. The action of the $\widehat{osp}(2|2)_{-2}$ WZNW model may be represented as the direct sum of three simple theories [55, 61, 174]:

$$\widehat{osp}(2|2)_{-2} = \widehat{su}(2)_1 + \frac{1}{4\pi}\int d^2\xi\,(\partial_\mu\varphi)^2 + \int d^2\xi\,\epsilon_{ab}\,\partial_\mu\chi^a\partial^\mu\chi^b, \tag{2.42}$$

where φ is a non-compact bosonic field, χ^a is a two-component symplectic fermion [136], and ϵ_{ab} is a two-component antisymmetric tensor. As expected in this supersymmetric theory, the bosonic sector ($c = 2$) and the symplectic fermions ($c = -2$) together yield a total central charge of zero. We note that the presence of the $\widehat{su}(2)_1$ WZNW model in the decomposition (2.42) of the $\widehat{osp}(2|2)_{-2}$ WZNW model reflects an underlying $\widehat{su}(2)_{-k/2}$ Kač–Moody subalgebra residing in the $\widehat{osp}(2|2)_k$ algebra [164]. The decomposition (2.42) may be simplified even further by noting that the $\widehat{su}(2)_1$ WZNW model admits the following free field representation:

$$\widehat{su}(2)_1 = \frac{1}{4\pi}\int d^2\xi\,(\partial_\mu\phi)^2, \tag{2.43}$$

where ϕ is a compact free boson; note that our choice of normalization is for latter convenience in equation (2.50). That is to say, the $\widehat{osp}(2|2)_{-2}$ WZNW model may be represented as the sum of a compact bosonic field ϕ ($c = 1$) a non-compact bosonic field φ ($c = 1$) and a two-component symplectic fermion ($c = -2$):

$$\widehat{osp}(2|2)_{-2} = \int d^2\xi\,\left[\frac{1}{4\pi}(\partial_\mu\phi)^2 + \frac{1}{4\pi}(\partial_\mu\varphi)^2 + \epsilon_{ab}\,\partial_\mu\chi^a\partial^\mu\chi^b\right]. \tag{2.44}$$

In view of the decomposition (2.44) one anticipates a representation of the $\widehat{osp}(2|2)_{-2}$ primary fields in terms of the primary fields of the models appearing on the right-hand side of equation (2.44) — namely vertex operators $e^{\alpha\phi}$ and $e^{\beta\varphi}$ and the primary fields of the $c = -2$ theory. As we shall discover in §2.7.1 – §2.7.3 this is indeed possible. The non-unitary $c = -2$ minimal model has received a great deal of attention in recent years as a theory of dense polymers [62, 63], as a celebrated example of a logarithmic conformal field theory [64, 65, 116, 129, 136, 137], and as a conformal ghost system [175]. The structure of this theory is rather rich and is known to consist of several sectors. As we shall see below, the so-called $\mathbb{Z}_2$ twisted sector of the $c = -2$ theory will play a crucial rôle here; this sector consists of a scalar primary field μ of conformal dimension $-1/8$ and tensor primary fields $\nu^{\pm}$ of conformal dimension $3/8$. The correlation functions and operator product expansions of these fields have been studied in reference [62] and more fully in reference [129]. In §2.7.1 we shall recast the four-point correlation functions of the $\widehat{osp}(2|2)_{-2}$ WZNW in a form which facilitates the elucidation of the desired operator correspondence. In §2.7.2 we shall discuss the $\mathbb{Z}_2$ twist operator correlation functions of the $c=-2$ model [129]. In §2.7.3 we shall compare the correlation functions (summarized in Table 2.1 and Table 2.2 of §2.7.1 and §2.7.2 respectively) and arrive at the aforementioned correspondence.

§2.7.1 The $\widehat{osp}(2|2)_{-2}$ WZNW model

In view of the large number of components of the generic $\widehat{osp}(2|2)_{-2}$ non-chiral four-point correlation function and the rather cumbersome and opaque invariant tensors (2.103) we shall refashion the correlator somewhat.[2] In particular the $c=-2$ operator correspondence will follow quite naturally. We study the four-point correlation function of the supersymmetric Q-matrix:

$$\mathcal{F}^{\alpha,\bar{\alpha}}(z_i,\bar{z}_i)=\langle Q^{\alpha_1,\bar{\alpha}_1}(z_1,\bar{z}_1)Q^{\alpha_2,\bar{\alpha}_2}(z_2,\bar{z}_2)Q^{\alpha_1,\bar{\alpha}_3}(z_3,\bar{z}_3)Q^{\alpha_4,\bar{\alpha}_4}(z_4,\bar{z}_4)\rangle, \tag{2.45}$$

where on the left hand side we use the symbol α to denote the ordered sequence of indices $\alpha_1,\alpha_2,\alpha_3,\alpha_4$. The indices α_i assume the values $1,2,3,4$ and label the basis states of the four-dimensional $[0,1/2]$ representation of $osp(2|2)$; in the notation of [53] states 1 and 4 are bosonic, whilst states 2 and 3 are fermionic. As may be seen from our more detailed studies in Appendix 2.B, this correlation function may be written in the (off-diagonal) form

$$\mathcal{F}^{\alpha,\bar{\alpha}}(z_i,\bar{z}_i)=-\mathcal{F}^{\alpha,(1)}(z_i)\mathcal{F}^{\bar{\alpha},(2)}(\bar{z}_i)-\mathcal{F}^{\alpha,(2)}(z_i)\mathcal{F}^{\bar{\alpha},(1)}(\bar{z}_i), \tag{2.46}$$

in which we have set the overall normalization of the four-point function to minus unity for latter convenience, and where

$$\mathcal{F}^{\alpha,(a)}(z_i)\equiv|z_{14}z_{23}|^{-1/4}\sum_{i=1}^{3}I_i^{\alpha}F_i^{(a)}(z). \tag{2.47}$$

Using the explicit form of the chiral blocks $F_i^{(a)}(z)$ appearing in equations (2.118), (2.121) and (2.122) together with the known transformation properties of the invariant tensors — see equations (2.143) and (2.145) — we deduce that

$$\mathcal{F}^{\alpha,(2)}(z_i)=-\tilde{\mathcal{P}}\mathcal{F}^{\tilde{\alpha},(1)}(\tilde{z}_i). \tag{2.48}$$

[2] Restricting our attention to four-point correlation functions pertaining to the four-dimensional $[0,1/2]$ representation we have a total of 4^8 components. Since the vast majority of these vanish however, and many are related by symmetry, it is desirable to distill them further.

Sector	α				$\mathcal{F}^{\alpha,(1)}(z_i)$
Bosonic	1144	4411			$(4\epsilon\gamma)^{-1}\,\Upsilon\,zK(z)$
	1414	4141			$(4\epsilon\gamma)^{-1}\,\Upsilon\,(1-z)K(z)$
	1441	4114			$-(4\epsilon\gamma)^{-1}\,\Upsilon\,K(z)$
Fermionic	2233	3322			$(4\epsilon\gamma)\,\Upsilon\,[2E(z)-(2-z)K(z)]$
	2323	3232			$(4\epsilon\gamma)\,\Upsilon\,[2E(z)-(1-z)K(z)]$
	2332	3223			$(4\epsilon\gamma)\,\Upsilon\,[2E(z)-K(z)]$
Mixed	1234	1324	2143	3142	$\pm\,\Upsilon\,[E(z)-(1-z)K(z)]$
	2413	3412	4231	4321	
	1243	1342	2134	3124	$\pm\,\Upsilon\,[K(z)-E(z)]$
	2431	3421	4213	4312	
	1423	1432	2314	3214	$\pm\,\Upsilon\,E(z)$
	2341	3241	4123	4132	

Table 2.1: Chiral correlation functions of the $\widehat{osp}(2|2)_{-2}$ WZNW model displaying regular behaviour in the vicinity of $z=0$ and logarithmic behaviour in the vicinity of $z=1$; $\Upsilon = [z_{14}z_{23}\,z(1-z)]^{-1/4}$ and $z = z_{12}z_{34}/z_{14}z_{32}$ — see Appendix 2.B for further details.

We use the tilde to denote the interchange of the coordinates or indices 2 and 3, and $\tilde{\mathcal{P}}$ to denote the fermionic parity of this permutation; note that this permutation induces the transformation $z \to 1-z$ in these functions. Substituting equation (2.48) into equation (2.46) we obtain the result

$$\mathcal{F}^{\alpha,\bar{\alpha}}(z_i,\bar{z}_i) = \tilde{\bar{\mathcal{P}}}\mathcal{F}^{\alpha,(1)}(z_i)\mathcal{F}^{\tilde{\bar{\alpha}},(1)}(\tilde{\bar{z}}_i) + \tilde{\mathcal{P}}\mathcal{F}^{\tilde{\alpha},(1)}(\tilde{z}_i)\mathcal{F}^{\bar{\alpha},(1)}(\bar{z}_i). \tag{2.49}$$

That is to say, one may build the full quota of single-valued and crossing symmetric non-chiral correlation functions of the $\widehat{osp}(2|2)_{-2}$ WZNW from our knowledge of the chiral functions $\mathcal{F}^{\alpha,(1)}(z_i)$ which display regular behaviour in the vicinity of $z=0$ and logarithmic behaviour in the vicinity of $z=1$. Using the explicit results for the chiral blocks (2.118), (2.121) and (2.122) together with the invariant tensors (2.103) we are able to gather these non-trivial functions in Table 2.1. We emphasize that the functions $\mathcal{F}^{\alpha,(1)}(z_i)$ are not simply the conformal blocks of the $\widehat{osp}(2|2)_{-2}$ WZNW model, but also embody the non-trivial tensorial structure of $osp(2|2)$ — see equation (2.47).

Sector	Correlator	Regular contribution
μ	$\langle\mu\mu\mu\mu\rangle$	$\Upsilon^{-1}\,K(z)$
ν	$\langle\nu^{\pm}\nu^{\pm}\nu^{\mp}\nu^{\mp}\rangle$	$\Upsilon^3\,z\,[2E(z)-(2-z)K(z)]$
	$\langle\nu^{\pm}\nu^{\mp}\nu^{\pm}\nu^{\mp}\rangle$	$\Upsilon^3\,(1-z)\;[2E(z)-(1-z)K(z)]$
	$\langle\nu^{\pm}\nu^{\mp}\nu^{\mp}\nu^{\pm}\rangle$	$\Upsilon^3\;[2E(z)-K(z)]$
Mixed	$\langle\mu\nu^{\pm}\nu^{\mp}\mu\rangle\times z_{14}^{-1/2}z_{23}^{+1/2}$ $\langle\nu^{\pm}\mu\mu\nu^{\mp}\rangle\times z_{14}^{+1/2}z_{23}^{-1/2}$	$\pm\,\Upsilon\,[E(z)-(1-z)K(z)]$
	$\langle\mu\nu^{\pm}\mu\nu^{\mp}\rangle\times z_{13}^{-1/2}z_{24}^{+1/2}$ $\langle\nu^{\pm}\mu\nu^{\mp}\mu\rangle\times z_{13}^{+1/2}z_{24}^{-1/2}$	$\pm\Upsilon\,[K(z)-E(z)]$
	$\langle\mu\mu\nu^{\pm}\nu^{\mp}\rangle\times z_{12}^{-1/2}z_{34}^{+1/2}$ $\langle\nu^{\pm}\nu^{\mp}\mu\mu\rangle\times z_{12}^{+1/2}z_{34}^{-1/2}$	$\pm\,\Upsilon\,E(z)$

Table 2.2: Chiral correlation functions of the $\mathbb{Z}_2$ twist operators of $c=-2$ displaying regular behaviour in the vicinity of $z=0$ and logarithmic behaviour in the vicinity of $z=1$; $\Upsilon=[z_{14}z_{23}\,z(1-z)]^{-1/4}$ and $z=z_{12}z_{34}/z_{14}z_{32}$ — see Gaberdiel and Kausch for further details [129] and note that our definition of the anharmonic ratio differs from theirs.

§2.7.2 $\mathbb{Z}_2$ Twist Operator Correlation Functions

The correlation functions of the twist operators μ and ν in the non-unitary $c=-2$ minimal model have been extensively studied by Gaberdiel and Kausch [129]. Their non-chiral four-point functions are built from linear combinations of their chiral counterparts so as to respect the stringent constraints of single-valuedness and crossing symmetry. As may be seen from §4 of their work, the contributions from either chiral sector are typically composed of two elliptic integral solutions posessing logarithmic branch cuts; one of these solutions displays regular behaviour in the vicinity of $z=0$ and logarithmic behaviour in the vicinity of $z=1$, whilst the other displays regular behaviour in the vicinity of $z=1$ and logarithmic behaviour in the vicinity of $z=0$. As in §2.7.1, in order to establish our correspondence (2.50) we find it convenient to focus on the chiral contributions to the four-point functions which exhibit regular behaviour in the vicinity of $z=0$. The chiral four-point functions most relevant to our discussion and displaying such regular behaviour in the vicinity of the origin are summarized in table 2.2.

§2.7.3 $\mathbb{Z}_2$ Twist Operator Correspondence

Comparing our correlation functions for the $\widehat{osp}(2|2)_{-2}$ WZNW model (summarized in Table 2.1) with those of Gaberdiel and Kausch for the $\mathbb{Z}_2$ twist fields of the $c = -2$ minimal model [129] (summarized in Table 2.2) we see a clear correspondence at the level of their elliptic integrals. In particular we see that the bosonic and fermionic states of the $\widehat{osp}(2|2)_{-2}$ WZNW model are naturally associated with the $\mathbb{Z}_2$ twist fields μ and ν_α respectively. The remaining powers of z_{ij} which distinguish Table 2.1 from Table 2.2 are provided by bosonic vertex operators as suggested by the decomposition (2.44). Explicitly, we establish the following equivalence between the chiral operators of the $\widehat{osp}(2|2)_{-2}$ WZNW model and the 'dressed' chiral operators of the $c = -2$ minimal model:[3]

$$Q_1 \sim \mathcal{A}^{-1}\, e^{i\phi}\, \mu, \quad Q_2 \sim \mathcal{A}\, e^{-\varphi}\, \nu^-, \quad Q_3 \sim \mathcal{A}\, e^{\varphi}\, \nu^+, \quad Q_4 \sim \mathcal{A}^{-1}\, e^{-i\phi}\, \mu, \tag{2.50}$$

where $\mathcal{A} = (4\epsilon\gamma)^{1/4}$ is a free parameter corresponding to the arbitrary relative normalizations of the $su(2)$ doublet (Q_1, Q_4) and the two singlets (Q_2, Q_3) in the four-dimensional representation of $osp(2|2)$ [53, 176]. With the normalization chosen in equation (2.44) the compact bosonic exponents $e^{\pm i\phi}$ have conformal dimension $h_\phi = 1/4$, and the non-compact bosonic exponents $e^{\pm\varphi}$ have $h_\varphi = -1/4$. It is thus straightforward to see that the dimensions add up to the correct value of $1/8$ — equation (2.101) — in the decomposition (2.50).

We have now highlighted the important rôle played by the $c = -2$ model in the chiral structure of the $\widehat{osp}(2|2)_{-2}$ WZNW model, and have provided a prescription for constructing the non-chiral correlation functions (2.49). Although there are many interesting and notable exceptions, we emphasize that in general, the non-chiral correlation functions of the $\widehat{osp}(2|2)_{-2}$ WZNW are *not* simply related to *single* non-chiral correlation functions of the $\mathbb{Z}_2$ twist operators dressed by non-chiral bosons. For example, it follows from our solution of the $\widehat{osp}(2|2)_{-2}$ Knizhnik–Zamolodchikov equations that

$$\langle Q^{2,\bar{2}}(1) Q^{3,\bar{3}}(2) Q^{2,\bar{2}}(3) Q^{3,\bar{3}}(4)\rangle = \mathcal{C}\{[(1-z)K - 2E][(1+\bar{z})\tilde{\bar{K}} - 2\tilde{\bar{E}}] + c.c.\}, \tag{2.51}$$

where $\mathcal{C} = (16\epsilon^2\gamma^2)^2\, |z_{14}z_{23}|^{-1/2}\, \Lambda$, and Λ is given by equation (2.128). We note that this is *not* simply related to a *single* correlation function of the non-chiral operators $\mu(z,\bar{z})$ and $\nu_{\alpha,\bar{\alpha}}(z,\bar{z})$. That is to say, no *single* non-chiral four-point function of the $\mathbb{Z}_2$ twist operators gives rise to this particular single-valued combination of the elliptic integrals. We invite the reader to verify this statement with the aid of §4 of Gaberdiel and

[3] We use the (overused and rather abused) symbol $\sim$ to emphasize that we have derived the correspondence from the chiral regular solutions and that it is valid up to phase.

Kausch [129]. It may be interesting to study the non-chiral aspects of this correspondence in greater detail.

§2.7.4 $\mathbb{Z}_N$ Twist Operators

In this section we discuss an extension of the twist operator correspondence to the general $su(N_c)$ problem. Evidence for such a correspondence may be seen in the replica and strong disorder approaches. As may be seen from equation (2.19), the replica diagonal four-point correlation function of the Q-matrix reads:

$$\langle Q_{1\bar{1}}(1)Q^\dagger_{1\bar{1}}(2)Q^\dagger_{1\bar{1}}(3)Q_{1\bar{1}}(4)\rangle \sim \\ |z_{14}z_{23}\, z(1-z)|^{-2/N^2}\left[K_N(z)K_N(1-\bar{z})+K_N(1-z)K_N(\bar{z})\right], \tag{2.52}$$

where for simplicity we have set $N_c = N$. We invite the reader to compare this four-point function with that of the $\mathbb{Z}_N$ twist operators of the $c = -2$ LCFT as discussed in the context of dense polymers — see equations (83) and (84) of reference [62], note the small typing error in the prefactor, and further note our different definition of the anharmonic ratio:

$$\langle \sigma_+(1)\sigma_-(2)\sigma_-(3)\sigma_+(4)\rangle \sim \\ |z_{14}z_{23}\, z(1-z)|^{2k/N(1-k/N)}\left[F_{k/N}(z)F_{k/N}(1-\bar{z})+F_{k/N}(1-z)F_{k/N}(\bar{z})\right]. \tag{2.53}$$

Here σ_+ stands for the $\mathbb{Z}_N$ twist operator $\sigma_{k/N}$, σ_- stands for the $\mathbb{Z}_N$ anti-twist operator, both of conformal dimension

$$h = -\frac{1}{2}\frac{k}{N}\left(1-\frac{k}{N}\right) \tag{2.54}$$

and

$$F_{k/N} = {}_2F_1[1-\tfrac{k}{N}, \tfrac{k}{N}; 1; z]. \tag{2.55}$$

For more details we refer the reader to §3.2 of [62]. In particular setting $k = 1$ in equations (2.53) and (2.55) one obtains

$$\langle \sigma_+(1)\sigma_-(2)\sigma_-(3)\sigma_+(4)\rangle \sim \\ |z_{14}z_{23}\, z(1-z)|^{2/N(1-1/N)}[K_N(z)K_N(1-\bar{z})+K_N(\bar{z})K_N(1-z)]. \tag{2.56}$$

It is readily seen that equations (2.52) and (2.56) coincide up to a simple prefactor to which we now turn our attention. Adopting the conventions of §2.7 we consider the four-point correlation functions of the exponentials of free bosonic fields

$$\nu_\alpha = e^{i\alpha\phi}, \tag{2.57}$$

governed by the action

$$S = \frac{1}{4\pi}\int d^2\xi\, \partial_\mu\phi\partial^\mu\phi, \tag{2.58}$$

and of conformal dimension $h = \alpha^2/4$. The four-point function of such fields is

$$\langle \nu_\alpha(1)\nu_{-\alpha}(2)\nu_{-\alpha}(3)\nu_\alpha(4)\rangle = |z_{14}z_{23}\, z(1-z)|^{-\alpha^2}. \tag{2.59}$$

The necessary compensating prefactor between equations (2.52) and (2.56) is obtained by setting $\alpha^2 = 2/N$. That is to say, one obtains the following (heuristic) identification between the Q-matrix and the 'dressed' $\mathbb{Z}_N$ twist operator of $c = -2$:

$$Q_{1\bar{1}} \sim e^{i\phi\sqrt{\frac{2}{N}}}\,\sigma_+, \quad Q^\dagger_{1\bar{1}} \sim e^{-i\phi\sqrt{\frac{2}{N}}}\,\sigma_- \tag{2.60}$$

Indeed, the conformal dimensions of the bosonic exponent and the $\mathbb{Z}_N$ twist operator (with k=1) add up to

$$\frac{1}{2N} - \frac{1}{2}\frac{1}{N}\left(1 - \frac{1}{N}\right) = \frac{1}{2N^2} \tag{2.61}$$

which is the known conformal dimension of the Q-matrix/LDoS [27] — see equations (2.17) and (2.27). That is to say we have established an explicit (although by no means exhaustive) connection between Dirac fermions in a random su(N) gauge field and $\mathbb{Z}_N$ twists in the $c = -2$ LCFT. This generalizes the 'dressing' of the $h = -1/8$ twist field of relevance in the $N = 2$ case [55, 61]. We note that we have not discussed the $N > 2$ analogue of 'dressing' the $h = 3/8$ twist field, but one anticipates a connection to the excited twist fields $\tau_{k/N}$ discussed in [62]. Moreover, the dressed $\mathbb{Z}_2$ twist operators are deemed responsible for the creation of half flux-quantum quasiholes in the bulk excitations of the Haldane–Rezayi Quantum Hall state — see section V of reference [66] — and one might expect a $\mathbb{Z}_N$ generalization to shed some light on the su(N) random gauge field problem.

§2.8 Convergence of Approaches

In §2.4–§2.6 we demonstrated that three different approaches to the random Dirac fermion problem yield identical results for the four-point correlation functions of the local density of states. In §2.7 we have further demonstrated that the supersymmetric approach inherits its non-trivial logarithmic structure from the $c = -2$ non-unitary min-

Approach	Fermions	Bosons	Active	Passive
Replicas	$N_c N_f$	0	$\widehat{u}(1) + \widehat{su}(N_f)_{N_c}$	$\widehat{su}(N_c)_{N_f}$
Strong	N_c	0	$\widehat{u}(1) + \widehat{su}(N_c)_1 + \widehat{su}(N_c)_{-2N_c}$	$(c = -2)^{N_c^2 - 1}$
SUSY	2	2	$\widehat{osp}(2\|2)_{-2}$	$\widehat{su}(2)_0$

Table 2.3: Current algebra approaches used to investigate the critical behaviour of the random non-Abelian Dirac fermion problem and the separation of the active (critical WZNW) degrees of freedom from the passive (massive or decoupled Jacobian) degrees of freedom. The central charges and the conformal dimensions may be seen to add up correctly in each approach.

imal model. In this section we shall discuss the convergence of the these approaches — summarized in table 2.3 — and the emergence of the $c = -2$ theory in a little more detail. Indeed, we shall point to a remarkable proliferation of $c = -2$ theories and effects. In table 2.3 we have highlighted the separation of the active (critical WZNW) degrees of freedom from the passive (massive or decoupled Jacobian) degrees of freedom in each approach.

An interesting aspect of this marriage of approaches is the agreement between the weak coupling (replica and supersymmetry) limit and the strong coupling limit. Although such an agreement might have been anticipated purely on physical grounds, its emergence is quite remarkable from a field-theoretic perspective. At the very least, it is quite surprising that such superficially different conformal field theories may nonetheless yield equivalent results, albeit for a specific subset of physically motivated correlation functions. In particular, the resultant central charge of the active degrees of freedom in the strong disorder approach differs from that in the replica and supersymmetric approaches, namely zero. As we have discussed in §2.5 — and indicate in table 2.3 — the active degrees of freedom in the strong disorder limit may be expressed as the sum of three different models:

$$S = \widehat{u}(1) + \widehat{su}(N_c)_1 + \widehat{su}(N_c)_{-2N_c}. \tag{2.62}$$

where the first two models describe N_c colours of free massless Dirac fermions, and the remaining model encodes the non-trivial logarithmic structure. We use the well known result for the central charge of the $\widehat{g}_k$ WZNW model [68]

$$c = \frac{k \dim \mathrm{g}}{k + g^{\vee}}, \tag{2.63}$$

where $g^\vee$ is the dual Coxeter number of the algebra g (equal to N for $su(N)$) together with the fact that a free boson $\widehat{u}(1)$ has central charge 1, to find the resultant central charge of (2.62):

$$c = 1 + \frac{N_c^2 - 1}{1 + N_c} + \frac{-2N_c(N_c^2 - 1)}{-2N_c + N_c} = 2N_c^2 + N_c - 2 \neq 0. \tag{2.64}$$

That is to say, the active degrees of freedom in the strong disorder approach yield a positive central charge, in contrast to the replica and supersymmetric degrees of freedom which yield a net zero central charge.

As we have discussed in §2.7, the supersymmetric approach to the disordered Dirac fermion problem reveals a hidden substructure which is inherited from the $c = -2$ minimal model. In view of the convergence of approaches outlined in this paper it follows quite naturally that the active degrees of freedom in the replica and strong disorder treatments also inherit non-trivial traits from the $c = -2$ model. In addition to the rather natural proliferation of elliptic integrals in the conformal blocks presented in this paper, we note that a relation between the $\widehat{su}(2)_{-4}$ WZNW model (which arises in the strong disorder treatment with $N_c = 2$) and the $c = -2$ theory has already been discussed in the string theory literature [177]: the $\widehat{su}(2)_{-4}$ WZNW model is cohomologically equivalent to an $N = 4$ supersymmetric bosonic string with $c = -2$ matter.[4]

A curious by-product of these investigations is that the presence of the $c = -2$ theory may apparently be seen in the 'passive' degrees of freedom outlined in table 2.3. Indeed, a number of $c = -2$ models (arising as Jacobians) explicitly decouple in the strong disorder approach and play a passive (or spectator) rôle in regards the LDoS. A possible explanation for the closely related structure of the active and passive theories indicated in table 2.3 may arise from the requirement of the mutual cancellation of their logarithmic singularities [54]. In particular we anticipate a relationship between the $\widehat{su}(2)_0$ WZNW model and the $c = -2$ theory and a correspondence analogous to equation (2.50). We note that the four-point functions of the $\widehat{su}(2)_0$ model have been studied in references [117, 122] and assume a simple form in terms of the complete elliptic integrals [54]. The investigation of the $\widehat{su}(2)_0$ WZNW model will be continued in reference [178].

As we close this section we comment briefly on the stability of the critical point with respect to variations of the disorder distribution. In the replica and supersymmetric approaches the gauge potential disorder is coupled directly to the massive sector and

[4] We note that this matter + string theory admits a description in terms of the $c = -2$ model dressed by $c = 28$ Liouville theory and $c = -26$ string ghosts. We remind the interested reader that this (and other) Liouville LCFTs [42, 111, 112] emerged in a very natural way in the closely related problem of prelocalization in disordered conductors [42, 45].

it is natural to assume that the critical theory is protected from such variations. In the strong disorder approach, however, the disorder variations are coupled directly to the critical $\widehat{su}(N_c)_{-2N_c}$ subsector, and its stability is far from obvious. It is therefore interesting to study the stability properties of this critical theory. A simple perturbation of the critical theory that one might consider is the deformation of the WZNW action by its kinetic term. This operator has scaling dimension zero and is strongly relevant. However, at level $k = -2N$ (or more generally $-2g^\vee$) this operator commutes with all the Kač–Moody currents and therefore does not affect the correlation functions — see Appendix 6 of reference [12] and note the different sign conventions for the level k. In a more general framework, the WZNW models at level $k = -2g^\vee$ arise quite naturally in both two-dimensional ($c = 0$) topological field theories possessing a non-Abelian current algebra [179], and in the strong disorder treatment of arbitrary WZNW models coupled to random vector potentials [46]. The critical level arises from a BRST (Becchi–Rouet–Stora–Tyutin) [180, 181] symmetry nilpotency condition and is required for the coexistence of a Kač–Moody algebra symmetry and a topological algebra symmetry [179]. We emphasize that the level of the underlying Kač–Moody algebra, the existence of a topological algebra and the stability of our strongly disordered theory are therefore intimately related.

§2.9 Discussion

We briefly summarize here a number of our main results:

- We have solved the random non-Abelian Dirac fermion problem by means of three different non-perturbative procedures based on the replica approach, supersymmetry, and in the limit of strong disorder. We have demonstrated that this *ménage à trois* of approaches yields identical results (for the four-point correlation functions of the local density of states) in this relatively simple, but quite non-trivial model of disorder.

- We have emphasized the special rôle played by the level k of the Kač–Moody algebra. As we have seen both here and in reference [54] the level $k = -2$ is rather special for the $\widehat{osp}(2|2)_k$ WZNW model which arises in the supersymmetric treatment of the random Dirac fermion problem: an entire conformal block decouples from the spectrum and the resulting theory is drastically simplified.

- We have established that the $c = -2$ minimal model plays an important rôle in the random non-Abelian Dirac fermion problem. We have found a rather simple and suggestive form for the supersymmetric critical action (2.44):

$$\widehat{osp}(2|2)_{-2} = u(1) + gl(1) + [c = -2].$$

We have highlighted the relevance of the twist operators of the $c = -2$ theory in the disorder averaged correlation functions of the Dirac fermion problem. The $c = -2$ symplectic fermions are ubiquitous in the construction of supersymmetric sigma models [50, 51]; this example indicates that this sector may well be responsible for many remarkable features of disordered critical points including the presence of logarithmic operators [64]. We draw attention to the fact that the $c = -2$ theory emerges in the theoretical description of dense polymers [62, 63]. We hope that this connection may ultimately yield a more intuitive picture of the random non-Abelian Dirac fermion problem and other disordered critical points.

- We have argued for the stability of the critical point with respect to variations in the disorder, and noted that the level of the underlying Kač–Moody algebra, the existence of a topological algebra and the stability of our strongly disordered theory are rather intimately linked. It is an interesting open question whether such considerations may yield Kač–Moody level selection mechanisms in models displaying lines of critical points [11].

Appendices to Chapter 2

2.A The $\widehat{su}(N)_k$WZNW model

In this appendix we consider the appearance of logarithms in the correlation functions of the $\widehat{su}(N)_k$ WZNW model. Following §4 of Knizhnik and Zamolodchikov [68] we compute the four-point functions

$$\mathcal{F}^{\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}}}(z_i,\bar{z}_i) = \langle g(z_1,\bar{z}_1)g^\dagger(z_2,\bar{z}_2)g^\dagger(z_3,\bar{z}_3)g(z_4,\bar{z}_4)\rangle, \tag{2.65}$$

of the field $g(z_i,\bar{z}_i) = g^{\alpha_i,\bar{\alpha}_i}(z_i,\bar{z}_i)$ transforming in the fundamental representation of SU(N) $\times$ SU(N). We use the symbol $\boldsymbol{\alpha}$ to denote the ordered sequence of indices $\alpha_1,\alpha_2,\alpha_3,\alpha_4$. Global conformal invariance restricts this correlation function to have the form

$$\mathcal{F}^{\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}}}(z_i,\bar{z}_i) = (z_{14}z_{23}\bar{z}_{14}\bar{z}_{23})^{-2h}F^{\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}}}(z,\bar{z}) \tag{2.66}$$

where z and $\bar{z}$ are the anharmonic ratios

$$z = \frac{z_{12}z_{34}}{z_{14}z_{32}}, \quad \bar{z} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{14}\bar{z}_{32}} \tag{2.67}$$

and the conformal dimension h of the field g is

$$h = \frac{N^2-1}{2N(N+k)} \tag{2.68}$$

The correlation function (2.66) admits the su$(N)\times$su(N) invariant decomposition

$$F^{\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}}}(z,\bar{z}) = \sum_{ij=1}^{2} I_i^{\boldsymbol{\alpha}}\bar{I}_j^{\bar{\boldsymbol{\alpha}}}F_{ij}(z,\bar{z}) \tag{2.69}$$

where the invariant tensors I_1 and I_2 are defined as

$$I_1^{\boldsymbol{\alpha}} = \delta^{\alpha_1,\alpha_2}\delta^{\alpha_3,\alpha_4}, \quad I_2^{\boldsymbol{\alpha}} = \delta^{\alpha_1,\alpha_3}\delta^{\alpha_2,\alpha_4} \tag{2.70}$$

with similar equations for $\bar{I}_1$ and $\bar{I}_2$. The four scalar functions F_{ij} satisfy the coupled first-order differential equations [68]

$$\frac{dF}{dz} = \left[\frac{1}{z}P + \frac{1}{z-1}Q\right]F, \quad \text{where} \quad F = \begin{pmatrix} F_{1j} \\ F_{2j} \end{pmatrix} \quad \forall j \tag{2.71}$$

where the matrices P and Q are given by

$$P = -\frac{1}{N(N+k)}\begin{pmatrix} N^2-1 & N \\ 0 & -1 \end{pmatrix}, \quad Q = -\frac{1}{N(N+k)}\begin{pmatrix} -1 & 0 \\ N & N^2-1 \end{pmatrix} \tag{2.72}$$

There are similar equations for the antiholomorphic dependence. Suppressing the antiholomorphic index j from the functions F_{1j} and F_{2j}, one may obtain the second-order differential equation satisfied by $F_1(z)$:

$$\begin{aligned} &N^2(N+k)^2z^2(1-z)^2F_1''(z)- \\ &\qquad N(N+k)z(1-z)\left[2-N(2N+k)+(3N^2+Nk-4)z\right]F_1'(z)+ \\ &\qquad\qquad \left[1-N^2-(N^4-6N^2-Nk+4)z+(N^4-5N^2+4)z^2\right]F_1(z)=0. \end{aligned} \tag{2.73}$$

The corresponding $F_2(z)$ may be obtained from the $F_1(z)$ solutions:

$$F_2(z) = -(N+k)zF_1'(z) + N^{-1}(1-z)^{-1}\left[1-N^2+(N^2-2)z\right]F_1(z). \tag{2.74}$$

Denoting the solutions to equations (2.73) and (2.74) as $F_1^{(a)}$ and $F_2^{(a)}$ respectively, the $F_{ij}(z,\bar{z})$ appearing in equation (2.69) may be written in the form

$$F_{ij}(z,\bar{z}) = \sum_{a,b=1}^{2} X_{ab}\, F_i^{(a)}(z)F_j^{(b)}(\bar{z}). \tag{2.75}$$

The coefficients X_{ab} are determined by single-valuedness and crossing symmetry. Upon the change of variables

$$F_1(z) = z^{-N/(N+k)}[z(1-z)]^{1/N(N+k)}G_1(z) \tag{2.76}$$

one obtains an equation of the hypergeometric form:

$$z(1-z)G_1'' + [\gamma - (\alpha+\beta+1)z]\,G_1' - \alpha\beta G_1 = 0. \tag{2.77}$$

where $\alpha = -1/(N+k)$, $\beta = 1/(N+k)$ and $\gamma = k/(N+k)$. For non-integer values of γ the solutions of these equations are given by Knizhnik and Zamolodchikov — see equations (4.10a) and (4.10b) of [68]. For integer values of γ, however, the solution involves *logarithms* — the roots of the indicial equation corresponding to (2.77) (namely 0 and $1-\gamma$) differ by an integer. Of particular importance in the study of the disordered Dirac fermion problem are the cases $N \to 0$ ($\gamma = 1$) appearing in the replica treatment

(§2.4) and $k = -2N$ ($\gamma = 2$) appearing in the strong disorder treatment (§2.5). We consider both of these cases below.

2.A.1 The $\widehat{u}(N)_k$ WZNW Model as $N \to 0$

As may be seen from equation (2.68) the conformal dimensions of the $\widehat{su}(N)_k$ WZNW model *diverge* as $N \to 0$, as do some of the prefactors in the chiral blocks (2.76); the $N \to 0$ limit of the $\widehat{su}(N)_k$ WZNW model *alone* fails to yield a well defined CFT. However, the replica approach to the random Dirac fermion problem instructs us to consider the $\widehat{u}(N)_k$ WZNW model. As we shall demonstrate below, the $\widehat{u}(N)_k$ WZNW model *does* have a well defined $N \to 0$ limit.

An arbitrary element u, of the group U(N), may be expressed as an element of SU(N), g, multiplied by a phase: $u = e^{i\alpha\varphi}g$. Using the Polyakov–Wiegmann identity for the WZNW model [89]

$$W_k[ab] = W_k[a] + W_k[b] + \frac{k}{2\pi}\int d^2\xi\, \mathrm{Tr}'(a^{-1}\bar{\partial}ab\partial b^{-1}) \tag{2.78}$$

with $a = e^{i\alpha\varphi}$ and $b = g$ one obtains[5]

$$W_k[u] = W_k[g] + \frac{\alpha^2 Nk}{8\pi}\int d^2\xi\, \partial_\mu\varphi\partial^\mu\varphi \tag{2.79}$$

The $\widehat{u}(N)_k$ WZNW model is therefore seen to be the $\widehat{su}(N)_k$ WZNW model augmented by a free scalar field of *prescribed* normalization. The conformal dimension of the field $e^{i\alpha\varphi}$ is *fixed* by this normalization to be[6] $h_\alpha = 1/(2Nk)$. The conformal dimension of the composite field $u = e^{i\alpha\varphi}g$ is thus

$$h_{\widehat{u}(N)_k} = \frac{1}{2Nk} + \frac{N^2-1}{2N(N+k)} \xrightarrow[N\to 0]{} \frac{1}{2k^2} \tag{2.80}$$

and is seen to have the finite replica limit $1/2k^2$. In addition, the (non-vanishing) holomorphic four-point correlation function of the U(1) fields $\mathcal{V}_\alpha = e^{i\alpha\varphi}$ is given by

$$\langle \mathcal{V}_\alpha(1)\mathcal{V}_{-\alpha}(2)\mathcal{V}_{-\alpha}(3)\mathcal{V}_\alpha(4)\rangle = (z_{14}z_{32})^{-1/Nk}[z(1-z)]^{-1/Nk} \tag{2.81}$$

The divergences occurring in the $\mathrm{SU(N)}_k$ results (2.66) and (2.76) as $N \to 0$ are seen to be compensated by those of the U(1) phase (2.81). In the replica ($N \to 0$) limit the

[5] We have used the facts that the current $g\partial g^{-1}$ residing in the su(N) *algebra* is traceless, thereby eliminating the second term in (2.78), that the WZNW term (2.13) vanishes for the U(1) element $e^{i\alpha\varphi}$, and that $\mathrm{Tr}' \equiv 2\mathrm{Tr}$.

[6] For a free scalar field governed by the action $S = \frac{1}{2}g\int d^2\xi\, \partial_\mu\varphi\partial^\mu\varphi$, the field $e^{i\alpha\varphi}$ has $h_\alpha = \alpha^2/8\pi g$ — see for example equations (5.73) and (6.60) of [23].

hypergeometric equation (2.77) reads

$$z(1-z)G_1'' + (1-z)G_1' + k^{-2}G_1 = 0 \tag{2.82}$$

and one is able to find the $N \to 0$ limit of the $\widehat{u}(N)_k$ conformal blocks:

$$\begin{aligned}
F_1^{(1)} &= \gamma \;{}_2F_1[-\tfrac{1}{k}, \tfrac{1}{k}; 1; z] && (2.83a)\\
F_1^{(2)} &= \gamma\,(1-z)\;{}_2F_1[1-\tfrac{1}{k}, 1+\tfrac{1}{k}; 2; 1-z] && (2.83b)\\
F_2^{(1)} &= \gamma\,\frac{z}{k}\;{}_2F_1[1-\tfrac{1}{k}, 1+\tfrac{1}{k}; 2; z] && (2.83c)\\
F_2^{(2)} &= \gamma\,k\;{}_2F_1[-\tfrac{1}{k}, \tfrac{1}{k}; 1; 1-z] && (2.83d)
\end{aligned}$$

where $\gamma = [z(1-z)]^{-1/k^2}$. We find it convenient to introduce generalizations of the complete elliptic integrals of the first and second kind [56]:

$$\begin{aligned}
K_k(z) &\equiv \frac{\pi}{k}\;{}_2F_1[1-\tfrac{1}{k}, \tfrac{1}{k}; 1; z] && (2.84a)\\
E_k(z) &\equiv \frac{\pi}{k}\;{}_2F_1[-\tfrac{1}{k}, \tfrac{1}{k}; 1; z] && (2.84b)
\end{aligned}$$

In terms of these functions (and rescaling) the $N \to 0$ $\widehat{u}(N)_k$ conformal blocks read:

$$\begin{aligned}
F_1^{(1)} &= \gamma\, E_k(z) && (2.85a)\\
F_1^{(2)} &= \gamma\, k\, [K_k(1-z) - E_k(1-z)] && (2.85b)\\
F_2^{(1)} &= \gamma\, [K_k(z) - E_k(z)] && (2.85c)\\
F_2^{(2)} &= \gamma\, k\, E_k(1-z) && (2.85d)
\end{aligned}$$

The full $U(N) \times U(N)$ invariant correlation function is built from single-valued combinations of these conformal blocks; it is straightforward to see that one can only have

$$F_{ij}(z,\bar{z}) = X_{12}\left[F_i^{(1)}(z)F_j^{(2)}(\bar{z}) + F_i^{(2)}(z)F_j^{(1)}(\bar{z})\right] \tag{2.86}$$

Substituting the explicit forms (2.85) into (2.86) one obtains

$$\begin{aligned}
F_{11} &= X_{12}\, k\, |\gamma|^2 \left[E_k(\tilde{\bar{K}}_k - \tilde{\bar{E}}_k) + \bar{E}_k(\tilde{K}_k - \tilde{E}_k)\right] && (2.87a)\\
F_{12} &= X_{12}\, k\, |\gamma|^2 \left[E_k\tilde{\bar{E}}_k + (\tilde{K}_k - \tilde{E}_k)(\bar{K}_k - \bar{E}_k)\right] && (2.87b)\\
F_{21} &= X_{12}\, k\, |\gamma|^2 \left[\bar{E}_k\tilde{E}_k + (\tilde{\bar{K}}_k - \tilde{\bar{E}}_k)(K_k - E_k)\right] && (2.87c)\\
F_{22} &= X_{12}\, k\, |\gamma|^2 \left[\tilde{E}_k(\bar{K}_k - \bar{E}_k) + \tilde{\bar{E}}_k(K_k - E_k)\right] && (2.87d)
\end{aligned}$$

where we have adopted the notation that $\tilde{f}(z) \equiv f(1-z)$, $\bar{f}(z) \equiv f(\bar{z})$ and $\tilde{\bar{f}}(z) \equiv f(1-\bar{z})$ for an arbitrary function f.

2.A.2 The $\widehat{su}(N)_{-2N}$ WZNW Model

As may be seen from equations (2.76) and (2.77) with $k = -2N$, the $F_1(z)$ chiral blocks for the $\widehat{su}(N)_{-2N}$ WZNW Model are given by

$$F_1(z) = z\,[z(1-z)]^{-1/N^2}\; G_1(z) \tag{2.88}$$

where G_1 satisfies the hypergeometric equation with $\alpha = 1/N$, $\beta = -1/N$, $\gamma = 2$:

$$z(1-z)G_1'' + (2-z)G_1' + N^{-2}G_1 = 0 \tag{2.89}$$

Solving this equation and applying the relation (2.74) enables one to obtain the full set of $\widehat{su}(N)_{-2N}$ chiral blocks:[7]

$$\begin{aligned}
F_1^{(1)} &= \Lambda\, z\; {}_2F_1[-\tfrac{1}{N}, \tfrac{1}{N}; 2; z] && \text{(2.90a)}\\
F_1^{(2)} &= \Lambda\, (1-z)^2\; {}_2F_1[1-\tfrac{1}{N}, 1+\tfrac{1}{N}; 3; 1-z] && \text{(2.90b)}\\
F_2^{(1)} &= -\frac{\Lambda z^2}{2N}\; {}_2F_1[1-\tfrac{1}{N}, 1+\tfrac{1}{N}; 3; z] && \text{(2.90c)}\\
F_2^{(2)} &= -2N\Lambda\,(1-z)\; {}_2F_1[-\tfrac{1}{N}, \tfrac{1}{N}; 2; 1-z] && \text{(2.90d)}
\end{aligned}$$

where $\Lambda = [z(1-z)]^{-1/N^2}$. In terms of generalized elliptic integrals the $\widehat{su}(N)_{-2N}$ chiral blocks read:

$$\begin{aligned}
F_1^{(1)} &= c\Lambda\,[[1+(N-1)z]\,E_N - (1-z)K_N] && \text{(2.91a)}\\
F_1^{(2)} &= 2Nc\Lambda\left[[1+(N-1)z]\,\tilde{E}_N - Nz\tilde{K}_N\right] && \text{(2.91b)}\\
F_2^{(1)} &= -c\Lambda\,[[N-(N-1)z]\,E_N - N(1-z)K_N] && \text{(2.91c)}\\
F_2^{(2)} &= -2Nc\Lambda\left[[N-(N-1)z]\,\tilde{E}_N - z\tilde{K}_N\right] && \text{(2.91d)}
\end{aligned}$$

where $c = N^2/[\pi(N^2-1)]$. We have used the Gauss recursion formula GR 9.137(13) [69]

$$\begin{aligned}
\gamma[\alpha-(\gamma-\beta)z]\; {}_2F_1[\alpha,\beta;\gamma;z] - \alpha\gamma(1-z)\; {}_2F_1[\alpha+1,\beta;\gamma;z]&\\
+\,(\gamma-\alpha)(\gamma-\beta)z\; {}_2F_1[\alpha,\beta;\gamma+1;z] = 0&
\end{aligned} \tag{2.92}$$

[7] We note that (2.90b) may be obtained by substituting $G_1(z) = z^{-1}(1-z)^2H_1(z)$ into (2.89) and replacing z by $1-z$ — this yields a hypergeometric equation. The result (2.90c) may be obtained by straightforward application of (2.74) to (2.90a). In deriving (2.90d) one may utilize the result $(N^2-1)z(1-z)\;{}_2F_1[2-\frac{1}{N}, 2+\frac{1}{N}; 4; z] + 3N^2(2-z)\;{}_2F_1[1-\frac{1}{N}, 1+\frac{1}{N}; 3; z] = 6N^2\;{}_2F_1[-\frac{1}{N}, \frac{1}{N}; 2; z]$ which is easily verified by means of the Gauss recursion relations.

with $\alpha = -1/N$, $\beta = 1/N$, $\gamma = 1$ to obtain (2.91a), and GR 9.137(6) [69]

$$\begin{aligned} &\gamma(\gamma+1)\ {}_2F_1[\alpha,\beta;\gamma;z] - \gamma(\gamma+1)\ {}_2F_1[\alpha,\beta;\gamma+1;z] \\ &\quad - \alpha\beta z\ {}_2F_1[\alpha+1,\beta+1;\gamma+2;z] = 0 \end{aligned} \tag{2.93}$$

with $\alpha = -1/N$, $\beta = 1/N$, $\gamma = 1$ to obtain (2.91b). One may obtain (2.91c) and (2.91d) by noting that $F_2^{(1)}(z) = -F_1^{(2)}(1-z)/2N$ and $F_2^{(2)}(z) = -2NF_1^{(1)}(1-z)$ as follows from (2.90). In particular we note that

$$N^2F_1^{(1)} + NF_2^{(1)} = c'\Lambda z E_N \tag{2.94a}$$
$$NF_1^{(1)} + N^2F_2^{(1)} = c'\Lambda(1-z)[K_N - E_N] \tag{2.94b}$$
$$N^2F_1^{(2)} + NF_2^{(2)} = -2Nc'\Lambda z[\tilde{K}_N - \tilde{E}_N] \tag{2.94c}$$
$$NF_1^{(2)} + N^2F_2^{(2)} = -2Nc'\Lambda(1-z)\tilde{E}_N \tag{2.94d}$$

where $c' \equiv cN(N^2-1)$ is simply another constant.

2.B The $\widehat{osp}(2|2)_k$WZNW model

2.B.1 Representations of the Lie Superalgebra $osp(2|2)$

The Lie superalgebra osp(2|2) $\sim$ spl(2|1) consists of four bosonic generators Q_3, Q_+, Q_-, B, and four fermionic generators W_+, W_-, V_+, V_-. The bosonic subalgebra is $sl(2) \oplus u(1)$:

$$[Q_3, Q_\pm] = \pm Q_\pm, \quad [Q_+, Q_-] = 2Q_3, \quad [B, Q_\pm] = 0, \quad [B, Q_3] = 0 \tag{2.95}$$

and the eigenvalues of Q_3 and B (called isospin and baryon number respectively) are used to classify the basis states of finite-dimensional representations [176, 182]. A representation $[b, q]$ contains at most four multiplets of states:

$$|b,q\rangle, \quad |b+\tfrac{1}{2}, q-\tfrac{1}{2}\rangle, \quad |b-\tfrac{1}{2}, q-\tfrac{1}{2}\rangle, \quad |b, q-1\rangle, \tag{2.96}$$

the states within a given multiplet $|b,q\rangle$ being labeled by their third component of isospin $|b,q,q_3\rangle$. In particular the $[0,\frac{1}{2}]$ representation is four-dimensional and contains a doublet and two singlets (the multiplet $|b,q-1\rangle$ being absent):

$$|0,\tfrac{1}{2},\tfrac{1}{2}\rangle, \quad |0,\tfrac{1}{2},-\tfrac{1}{2}\rangle, \quad |\tfrac{1}{2},0,0\rangle, \quad |-\tfrac{1}{2},0,0\rangle. \tag{2.97}$$

Following [53] we denote these states as $|1\rangle$, $|4\rangle$, $|3\rangle$ and $|2\rangle$ respectively. In addition to their isospin and baryon number, the basis states carry a grading denoted by $\varepsilon = \pm 1$. States $|1\rangle$ and $|4\rangle$ are even/bosonic and have $\varepsilon = 0$, whilst states $|2\rangle$ and $|3\rangle$ are odd/fermionic and have $\varepsilon = 1$. On this basis of states one may construct 4×4 matrix representation $[0, \frac{1}{2}]$ of the generators of osp(2|2).

2.B.2 Knizhhnik–Zamolodchikov Equations

Let us study the four-point function of the supersymmetric Q-matrix:

$$\mathcal{F}^{\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}}}(z_i,\bar{z}_i) = \langle Q^{\alpha_1,\bar{\alpha}_1}(z_1,\bar{z}_1)Q^{\alpha_2,\bar{\alpha}_2}(z_2,\bar{z}_2)Q^{\alpha_1,\bar{\alpha}_3}(z_3,\bar{z}_3)Q^{\alpha_4,\bar{\alpha}_4}(z_4,\bar{z}_4)\rangle, \tag{2.98}$$

where on the left hand side we use the symbol $\boldsymbol{\alpha}$ to denote the ordered sequence of indices $\alpha_1, \alpha_2, \alpha_3, \alpha_4$. Global conformal invariance restricts this correlation function to have the form

$$\mathcal{F}^{\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}}}(z_i,\bar{z}_i) = (z_{14}z_{23}\bar{z}_{14}\bar{z}_{23})^{-2h}F^{\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}}}(z,\bar{z}) \tag{2.99}$$

where z and $\bar{z}$ are the anharmonic ratios

$$z = \frac{z_{12}z_{34}}{z_{14}z_{32}}, \quad \bar{z} = \frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{14}\bar{z}_{32}} \tag{2.100}$$

and the conformal dimension h of the field ϕ is

$$h = \frac{1}{4-2k} \tag{2.101}$$

The correlation function (2.99) has the osp(2|2)×osp(2|2) invariant decomposition

$$F^{\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}}}(z,\bar{z}) = \sum_{ij=1}^{3} I_i^{\boldsymbol{\alpha}}\bar{I}_j^{\bar{\boldsymbol{\alpha}}}F_{ij}(z,\bar{z}) \tag{2.102}$$

The tensors I and $\bar{I}$ are given in appendix A of [53]:

$$\begin{aligned}
I_1 &= (1144) + (1234)4\epsilon\gamma + (1324)4\epsilon\gamma - (1414) + (2143)4\epsilon\gamma \\
&+ (2233)16\epsilon^2\gamma^2 + (2323)16\epsilon^2\gamma^2 - (2413)4\epsilon\gamma + (3142)4\epsilon\gamma \\
&+ (3232)16\epsilon^2\gamma^2 + (3322)16\epsilon^2\gamma^2 - (3412)4\epsilon\gamma - (4141) - (4231)4\epsilon\gamma \\
&- (4321)4\epsilon\gamma + (4411)
\end{aligned} \tag{2.103a}$$

$$\begin{aligned}
I_2 &= (1234)4\epsilon\gamma - (1243)4\epsilon\gamma + (1324)4\epsilon\gamma - (1342)4\epsilon\gamma \\
&- (1414) + (1441) - (2134)4\epsilon\gamma + (2143)4\epsilon\gamma + (2233)32\epsilon^2\gamma^2 + (2323)16\epsilon^2\gamma^2 \\
&+ (2332)16\epsilon^2\gamma^2 - (2413)4\epsilon\gamma + (2431)4\epsilon\gamma - (3124)4\epsilon\gamma + (3142)4\epsilon\gamma \\
&+ (3223)16\epsilon^2\gamma^2 + (3232)16\epsilon^2\gamma^2 + (3322)32\epsilon^2\gamma^2 - (3412)4\epsilon\gamma + (3421)4\epsilon\gamma \\
&+ (4114) - (4141) + (4213)4\epsilon\gamma - (4231)4\epsilon\gamma + (4312)4\epsilon\gamma - (4321)4\epsilon\gamma
\end{aligned} \tag{2.103b}$$

$$\begin{aligned}
I_3 &= (1234) - (1243) + (1324) - (1342) + (1423) + (1432) - (2134) \\
&+ (2143) + (2233)8\epsilon\gamma + (2314) + (2323)8\epsilon\gamma + (2332)8\epsilon\gamma - (2341) \\
&- (2413) + (2431) - (3124) + (3142) + (3214) + (3223)8\epsilon\gamma + (3232)8\epsilon\gamma \\
&- (3241) + (3322)8\epsilon\gamma - (3412) + (3421) - (4123) - (4132) + (4213) \\
&- (4231) + (4312) - (4321)
\end{aligned} \tag{2.103c}$$

and the nine scalar functions F_{ij} satisfy the coupled first-order differential equations [53]

$$\frac{dF}{dz} = \left[\frac{1}{z}P + \frac{1}{z-1}Q\right]F, \quad \text{where} \quad F = \begin{pmatrix} F_{1j} \\ F_{2j} \\ F_{3j} \end{pmatrix} \quad \forall j \tag{2.104}$$

and the matrices P and Q are given by,

$$P = \frac{1}{2-k}\begin{pmatrix} 1 & 0 & 0 \\ -2 & -3 & -\frac{1}{2\epsilon\gamma} \\ 4\epsilon\gamma & 8\epsilon\gamma & 1 \end{pmatrix}, \quad Q = \frac{1}{2-k}\begin{pmatrix} -1 & 0 & -\frac{1}{2\epsilon\gamma} \\ 2 & 1 & \frac{1}{2\epsilon\gamma} \\ -4\epsilon\gamma & -4\epsilon\gamma & -1 \end{pmatrix}. \tag{2.105}$$

There are similar equations for the antiholomorphic dependence. Suppressing the antiholomorphic index j from the functions $F_{1j} \cdots F_{3j}$, one may reduce this first-order matrix differential equation to the following set of equations [53]

$$x^3z^3(1-z)^3\frac{d^3F_3(z)}{dz^3}+x^2(1+2x)z^2(1-z)^2(1-2z)\frac{d^2F_3(z)}{dz^2}+$$
$$xz(1-z)[-1-x+2xz-2x(2+x)z(1-z)]\frac{dF_3(z)}{dz}+$$
$$[-1-x+2z+2xz(1-z)]F_3(z)=0 \tag{2.106a}$$

$$F_2(z)=-\frac{1}{4\epsilon\gamma xz(1-z)}\left[x^2D^2F_3(z)+2x(1-z)DF_3(z)+(1-2z)F_3(z)\right] \tag{2.106b}$$
$$F_1(z)=\frac{1}{4\epsilon\gamma}\left[xDF_3(z)-F_3(z)\right]+(z-2)F_2(z) \tag{2.106c}$$

where $D=z(1-z)d/dz$ and $x=2-k$. Equation (2.106a) has three independent solutions which we denote $F_3^{(a)}$ where $a=1,2,3$. Equations (2.106b) and (2.106c) yield the corresponding solutions $F_2^{(a)}$ and $F_1^{(a)}$. The nine scalar functions F_{ij} appearing in equation (2.102) may be expressed as a linear combination of these nine functions (the so-called chiral/current blocks.)

$$F_{ij}(z,\bar{z})=\sum_{a,b=1}^{3}X_{ab}F_i^{(a)}(z)F_j^{(b)}(\bar{z}) \tag{2.107}$$

The values of the coefficients X_{ab} are determined by single valuedness (monodromy invariance) and crossing symmetry to be discussed later.

Factorization at Level $k=1$

At level $k=1$ ($x=1$) the Knizhnik–Zamolodchikov equation (2.106a) takes the form

$$z^3(1-z)F_3^{'''}+3z^2(1-2z)F_3^{''}-2z(1+3z)F_3^{'}-2F_3=0 \tag{2.108}$$

This may be factorized to read

$$\mathcal{D}^{(2)}\mathcal{D}^{(1)}F_3=0 \tag{2.109}$$

where

$$\mathcal{D}^{(1)} = z\frac{d}{dz}+1 \tag{2.110}$$
$$\mathcal{D}^{(2)} = z^2(1-z)\frac{d^2}{dz^2}-3z^2\frac{d}{dz}-2 \tag{2.111}$$

It is straightforward to see that the equation $\mathcal{D}^{(1)}F_3 = 0$, and thus (2.109) has the solution

$$F_3^{(1)} = \frac{1}{z} \tag{2.112}$$

Applying the formulae (2.106b) and (2.106c) to this single solution of the Knizhnik–Zamolodchikov equation (2.109) one obtains:

$$F_2^{(1)} = \frac{2z-1}{4\epsilon\gamma z(1-z)}, \quad F_1^{(1)} = \frac{z-2}{4\epsilon\gamma(1-z)} \tag{2.113}$$

Analysis of the full set of solutions of (2.109) reveals that it is only these blocks which enter the physical correlator - a fact which is intimately related to the factorization of the Knizhhnik–Zamolodchikov equation (2.109). In the $su(2)$ WZNW model the factorization properties of the differential operators are related to a finite closure of the underlying operator algebra [183].

Factorization at Level $k = -2$

At level $k = -2$ ($x = 4$) the differential equation (2.106a) takes the form

$$\begin{aligned} &[4z(1-z)]^3 F_3''' + 9[4z(1-z)]^2(1-2z)F_3'' + \\ &\qquad 4z(1-z)[-5+8z-48z(1-z)]F_3' + \\ &\qquad\qquad [-5+2z+8z(1-z)]F_3 = 0 \end{aligned} \tag{2.114}$$

This equation may be factorized to read

$$\mathcal{D}^{(1)}\mathcal{D}^{(2)}F_3 = 0 \tag{2.115}$$

where this time the differential operators $\mathcal{D}^{(1)}$ and $\mathcal{D}^{(2)}$ take the form

$$\mathcal{D}^{(1)} = 4z(1-z)\frac{d}{dz} - (5-6z) \tag{2.116}$$

$$\mathcal{D}^{(2)} = [4z(1-z)]^2\frac{d^2}{dz^2} + 8z(1-z)(3-4z)\frac{d}{dz} + (1-4z) \tag{2.117}$$

In the light of the factorization which occurred at $k = 1$ one might expect that the subset of solutions obtained from $\mathcal{D}^{(2)}F_3 = 0$, which are closed under analytic continuation on the Riemann sphere, might serve as a reduced basis on which to perform the conformal bootstrap. This indeed turns out to be the case. It is easily seen that the equation $\mathcal{D}^{(2)}F_3 = 0$, and therefore the full Knizhnik–Zamolodchikov equation (2.115) admits the two solutions[8]

$$F_3^{(1)}(z) = \frac{E(z)}{[z(1-z)]^{1/4}} \qquad F_3^{(2)}(z) = \frac{E(1-z) - K(1-z)}{[z(1-z)]^{1/4}} \tag{2.118}$$

where $K(z)$ is the complete elliptic integral of the first kind and $E(z)$ is the complete elliptic integral of the second kind:[9]

$$E(z) = \int_0^1 \frac{\sqrt{1-zx^2}}{\sqrt{1-x^2}}dx \qquad K(z) = \int_0^1 \frac{1}{\sqrt{(1-x^2)(1-zx^2)}}dx \tag{2.119}$$

The representation (2.118) is particularly useful owing to the very simple manner in which the elliptic integrals behave under differentiation with respect to the parameter z:

$$\frac{dE(z)}{dz} = \frac{E(z) - K(z)}{2z} \qquad \frac{dK(z)}{dz} = \frac{E(z) - (1-z)K(z)}{2z(1-z)} \tag{2.120}$$

Applying (2.106b) and (2.106c) to these solutions yields:

$$F_2^{(1)} = -\frac{K(z)}{4\epsilon\gamma[z(1-z)]^{1/4}} \qquad F_1^{(1)} = \frac{zK(z)}{4\epsilon\gamma[z(1-z)]^{1/4}} \tag{2.121}$$

$$F_2^{(2)} = \frac{K(1-z)}{4\epsilon\gamma[z(1-z)]^{1/4}} \qquad F_1^{(2)} = -\frac{zK(1-z)}{4\epsilon\gamma[z(1-z)]^{1/4}} \tag{2.122}$$

As we shall subsequently demonstrate, one may satisfy the demands of single-valuedness and crossing symmetry on the subspace of functions (2.118), (2.121) and (2.122). Once again this is intimately connected with the factorization of the Knizhnik–Zamolodchikov equation (2.115). Having found closed form expressions for the chiral blocks, one must now construct the physical correlation functions to be single-valued on the whole Riemann sphere. It is enough to ensure this property at the two singular points $z = 0$ and $z = 1$. The blocks $F_i^{(1)}(z)$ are regular at $z = 0$ and logarithmic at $z = 1$, whilst the blocks $F_i^{(2)}(z)$ are logarithmic at $z = 0$ and regular at $z = 1$. It is straightforward to see that in this subspace of functions one can only have[10]

$$F_{ij}(z,\bar{z}) = X_{12}\left[F_i^{(1)}(z)F_j^{(2)}(\bar{z}) + F_i^{(2)}(z)F_j^{(1)}(\bar{z})\right] \tag{2.123}$$

[8]Upon the change of variables $F_3 = [z(1-z)]^{-1/4}H(z)$, the equation $\mathcal{D}^{(2)}F_3 = 0$ reduces to the canonical form of the hypergeometric equation $z(1-z)H'' + [c-(a+b+1)z]H' - abH = 0$ with $a = -1/2$, $b = 1/2$, $c = 1$. This has solutions $H^{(1)} = {}_2F_1\left[-\frac{1}{2},\frac{1}{2};1;z\right]$ and $H^{(2)} = (1-z)\,{}_2F_1\left[\frac{1}{2},\frac{3}{2};2;1-z\right]$. Using the well known fact that ${}_2F_1\left[-\frac{1}{2},\frac{1}{2};1;z\right] = \frac{2}{\pi}E(z)$ together with it's derivative, the result follows.

[9]Note that many texts on the theory of elliptic integrals denote the parameter z by k^2 - the so-called modulus. This is purely a mater of convention.

[10]A detailed proof of this statement in terms of monodromy matrices is given in appendix 2.B.3.

One may gather the explicit expressions for the F_{ij} at level $k = -2$ in the Hermitian matrix:

$$F_{ij} = \begin{pmatrix} |z|^2 F_{22} & -zF_{22} & -zF_{23} \\ -\bar{z}F_{22} & F_{22} & F_{23} \\ -\bar{z}\bar{F}_{23} & \bar{F}_{23} & F_{33} \end{pmatrix} \tag{2.124}$$

in which we have singled out the elements

$$F_{22} = -\Lambda \left[K\tilde{\bar{K}} + \bar{K}\tilde{K}\right] \tag{2.125}$$

$$F_{23} = 4\epsilon\gamma\Lambda \left[K\tilde{\bar{K}} + \tilde{K}\bar{E} - K\tilde{\bar{E}}\right] \tag{2.126}$$

$$F_{33} = (4\epsilon\gamma)^2\Lambda \left[E\tilde{\bar{E}} - E\tilde{\bar{K}} + \bar{E}\tilde{E} - \bar{E}\tilde{K}\right] \tag{2.127}$$

where

$$\Lambda = \frac{X_{12}}{(4\epsilon\gamma)^2|z(1-z)|^{1/2}} \tag{2.128}$$

and where we have adopted the notation that $\tilde{f}(z) \equiv f(1-z)$, $\bar{f}(z) \equiv f(\bar{z})$ and $\tilde{\bar{f}}(z) \equiv f(1-\bar{z})$ for an arbitrary function f. One may also demonstrate that this combination is consistent with the crossing symmetry constraints on the correlator:

$$F^{\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}}}(z,\bar{z}) = \tilde{\mathcal{P}}\tilde{\bar{\mathcal{P}}}F^{\tilde{\boldsymbol{\alpha}},\tilde{\bar{\boldsymbol{\alpha}}}}(1-z,1-\bar{z}), \tag{2.129a}$$

$$F^{\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}}}(z,\bar{z}) = z^{-2\Delta}\bar{z}^{-2\Delta}\hat{\mathcal{P}}\hat{\bar{\mathcal{P}}}F^{\hat{\boldsymbol{\alpha}},\hat{\bar{\boldsymbol{\alpha}}}}(1/z,1/\bar{z}) \tag{2.129b}$$

where $\tilde{\boldsymbol{\alpha}}$ denotes the permuted sequence of indices $\alpha_1, \alpha_3, \alpha_2, \alpha_4$, $\hat{\boldsymbol{\alpha}}$ denotes the sequence α_1,α_4, α_3,α_2, and $\mathcal{P}$ denotes the parity of the permutation:

$$\tilde{\mathcal{P}} = (-1)^{\varepsilon_{\alpha_2}\varepsilon_{\alpha_3}}, \qquad \hat{\mathcal{P}} = (-1)^{\varepsilon_{\alpha_2}(\varepsilon_{\alpha_3}+\varepsilon_{\alpha_4})+\varepsilon_{\alpha_3}\varepsilon_{\alpha_4}} \tag{2.130}$$

The proof of this requires the use of the analytic continuation formulae of the elliptic integrals and is presented in appendix 2.B.4.

2.B.3 Monodromy Invariance

A monodromy transformation of a function of z consists in letting z circulate around some other point (typically a singular point). We define

$$\mathcal{C}_0\, F(z,\bar{z}) \;=\; \lim_{t\to 1^-} F(ze^{2i\pi t}, \bar{z}e^{-2i\pi t}) \tag{2.131}$$

$$\mathcal{C}_1\, F(z,\bar{z}) \;=\; \lim_{t\to 1^-} F(1+(z-1)e^{2i\pi t}, 1+(\bar{z}-1)e^{-2i\pi t}) \tag{2.132}$$

Using the standard analytic continuation formulae for the hypergeometric series, it is easily seen that the elliptic integrals have the following nontrivial monodromy properties:

$$\mathcal{C}_0 K(1-z) \;=\; K(1-z) - 2iK(z) \tag{2.133}$$

$$\mathcal{C}_0 E(1-z) \;=\; E(1-z) + 2i[E(z) - K(z)] \tag{2.134}$$

$$\mathcal{C}_1 K(z) \;=\; K(z) - 2iK(1-z) \tag{2.135}$$

$$\mathcal{C}_1 E(z) \;=\; E(z) + 2i[E(1-z) - K(1-z)] \tag{2.136}$$

together with the trivial transformations $\mathcal{C}_0 K(z) = K(z)$, $\mathcal{C}_0 E(z) = E(z)$, $\mathcal{C}_1 K(1-z) = K(1-z)$, $\mathcal{C}_1 E(1-z) = E(1-z)$. Using these results it is straightforward to see that

$$\mathcal{C}_0\, F_i^{(a)}(z) \;=\; (g_0)_{ab}\, F_i^{(b)}(z), \quad i = 1,2,3 \tag{2.137}$$

$$\mathcal{C}_1\, F_i^{(a)}(z) \;=\; (g_1)_{ab}\, F_i^{(b)}(z), \quad i = 1,2,3 \tag{2.138}$$

where (on this reduced subspace) the matrices g_0 and g_1 are given by

$$g_0 = \begin{pmatrix} -i & 0 \\ 2 & -i \end{pmatrix}, \quad g_1 = \begin{pmatrix} -i & 2 \\ 0 & -i \end{pmatrix}. \tag{2.139}$$

Under the monodromy transformation $\mathcal{C}_0$, the combination

$$F_{ij}(z,\bar{z}) = \sum_{a,b=1}^{2} X_{ab} F_i^{(a)}(z) F_j^{(b)}(\bar{z}) \tag{2.140}$$

transforms in the following manner

$$\mathcal{C}_0 F_{ij}(z,\bar{z}) = F_i^{(1)}(z)F_j^{(1)}(\bar{z})\,[X_{11} - 2i(X_{12} - X_{21}) + 4X_{22}] + F_i^{(2)}(z)F_j^{(2)}(\bar{z})\,[X_{22}]$$
$$+\, F_i^{(1)}(z)F_j^{(2)}(\bar{z})\,[X_{12} + 2iX_{22}] + F_i^{(2)}(z)F_j^{(1)}(\bar{z})\,[X_{21} - 2iX_{22}]$$

Invariance under the monodromy transformation $\mathcal{C}_0$ thus requires $X_{12} = X_{21}$, and $X_{22} = 0$. That is to say

$$F_{ij}(z,\bar{z}) = X_{11}F_i^{(1)}(z)F_j^{(1)}(\bar{z}) + X_{12}\left[F_i^{(1)}(z)F_j^{(2)}(\bar{z}) + F_i^{(2)}(z)F_j^{(1)}(\bar{z})\right]$$

Under the monodromy transformation $\mathcal{C}_1$ this simplified function transforms as

$$\begin{aligned}\mathcal{C}_1 F_{ij}(z,\bar{z}) = {} & F_i^{(1)}(z)F_j^{(1)}(\bar{z})\,[X_{11}] + F_i^{(1)}(z)F_j^{(2)}(\bar{z})\,[X_{12} - 2iX_{11}] \\ & + F_i^{(2)}(z)F_j^{(1)}(\bar{z})\,[X_{12} + 2iX_{11}] + F_i^{(2)}(z)F_j^{(2)}(\bar{z})\,[4X_{11}]\end{aligned}$$

Invariance under the monodromy transformation $\mathcal{C}_1$ therefore imposes the additional constraint $X_{11} = 0$. Hence monodromy invariance restricts $F_{ij}(z,\bar{z})$ to have the form

$$F_{ij}(z,\bar{z}) = X_{12}\left[F_i^{(1)}(z)F_j^{(2)}(\bar{z}) + F_i^{(2)}(z)F_j^{(1)}(\bar{z})\right]. \tag{2.141}$$

as stated in the text.

2.B.4 Crossing Symmetry

Invariance under $z \to 1 - z$

Crossing symmetry requires that

$$F^{\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}}}(z,\bar{z}) = \tilde{\mathcal{P}}\tilde{\bar{\mathcal{P}}}F^{\tilde{\boldsymbol{\alpha}},\tilde{\bar{\boldsymbol{\alpha}}}}(1-z,1-\bar{z}) \tag{2.142}$$

where $\boldsymbol{\alpha}$ denotes the sequence of indices $\alpha_1,\alpha_2,\alpha_3,\alpha_4$, $\tilde{\boldsymbol{\alpha}}$ denotes the permuted sequence of indices $\alpha_1,\alpha_3,\alpha_2,\alpha_4$, $\tilde{\mathcal{P}} = (-1)^{\varepsilon_{\alpha_2}\varepsilon_{\alpha_3}}$ denotes the parity of the interchange in the holomorphic sector (ε_α is 0 for bosons and 1 for fermions.) Introducing the following tensor [53]

$$J_i^{\boldsymbol{\alpha}} = \tilde{\mathcal{P}}I_i^{\tilde{\boldsymbol{\alpha}}} \tag{2.143}$$

the crossing symmetry constraint may be written

$$\sum_{i,j=1}^{3} I_i^{\boldsymbol{\alpha}}\bar{I}_j^{\bar{\boldsymbol{\alpha}}}F_{ij}(z,\bar{z}) = \sum_{i,j=1}^{3} J_i^{\boldsymbol{\alpha}}\bar{J}_j^{\bar{\boldsymbol{\alpha}}}F_{ij}(1-z,1-\bar{z}) \tag{2.144}$$

The tensor J admits the following decomposition [53]

$$J_i^{\boldsymbol{\alpha}} = C_1^{ij}I_j^{\boldsymbol{\alpha}} \quad C_1 = \begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & -4\epsilon\gamma \\ 0 & 0 & -1 \end{pmatrix} \tag{2.145}$$

Substituting this decomposition into equation (2.144) and equating the coefficients of $I_i\bar{I}_j$ on both sides, one finds the following nine identities which must be satisfied by the $F_{ij}(z,\bar{z})$ if this crossing symmetry is to be satisfied. Denoting $F_{ij}(z,\bar{z})$ by F_{ij}, and $F_{ij}(1-z,1-\bar{z})$ by $\tilde{F}_{ij}$ these are as follows:

$$\begin{aligned}
F_{11} &= \tilde{F}_{11}+\tilde{F}_{12}+\tilde{F}_{21}+\tilde{F}_{22} && \text{(2.146a)}\\
F_{12} &= -\tilde{F}_{12}-\tilde{F}_{22} && \text{(2.146b)}\\
F_{13} &= 4\epsilon\gamma(\tilde{F}_{12}+\tilde{F}_{22})+\tilde{F}_{13}+\tilde{F}_{23} && \text{(2.146c)}\\
F_{21} &= -\tilde{F}_{21}-\tilde{F}_{22} && \text{(2.146d)}\\
F_{22} &= \tilde{F}_{22} && \text{(2.146e)}\\
F_{23} &= -4\epsilon\gamma\tilde{F}_{22}-\tilde{F}_{23} && \text{(2.146f)}\\
F_{31} &= 4\epsilon\gamma(\tilde{F}_{21}+\tilde{F}_{22})+\tilde{F}_{31}+\tilde{F}_{32} && \text{(2.146g)}\\
F_{32} &= -4\epsilon\gamma\tilde{F}_{22}-\tilde{F}_{32} && \text{(2.146h)}\\
F_{33} &= 4\epsilon\gamma(4\epsilon\gamma\tilde{F}_{22}+\tilde{F}_{23}+\tilde{F}_{32})+\tilde{F}_{33} && \text{(2.146i)}
\end{aligned}$$

It is straightforward to show that these relations are indeed satisfied.

Invariance under $z \to 1/z$

Crossing symmetry requires that

$$F^{\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}}}(z,\bar{z}) = z^{-2\Delta}\bar{z}^{-2\Delta}\hat{\mathcal{P}}\hat{\bar{\mathcal{P}}}F^{\hat{\boldsymbol{\alpha}},\hat{\bar{\boldsymbol{\alpha}}}}(1/z,1/\bar{z}) \tag{2.147}$$

where $\boldsymbol{\alpha}$ denotes the sequence of indices $\alpha_1,\alpha_2,\alpha_3,\alpha_4$, $\hat{\boldsymbol{\alpha}}$ denotes the permuted sequence of indices $\alpha_1,\alpha_4,\alpha_3,\alpha_2$, $\hat{\mathcal{P}} = (-1)^{\varepsilon_{\alpha_2}(\varepsilon_{\alpha_3}+\varepsilon_{\alpha_4})+\varepsilon_{\alpha_3}\varepsilon_{\alpha_4}}$ denotes the parity of the interchange in the holomorphic sector (ε_α is 0 for bosons and 1 for fermions.) Introducing the following tensor [53]

$$K_i^{\boldsymbol{\alpha}} = \hat{\mathcal{P}}I_i^{\hat{\boldsymbol{\alpha}}} \tag{2.148}$$

the crossing symmetry constraint may be written

$$\sum_{i,j=1}^{3} I_i^{\boldsymbol{\alpha}}\bar{I}_j^{\bar{\boldsymbol{\alpha}}}F_{ij}(z,\bar{z}) = z^{-1/x}\bar{z}^{-1/x}\sum_{i,j=1}^{3} K_i^{\boldsymbol{\alpha}}\bar{K}_j^{\bar{\boldsymbol{\alpha}}}F_{ij}(1/z,1/\bar{z}) \tag{2.149}$$

The tensor K admits the following decomposition [53]

$$K_i^{\boldsymbol{\alpha}} = C_2^{ij} I_j^{\boldsymbol{\alpha}} \quad C_2 = \begin{pmatrix} 0 & 1 & -4\epsilon\gamma \\ 1 & 0 & -4\epsilon\gamma \\ 0 & 0 & -1 \end{pmatrix} \tag{2.150}$$

Substituting this decomposition into equation (2.149) and equating the coefficients of $I_i \bar{I}_j$ on both sides, one finds the following nine identities which must be satisfied by the $F_{ij}(z,\bar{z})$ if this crossing symmetry is to be satisfied. Denoting $F_{ij}(z,\bar{z})$ by F_{ij}, and $F_{ij}(1/z, 1/\bar{z})$ by $\hat{F}_{ij}$ these are as follows:

$$\begin{aligned}
F_{11} &= |z|^{-2/x}\hat{F}_{22} && (2.151a)\\
F_{12} &= |z|^{-2/x}\hat{F}_{21} && (2.151b)\\
F_{13} &= |z|^{-2/x}[-4\epsilon\gamma(\hat{F}_{21}+\hat{F}_{22}) - \hat{F}_{23}] && (2.151c)\\
F_{21} &= |z|^{-2/x}\hat{F}_{12} && (2.151d)\\
F_{22} &= |z|^{-2/x}\hat{F}_{11} && (2.151e)\\
F_{23} &= |z|^{-2/x}[-4\epsilon\gamma(\hat{F}_{11}+\hat{F}_{12}) - \hat{F}_{13}] && (2.151f)\\
F_{31} &= |z|^{-2/x}[-4\epsilon\gamma(\hat{F}_{12}+\hat{F}_{22}) - \hat{F}_{32}] && (2.151g)\\
F_{32} &= |z|^{-2/x}[-4\epsilon\gamma(\hat{F}_{11}+\hat{F}_{21}) - \hat{F}_{31}] && (2.151h)\\
F_{33} &= |z|^{-2/x}[16\epsilon^2\gamma^2(\hat{F}_{11}+\hat{F}_{12}+\hat{F}_{21}+\hat{F}_{22}) + \\
&\qquad 4\epsilon\gamma(\hat{F}_{13}+\hat{F}_{23}+\hat{F}_{31}+\hat{F}_{32}) + \hat{F}_{33}] && (2.151i)
\end{aligned}$$

In order to demonstrate that these identities are satisfied we shall make use of the following rather simple transformation laws of the elliptic integrals under $z \to 1/z$:[11]

$$\begin{aligned}
K(1/z) &= z^{+1/2}\left[K(z) + iK(1-z)\right], \quad \Im m z < 0 && (2.152a)\\
E(1/z) &= z^{-1/2}\left[D(z) - iD(1-z)\right], \quad \Im m z < 0 && (2.152b)\\
K(1-1/z) &= z^{+1/2}K(1-z) && (2.152c)\\
E(1-1/z) &= z^{-1/2}E(1-z) && (2.152d)
\end{aligned}$$

where

$$D(z) = E(z) - (1-z)K(z). \tag{2.153}$$

[11] We note that replacing i by $-i$ in equations (2.152a) and (2.152b) changes the domain of validity of the transformation from $\Im m z < 0$ to $\Im m z > 0$. Since E and K appear together in the F_{ij} it is essential that their transformations be defined in the same region of the complex plane.

These are easily obtained using the standard analytic continuation formulae for ordinary hypergeometric functions. Recalling the form of F_{22} appearing in equation (2.125) namely

$$F_{22} = -\Lambda \left[K(1-z)K(\bar{z}) + K(z)K(1-\bar{z}) \right] \tag{2.154}$$

it is straightforward to see how it transforms under $z \to 1/z$:

$$\hat{F}_{22} = |z|^{5/2} F_{22} \tag{2.155}$$

Since $F_{11} = |z|^2 F_{22}$ from equation (2.124) it follows that that constraint (2.151a) is satisfied. Replacing now z by $1/z$ in (2.151a) one may infer the validity of (2.151e). Further, since $F_{21} = -\bar{z}F_{22}$ from equation (2.124) it follows by reciprocity that

$$\hat{F}_{21} = -\bar{z}^{-1}\hat{F}_{22}. \tag{2.156}$$

Substituting for $\hat{F}_{22}$ using (2.155) and using the fact that $F_{12} = -zF_{22}$ one obtains

$$\hat{F}_{21} = |z|^{1/2} F_{12} \tag{2.157}$$

That is to say, constraint (2.151b) is satisfied. By reciprocity we see that constraint (2.151d) is also satisfied. Recalling now the form of F_{23} appearing in equation (2.126) namely

$$F_{23} = 4\epsilon\gamma\Lambda \left[K\tilde{\bar{K}} + \tilde{K}\bar{E} - K\tilde{\bar{E}} \right] \tag{2.158}$$

it is straightforward to see how it transforms under $z \to 1/z$:

$$\begin{aligned}
\hat{F}_{23} &= 4\epsilon\gamma|z|^{3/2}\Lambda \Big[z^{1/2}\left(K + i\tilde{K}\right)\bar{z}^{1/2}\tilde{\bar{K}} + \\
&\qquad z^{1/2}\tilde{K}\bar{z}^{-1/2}\left(\bar{D} + i\tilde{\bar{D}}\right) - z^{1/2}\left(K + i\tilde{K}\right)\bar{z}^{-1/2}\tilde{\bar{E}} \Big] \\
&= 4\epsilon\gamma|z|^{1/2}\Lambda \Big[|z|^2\left(K + i\tilde{K}\right)\tilde{\bar{K}} + z\tilde{K}\left(\bar{E} - (1-\bar{z})\bar{K}\right) \\
&\qquad + iz\tilde{K}\left(\tilde{\bar{E}} - \bar{z}\tilde{\bar{K}}\right) - z\left(K + i\tilde{K}\right)\tilde{\bar{E}} \Big] \\
&= 4\epsilon\gamma|z|^{1/2}\Lambda \left[|z|^2\left(K\tilde{\bar{K}} + \tilde{K}\bar{K}\right) + z\left(\tilde{K}\bar{E} - \tilde{K}\bar{K} - K\tilde{\bar{E}}\right) \right] \\
&= 4\epsilon\gamma|z|^{1/2}\Lambda \left[|z|^2\left(K\tilde{\bar{K}} + \tilde{K}\bar{K}\right) - z\left(K\tilde{\bar{K}} + \tilde{K}\bar{K}\right) + z\left(K\tilde{\bar{K}} + \tilde{K}\bar{E} - K\tilde{\bar{E}}\right) \right] \\
&= 4\epsilon\gamma|z|^{1/2}\left[-|z|^2F_{22} + zF_{22} + \frac{z}{4\epsilon\gamma}F_{23} \right] \\
&= |z|^{1/2}\left[-4\epsilon\gamma(F_{11} + F_{12}) - F_{13} \right]
\end{aligned}$$

Replacing z by $1/z$ on both sides of this equation one obtains the relation (2.151f). Taking now the complex conjugate of (2.151f) and using the Hermiticity property $\bar{F}_{ij} = F_{ji}$, one obtains (2.151h). Further, rearranging (2.159) for F_{13} and replacing F_{11} and F_{12} using (2.151a) and (2.151b) respectively one obtains (2.151c). Taking now the complex conjugate of (2.151c) and using Hermiticity, one obtains (2.151g). We consider finally how F_{33}, namely

$$F_{33} = (4\epsilon\gamma)^2\Lambda\left[E(\tilde{\bar{E}} - \tilde{\bar{K}}) + \text{c.c.}\right] \tag{2.159}$$

behaves under the transformation $z \to 1/z$:

$$\begin{aligned}
\hat{F}_{33} &= (4\epsilon\gamma)^2|z|^{3/2}\Lambda\left[z^{-1/2}(D - i\tilde{D})(\bar{z}^{-1/2}\tilde{\bar{E}} - \bar{z}^{1/2}\tilde{\bar{K}}) + c.c.\right]\\
&= (4\epsilon\gamma)^2|z|^{1/2}\Lambda\left[(D - i\tilde{D})(\tilde{\bar{E}} - \bar{z}\tilde{\bar{K}}) + c.c.\right]\\
&= (4\epsilon\gamma)^2|z|^{1/2}\Lambda\left[E\tilde{\bar{E}} - (1-z)K\tilde{\bar{E}} - z\bar{E}\tilde{K} + \bar{z}(1-z)K\tilde{\bar{K}} + c.c.\right]\\
&= (4\epsilon\gamma)^2|z|^{1/2}\Lambda\left[E\tilde{\bar{E}} - \bar{E}\tilde{K} + (1-z)(\bar{E}\tilde{K} - K\tilde{\bar{E}} + K\tilde{\bar{K}}) - |1-z|^2K\tilde{\bar{K}} + c.c.\right]\\
&= |z|^{1/2}\left[(4\epsilon\gamma)^2|1-z|^2F_{22} + 4\epsilon\gamma\left[(1-z)F_{23} + c.c.\right] + F_{33}\right]\\
&= |z|^{1/2}\left[(4\epsilon\gamma)^2(F_{11} + F_{12} + F_{21} + F_{22}) + 4\epsilon\gamma(F_{13} + F_{23} + F_{31} + F_{32}) + F_{33}\right]
\end{aligned}$$

which may be seen to be the relation (2.151i) with z replaced by $1/z$. This completes our proof of crossing symmetry in the reduced subspace.

2.B.5 Ancillary Differential Equation

As discussed earlier, the conformal blocks $F_3^{(a)}$ are obtained via differentiation of some ancillary functions $G_3^{(a)}$ which satisfy the equation

$$\begin{aligned}
&x^3z^3(1-z)^3G_3^{'''}(z) + x^2(3x+1)z^2(1-z)^2(1-2z)G_3^{''}(z) +\\
&\quad z(1-z)[x^2(x+1)(1-6z(1-z)) - x]G_3^{'}(z) - (1-2z)G_3(z) = 0.
\end{aligned} \tag{2.160}$$

One may solve this equation using the method of Frobenius. Namely one substitutes a power series of the form:

$$\begin{aligned}
G_3 &= \sum_{n=0}^{\infty} a_n \left[z(1-z)\right]^{n+\alpha}, \\
G_3' &= \sum_{n=0}^{\infty} a_n (n+\alpha)(1-2z)\left[z(1-z)\right]^{n+\alpha-1}, \\
G_3'' &= \sum_{n=0}^{\infty} a_n (n+\alpha)[z(1-z)]^{n+\alpha-2}\big\{(n+\alpha-1) - 2(2n+2\alpha-1)z(1-z)\big\}, \\
G_3''' &= (1-2z)\sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1)[z(1-z)]^{n+\alpha-3} \times \\
&\qquad\qquad\qquad\qquad \big\{(n+\alpha-2) - 2(2n+2\alpha-1)z(1-z)\big\}.
\end{aligned}$$

Substituting these relations into equation (2.160), dividing through by $(1-2z)$, and equating the coefficient of $[z(1-z)]^{\alpha+p}$ to zero one obtains the recurrence relation

$$\begin{gathered}
a_p\big\{x^3(p+\alpha)(p+\alpha-1)(p+\alpha-2) + x^2(3x+1)(p+\alpha)(p+\alpha-1) + \\
[x^2(x+1)-x](p+\alpha) - 1\big\} \\
+a_{p-1}\big\{-2x^3(p+\alpha-1)(p+\alpha-2)(2p+2\alpha-3) \\
-2x^2(3x+1)(p+\alpha-1)(2p+2\alpha-3) - 6x^2(x+1)(p+\alpha-1)\big\} = 0
\end{gathered}$$

This is straightforwardly factorized to read:

$$\begin{aligned}
&a_p(-1+\alpha x+px)(1+\alpha x+px)^2 \\
&\qquad - 2a_{p-1}(-1+\alpha+p)(\alpha+p)x^2(2-x+2\alpha x+2px) = 0
\end{aligned}$$

That is to say

$$\frac{a_p}{a_{p-1}} = \frac{4(p+\alpha)(p+\alpha-1)(p+\alpha+\frac{1}{x}-\frac{1}{2})}{(p+\alpha-\frac{1}{x})(p+\alpha+\frac{1}{x})^2} \tag{2.161}$$

By hypothesis, $a_0 = 1$, and is the first non-vanishing term. We find

$$a_n = \frac{4^n(\alpha+1)_n(\alpha)_n(\alpha+\frac{1}{x}+\frac{1}{2})_n}{(\alpha+1-\frac{1}{x})_n(\alpha+1+\frac{1}{x})_n(\alpha+1+\frac{1}{x})_n} \tag{2.162}$$

Consistency with the above relation implies the so-called indicial roots $\alpha = 1/x$, $-1/x$, $-1/x$. In particular, this method of solution has only yielded two independent solutions, corresponding to $\alpha = -1/x$ and $\alpha = 1/x$:

$$G_3^{(1)} = [z(1-z)]^{-\frac{1}{x}}\ {}_3F_2\left[\tfrac{1}{2}, -\tfrac{1}{x}, 1-\tfrac{1}{x}; 1, 1-\tfrac{2}{x}; 4z(1-z)\right] \tag{2.163}$$

$$G_3^{(3)} = [z(1-z)]^{\frac{1}{x}}\ {}_3F_2\left[\tfrac{1}{x}, 1+\tfrac{1}{x}, \tfrac{1}{2}+\tfrac{2}{x}; 1+\tfrac{2}{x}; 1+\tfrac{2}{x}; 4z(1-z)\right] \tag{2.164}$$

In order to find a third linearly independent solution, we apply the following rather general method outlined in §16.3 of [156] for example. The essential point is that the series defining G_3 in (2.161), with a_n given by (2.162) is an analytic function of α and may be differentiated any number of times with respect to α. Moreover, such functions also satisfy the original differential equation. A third solution is given by

$$G_3^{(2)} = \frac{d}{d\alpha}\sum_{n=0}^{\infty} a_n[z(1-z)]^{n+\alpha}\bigg|_{\alpha=-1/x} \tag{2.165}$$

$$= G_3^{(1)}\ln[4z(1-z)] + [z(1-z)]^{-\frac{1}{x}}\sum_{n=0}^{\infty}[z(1-z)]^n\,\frac{da_n}{d\alpha}\bigg|_{\alpha=-1/x} \tag{2.166}$$

Chapter 3

Legendre's Relation

§3.1 Introduction

In chapter 2, we employed the powerful techniques of non-Abelian bosonization to study the random non-Abelian Dirac fermion problem. In the replica approach, we used the fact that a theory of N_f 'flavours' and N_c 'colours' of free massless Dirac fermions is equivalent to three bosonic WZNW models carrying the charge, flavour, and colour degrees of freedom [93, 184]:

$$T_{\mathcal{D}_{N_f N_c}} = T_{\widehat{u}(1)} + T_{\widehat{su}(N_f)_{N_c}} + T_{\widehat{su}(N_c)_{N_f}}.$$

That is to say, the energy-momentum tensor of the fermionic theory *separates* into three commuting pieces, each pertaining to a different WZNW model. In the supersymmetric approach, we employed the fact that a theory of two 'colours' of free massless Dirac fermions, together with their supersymmetric 'flavour' partners, is equivalent to two WZNW models [52]:

$$T_{\widehat{osp}(4|4)_1} = T_{\widehat{osp}(2|2)_{-2}} + T_{\widehat{su}(2)_0},$$

wherein the $\widehat{osp}(2|2)_{-2}$ model carries the charge and flavour degrees of freedom, and the $\widehat{su}(2)_0$ model carries the colour degrees of freedom. The virtue of the WZNW description in the random non-Abelian Dirac fermion problem stems from this *separation* of degrees of freedom; the disorder only couples to the 'colour' subsector and may be integrated out. Indeed, this separation is behind many notable applications of the WZNW models in the study of quantum spin chains [93], the multi-channel Kondo effect [185], in $1+1$ dimensional QCD [186], and in the spin quantum Hall effect [52].

However, at the level of correlation functions, a subtlety lurks. In particular, as we have discussed in chapter 2, the $\widehat{osp}(2|2)_{-2}$ and $\widehat{su}(2)_0$ WZNW models are known to contain logarithmic operators [53–55, 117, 122] whereas the $\widehat{osp}(4|4)_1$ WZNW model is known have power law correlation functions [52]. In what sense can the supersymmetric equivalence hold? Indeed, this very concern has been raised in a revised version of [52]. In reference [54] we demonstrated that it is in fact possible to combine these logarithms, and achieve a reversion to power law behaviour. Our ability to recover such simple behaviour relied on a remarkable identity in the theory of elliptic integrals known as Legendre's relation [70, 187]. In this chapter we shall study this subject in a little more detail. In particular, we shall demonstrate that the logarithmic construction employed in [54] may be understood quite naturally from the non-logarithmic work of Fuchs entitled *Free Fermions and WZW Theories on Nonsimple Groups* [171]. We draw attention to the fact that the Knizhnik–Zamolodchikov equations for such nonsimple groups may yield *two* independent solutions, both of which satisfy the constraints of single-valuedness and crossing symmetry. In particular, logarithms may survive in one solution, and be removed in the other.

The structure of this chapter is as follows: in §3.2 we study the replica decomposition. In §3.2.1 we focus on its non-logarithmic aspects. We discuss the monodromy and braiding arguments of Fuchs [171], and how they may lead to power law behaviour. In §3.2.2 we consider the logarithmic aspects of the replica decomposition. We obtain a corollary to Legendre's relation, and recover the chiral blocks of the non-logarithmic case. In §3.3 we study the supersymmetric decomposition. We demonstrate how Legendre's relation [70, 187] enables one to braid the chiral blocks of the $\widehat{su}(2)_0$ and $\widehat{osp}(2|2)_{-2}$ WZNW models so as to obtain simple power laws. We demonstrate that this solution is a single-valued, crossing symmetric solution of the relevant Knizhnik–Zamolodchikov equations. Finally, we present concluding remarks and technical appendices.

§3.2 Replica Approach

As we have noted previously, the replica approach to the Dirac fermion problem is based on the energy-momentum tensor decomposition:

$$T_{\mathcal{D}_{N_f N_c}} = T_{\widehat{u}(1)} + T_{\widehat{su}(N_f)_{N_c}} + T_{\widehat{su}(N_c)_{N_f}}. \tag{3.1}$$

A proof of this decomposition may be found in reference [93]. In particular, the central charge of the $\widehat{su}(N)_k$ WZNW model is given by equation (3.15) of reference [68]

$$c = \frac{k(N^2-1)}{N+k} \tag{3.2}$$

and it is readily verified that (3.1) has the central charge of $N_f N_c$ Dirac fermions:

$$N_f N_c = 1 + \frac{N_c(N_f^2-1)}{N_f+N_c} + \frac{N_f(N_c^2-1)}{N_c+N_f}. \tag{3.3}$$

Moreover, the conformal dimensions of the $\widehat{su}(N)_k$ WZNW model fundamental primary fields are given by equation (4.11) of reference [68]

$$h = \frac{N^2-1}{2N(N+k)} \tag{3.4}$$

and it is readily verified that (3.1) has the conformal dimension of a Dirac fermion:

$$\frac{1}{2} = \frac{1}{2N_f N_c} + \frac{N_f^2-1}{2N_f(N_f+N_c)} + \frac{N_c^2-1}{2N_c(N_c+N_f)}. \tag{3.5}$$

We remind the reader that the conformal dimension $h = 1/2N_f N_c$ of the $\widehat{u}(1)$ piece is given in equation (2.80). The equality (3.5) ensures the coincide of the two- and three-point functions. However, the four-point correlation functions of the $\widehat{su}(N)_k$ WZNW model are, in general, single-valued combinations of non-trivial hypergeometric functions — see equations (3.11) below. It is quite clear that one *cannot* simply multiply the *non-chiral* correlation functions of the three WZNW models (3.1) and obtain the *non-chiral* correlation functions of (coloured–flavoured) Dirac fermions [93, 171]. We emphasize that this statement is true irrespective of the presence of logarithms in the non-chiral WZNW correlation functions. This apparent paradox was resolved by Fuchs, who demonstrated that the monodromy problem of the $\widehat{u}(1) + \widehat{su}(N_f)_{N_c} + \widehat{su}(N_c)_{N_f}$ Knizhnik–Zamolodchikov equations actually has *two* independent solutions [171]. The first of these solutions corresponds to the product of the non-chiral WZNW correlation functions. The second of these solutions involves a subtle chiral interplay between the $\widehat{su}(N_f)_{N_c}$ and $\widehat{su}(N_c)_{N_f}$ WZNW models and reproduces the Dirac fermion correlation functions. We discuss this procedure below.

In analogy with equations (2.66), (2.69) and (2.75), one may write the solutions to the $\widehat{u}(1) + \widehat{su}(N_f)_{N_c} + \widehat{su}(N_c)_{N_f}$ Knizhnik–Zamolodchikov equations in the form

$$\mathcal{F}^{\alpha_f \alpha_c \bar{\alpha}_f \bar{\alpha}_c}(z_i, \bar{z}_i) = |z_{14} z_{23}|^{-4h} \sum_{i,j,\bar{i},\bar{j}=1}^{2} I_i^{\alpha_f} I_j^{\alpha_c} I_{\bar{i}}^{\bar{\alpha}_f} I_{\bar{j}}^{\bar{\alpha}_c} F_{i j \bar{i} \bar{j}}(z, \bar{z}), \tag{3.6}$$

where $h = 1/2$, and where $I_i^{\alpha_f}$ and $I_j^{\alpha_c}$ are the invariant tensors of $su(N_f)$ and $su(N_c)$ respectively. The $F_{i j \bar{i} \bar{j}}(z, \bar{z})$ are single-valued combinations of the $\widehat{u}(1)$, $\widehat{su}(N_f)_{N_c}$ and

$\widehat{su}(N_c)_{N_f}$ chiral blocks:

$$F_{i\,j\,\bar{i}\,\bar{j}}(z,\bar{z}) = |z(1-z)|^{-2/N_f N_c} \sum_{a,b,c,d=1}^{2} X_{abcd}\, F^{(a)}_{i,\hat{f}_c}(z) F^{(b)}_{j,\widehat{c}_f}(z) F^{(c)}_{\bar{i},\hat{f}_c}(\bar{z}) F^{(d)}_{\bar{j},\widehat{c}_f}(\bar{z}), \tag{3.7}$$

where we have employed the $\widehat{u}(1)$ chiral blocks as given by equation (2.81), and the shorthand notation $\hat{f}_c \equiv \widehat{su}(N_f)_{N_c}$. As noted by Fuchs, equation (3.7) admits *two* independent single-valued combinations. The first of these is the straightforward product of the three non-chiral WZNW models:

$$F_{i\,j\,\bar{i}\,\bar{j}}(z,\bar{z}) = |\widehat{u}(z)|^2\, F^{\hat{f}_c}_{i\,\bar{i}}(z,\bar{z})\, F^{\hat{c}_f}_{j\,\bar{j}}(z,\bar{z}), \tag{3.8}$$

where $\widehat{u}(z) = [z(1-z)]^{-1/N_f N_c}$, and where $F^{\hat{f}_c}_{i\,\bar{i}}(z,\bar{z})$ corresponds to the usual single-valued combination of holomorphic and anti-holomorphic chiral blocks in the $\widehat{su}(N_f)_{N_c}$ WZNW model — see equation (2.75). The second solution may be written in the form

$$F_{i\,j\,\bar{i}\,\bar{j}}(z,\bar{z}) = E_{ij}(z)\, E_{\bar{i}\,\bar{j}}(\bar{z}), \tag{3.9}$$

where the functions

$$E_{ij}(z) = \widehat{u}(z) \sum_{a,b=1}^{2} X_{ab}\, F^{(a)}_{i,\hat{f}_c}(z) F^{(b)}_{j,\hat{c}_f}(z), \tag{3.10}$$

are *single-valued* combinations of the $\widehat{u}(1)$, $\widehat{su}(N_f)_{N_c}$ and $\widehat{su}(N_c)_{N_f}$ *holomorphic* chiral blocks. In reference [54] we introduced the term 'embedding block' to refer to such single-valued chiral combinations of *different* WZNW model chiral blocks. It the existence of these chiral embedding blocks which enables one to convert non-trivial hypergeometric functions (including logarithms) into free fields. We shall illustrate this reduction in the subsequent sections.

§3.2.1 Non-Logarithmic Case

In this section we shall examine the embedding blocks (3.10) of the replica decomposition (3.1) in the cases where the WZNW chiral blocks are free from logarithms. We shall adopt the simpler notation $N_f = N$ and $N_c = k$. The chiral blocks of the $\widehat{su}(N)_k$

WZNW model read [68, 171]:

$$F^{(1)}_{1,\,\widehat{su}(N)_k}(z) = z^{-\beta+\alpha/N}(1-z)^{\alpha/N}\ {}_2F_1[\alpha,-\alpha;1-\beta;z], \tag{3.11a}$$
$$F^{(2)}_{1,\,\widehat{su}(N)_k}(z) = z^{\alpha/N}(1-z)^{1-\beta+\alpha/N}\ {}_2F_1[1+\alpha,1-\alpha;1+\beta;z], \tag{3.11b}$$
$$F^{(1)}_{2,\,\widehat{su}(N)_k}(z) = k^{-1}z^{1-\beta+\alpha/N}(1-z)^{\alpha/N}\ {}_2F_1[1+\alpha,1-\alpha;2-\beta;z], \tag{3.11c}$$
$$F^{(2)}_{2,\,\widehat{su}(N)_k}(z) = -Nz^{\alpha/N}(1-z)^{-\beta+\alpha/N}\ {}_2F_1[\alpha,-\alpha;\beta;z], \tag{3.11d}$$

where $\alpha = 1/(N+k)$ and $\beta = N/(N+k)$. The explicit monodromy calculations of Fuchs lead one to consider the following embedding blocks:

$$E_{ij}(z) = \widehat{u}(z)\left[F^{(1)}_{i,\,\widehat{su}(N)_k}(z)F^{(1)}_{j,\,\widehat{su}(k)_N}(z) + (Nk)^{-1}F^{(2)}_{i,\,\widehat{su}(N)_k}(z)F^{(2)}_{j,\,\widehat{su}(k)_N}(z)\right], \tag{3.12}$$

where $\widehat{u}(z) = [z(1-z)]^{-1/Nk}$. Rather than repeat these manipulations, we shall simply calculate these embedding blocks below; the reader may then observe that they are indeed single-valued. It is readily seen from the chiral blocks (3.11) that

$$F^{(1)}_{1,\,\widehat{su}(k)_N}(z) = -N^{-1}z^{a-1}(1-z)^a F^{(2)}_{2,\,\widehat{su}(N)_k}(z), \tag{3.13a}$$
$$F^{(2)}_{1,\,\widehat{su}(k)_N}(z) = kz^{a-1}(1-z)^a F^{(1)}_{2,\,\widehat{su}(N)_k}(z), \tag{3.13b}$$

where $a = (N-k+N^2k)/Nk(N+k)$. Substituting these relations into equation (3.12) it immediately follows that $E_{12}(z) = E_{21}(z) = 0$. Further, under a level–rank transformation $N \leftrightarrow k$, it is readily seen that $\alpha \to \alpha$ and $\beta \to 1-\beta$. Embracing these relations and focusing on the cases $E_{11}(z)$ and $E_{22}(z)$, it is readily seen that the hypergeometric functions only appear in the combination,

$$\begin{aligned}{}_2F_1[\alpha,-\alpha;\beta;z]\ \ {}_2F_1[\alpha,-\alpha;1-\beta;z] + (Nk)^{-1}z(1-z)\times&\\ {}_2F_1[1+\alpha,1-\alpha;1+\beta;z]\ {}_2F_1[1+\alpha,1-\alpha;2-\beta;z] = 1.&\end{aligned} \tag{3.14}$$

That is to say, all the non-trivial hypergeometric dependence is removed. We complete Fuchs' perturbative verification of this 'level–rank' relation (3.14) in appendix 3.A. Employing this level–rank relation, one one may write the full quota of embedding blocks as follows:

$$E_{11} = \frac{1}{z}, \quad E_{12} = 0, \quad E_{21} = 0, \quad E_{22} = \frac{1}{1-z}. \tag{3.15}$$

Indeed, as promised, they are single-valued functions. The full correlation function (3.6) reads

$$\mathcal{F}^{\alpha\beta\bar{\alpha}\bar{\beta}}(z_i,\bar{z}_i) = \frac{1}{|z_{14}z_{23}|^2}\left[\frac{I_1^\alpha I_1^\beta}{z} + \frac{I_2^\alpha I_2^\beta}{1-z}\right]\times\left[\frac{I_1^{\bar{\alpha}} I_1^{\bar{\beta}}}{\bar{z}} + \frac{I_2^{\bar{\alpha}} I_2^{\bar{\beta}}}{1-\bar{z}}\right], \tag{3.16}$$

and is readily identified as the correlation function of free Dirac fermions carrying both 'colour' and 'flavour' indices:

$$\mathcal{F}^{\alpha\beta\bar{\alpha}\bar{\beta}}(z_i, \bar{z}_i) = |\langle \psi_{\alpha_1\beta_1}(z_1)\psi^{\dagger}_{\alpha_2\beta_2}(z_2)\psi^{\dagger}_{\alpha_3\beta_3}(z_3)\psi_{\alpha_4\beta_4}(z_4)\rangle|^2, \tag{3.17}$$

where the fermion OPEs take the form

$$\psi_{\alpha_i\beta_i}(z_i)\psi^{\dagger}_{\alpha_j\beta_j}(z_j) \sim \frac{\delta_{\alpha_i,\alpha_j}\delta_{\beta_i,\beta_j}}{z_i - z_j}. \tag{3.18}$$

§3.2.2 Logarithmic Case

In the light of the simplifications in §3.2.1, in is natural to investigate what happens to the replica decomposition (3.1) in the 'replica limit', as either N or k go to zero. We emphasize that we are *not* performing a disorder average here. In addition, we stress that it is the rôle of the invariant tensors to keep track of how many Dirac fermions we have, and *not* the chiral blocks; the level–rank relation (3.14) is (formally) valid for *any* values of N and k. We expect (and indeed shall demonstrate) that the WZNW logarithms cancel to yield the Dirac embedding blocks (3.15) again.

The $\widehat{u}(0)_N$ WZNW model chiral blocks have been studied in appendix 2.A — see equations (2.85). For convenience we reproduce them here:

$$\begin{aligned}
F^{(1)}_{1,\widehat{u}(0)_N}(z) &= \gamma\, E_N(z) &\quad (3.19\text{a})\\
F^{(2)}_{1,\widehat{u}(0)_N}(z) &= \gamma\, N\, [K_N(1-z) - E_N(1-z)] &\quad (3.19\text{b})\\
F^{(1)}_{2,\widehat{u}(0)_N}(z) &= \gamma\, [K_N(z) - E_N(z)] &\quad (3.19\text{c})\\
F^{(2)}_{2,\widehat{u}(0)_N}(z) &= \gamma\, N\, E_N(1-z) &\quad (3.19\text{d})
\end{aligned}$$

where $\gamma = [z(1-z)]^{-1/N^2}$. In order to compute the $\widehat{su}(N)_0$ chiral blocks we return to our discussion of the $\widehat{su}(N)_k$ WZNW model in appendix 2.A. From equations (2.76) and (2.77) it follows that

$$F_1(z) = z^{-1}[z(1-z)]^{1/N^2} G_1(z), \tag{3.20}$$

where $G_1(z)$ satisfies an equation of the hypergeometric form:

$$z(1-z)G_1'' + [\gamma - (\alpha+\beta+1)z]\, G_1' - \alpha\beta G_1 = 0, \tag{3.21}$$

where $\alpha = -1/N$, $\beta = 1/N$ and $\gamma = 0$. The fact that $\gamma = 0$ is a clear signal that we have logarithmic solutions. Indeed, it is straightforward to verify (by means of the differentiation formulae A.21 for example) that equation (3.21) admits the two solutions

$$G_1^{(1)} = E_N(z) - K_N(z), \tag{3.22a}$$
$$G_1^{(2)} = E_N(1-z). \tag{3.22b}$$

The chiral blocks of the $\widehat{su}(N)_0$ WZNW model may thus be written in the form:

$$F^{(1)}_{1,\widehat{su}(N)_0}(z) = z^{-1}\,\gamma^{-1}\,[E_N(z) - K_N(z)] \tag{3.23a}$$
$$F^{(2)}_{1,\widehat{su}(N)_0}(z) = z^{-1}\,\gamma^{-1}\,E_N(1-z) \tag{3.23b}$$
$$F^{(1)}_{2,\widehat{su}(N)_0}(z) = (1-z)^{-1}\,\gamma^{-1}\,E_N(z) \tag{3.23c}$$
$$F^{(2)}_{2,\widehat{su}(N)_0}(z) = (1-z)^{-1}\,\gamma^{-1}\,[E_N(1-z) - K_N(1-z)] \tag{3.23d}$$

where $\gamma = [z(1-z)]^{-1/N^2}$ again. We now consider the embedding blocks

$$E_{ij}(z) = \sum_{a,b=1}^{2} X_{ab}\,F^{(a)}_{i,\widehat{u}(o)_N}(z)\,F^{(b)}_{j,\widehat{su}(N)_0}(z) \tag{3.24}$$

which are a simple braiding of the (holomorphic) $\widehat{u}(0)_N$ chiral blocks, $F^{(a)}_{i,\widehat{u}(0)_N}(z)$, and the (holomorphic) $\widehat{su}(N)_0$ chiral blocks, $F^{(a)}_{i,\widehat{su}(N)_0}(z)$. We remind the reader that the $\widehat{u}(1)$ contribution which appeared in equation (3.10) has been absorbed into the $\widehat{u}(o)_N$ WZNW model. The X_{ab} are, as yet, undetermined coefficients which we shall determine by imposing single-valuedness in the vicinity of $z=0$ and $z=1$. In order to impose the condition of single-valuedness on the embedding blocks, we consider the monodromy transformations of its constituent chiral blocks. A monodromy transformation of a function of z consists in letting z circulate around some other point (typically a singular point). We define

$$\mathcal{C}_0\,F(z,\bar z) \;=\; \lim_{t\to 1^-} F(ze^{2i\pi t}, \bar z e^{-2i\pi t}) \tag{3.25}$$
$$\mathcal{C}_1\,F(z,\bar z) \;=\; \lim_{t\to 1^-} F(1+(z-1)e^{2i\pi t}, 1+(\bar z-1)e^{-2i\pi t}) \tag{3.26}$$

Using the analytic continuation formulae for the hypergeometric functions it is straightforward to see that

$$\mathcal{C}_0\,F^{(a)}_{i,\mathcal{A}}(z) \;=\; (g_0^{\mathcal{A}})_{ab}\,F^{(b)}_{i,\mathcal{A}}(z), \quad i=1,2 \tag{3.27a}$$
$$\mathcal{C}_1\,F^{(a)}_{i,\mathcal{A}}(z) \;=\; (g_1^{\mathcal{A}})_{ab}\,F^{(b)}_{i,\mathcal{A}}(z), \quad i=1,2 \tag{3.27b}$$

where the algebra index $\mathcal{A}$ is either $\widehat{u}(0)_N$ or $\widehat{su}(N)_0$. The $\widehat{u}(0)_N$ monodromy matrices are given by:

$$g_0^{\widehat{u}(0)_N} = e^{-2\pi i/N^2}\begin{pmatrix} 1 & 0 \\ -Na_N & 1 \end{pmatrix}, \quad g_1^{\widehat{u}(0)_N} = e^{-2\pi i/N^2}\begin{pmatrix} 1 & -a_N/N \\ 0 & 1 \end{pmatrix} \tag{3.28}$$

where $a_N = 2i\sin(\pi/N)$. The $\widehat{su}(N)_0$ monodromy matrices are given by:

$$g_0^{\widehat{su}(N)_0} = e^{2\pi i/N^2}\begin{pmatrix} 1 & 0 \\ a_N & 1 \end{pmatrix}, \quad g_1^{\widehat{su}(N)_0} = e^{2\pi i/N^2}\begin{pmatrix} 1 & a_N \\ 0 & 1 \end{pmatrix} \tag{3.29}$$

where once again $a_N = 2i\sin(\pi/N)$. Demanding that the embedding blocks be invariant under both $\mathcal{C}_0$ and $\mathcal{C}_1$, requires that $X_{11} = X_{22} = 0$ and $X_{12} = NX_{21}$. In other words,

$$E_{ij}(z) = \mathcal{N}\left[F^{(1)}_{i,\widehat{u}(0)_N}(z)F^{(2)}_{j,\widehat{su}(N)_0}(z) + (N)^{-1}F^{(2)}_{i,\widehat{u}(0)_N}(z)F^{(1)}_{j,\widehat{su}(N)_0}(z)\right]. \tag{3.30}$$

where the overall normalization $\mathcal{N} = [\pi/N\sin(\pi/N)]^{-1}$ is chosen for later convenience. We note that the off-diagonal structure in the upper indices (1) and (2), as compared to equation (3.12), is a direct consequence of the presence of logarithms. Since there are two invariant tensors in each of the $\widehat{u}(0)_N$ and $\widehat{su}(N)_0$ sectors this defines a total of four functions $E_{ij}(z)$. In our explicit evaluation of these embedding blocks, we are able to utilize a corollary to an elegant identity in the theory of elliptic integrals which is now termed Legendre's relation [70, 187]:

$$E_N(z)K_N(1-z) + E_N(1-z)K_N(z) - K_N(z)K_N(1-z) = \frac{\pi}{N}\sin\left(\frac{\pi}{N}\right) \tag{3.31}$$

Straightforward substitution of the $\widehat{u}(0)_N$ and $\widehat{su}(N)_0$ chiral blocks into (3.30) reveals that the elliptic integrals always appear in the manner prescribed by the (generalized) Legendre relation (3.31) — not only are the embedding blocks single-valued, but they do not contain any elliptic integrals. We give a proof of the generalized Legendre relation in appendix 3.B. We note that we have not been able to find this rather simple result in any of the common treatises on special functions [69, 70, 188–192]. The embedding blocks in the replica limit take the form:

$$E_{11} = \frac{1}{z}, \quad E_{12} = 0, \quad E_{21} = 0, \quad E_{22} = \frac{1}{1-z}, \tag{3.32}$$

and coincide with the Dirac blocks in equation (3.15). We see that the cancellation of the WZNW logarithms is natural, and the Legendre relation (3.31) plays the rôle of the level–rank relation (3.14).

§3.3 Supersymmetric Approach

In this section we shall investigate the supersymmetric decomposition

$$T_{\widehat{osp}(4|4)_1} = T_{\widehat{su}(2)_0} + T_{\widehat{osp}(2|2)_{-2}}, \tag{3.33}$$

A proof of this decomposition may be found in reference [52]. We note that all of these theories have central charge equal to zero. Moreover, It is readily verified that (3.33) yields the conformal dimension of a Dirac fermion or bosonic ghost:

$$\frac{1}{2} = \left.\frac{N^2-1}{2N(N+k)}\right|_{\substack{k=0\\N=2}} + \left.\frac{1}{4-2k}\right|_{k=-2}. \tag{3.34}$$

We define the embedding blocks

$$E_{ij}(z) = \sum_{a,b=1}^{2} X_{ab}\, F^{(a)}_{i,\,\mathrm{su}}(z) F^{(b)}_{j,\,\mathrm{osp}}(z) \tag{3.35}$$

which are a simple braiding of the (holomorphic) $\widehat{su}(2)_0$ chiral blocks, $F^{(a)}_{i,\,\mathrm{su}}(z)$, and the (holomorphic) $\widehat{osp}(2|2)_{-2}$ chiral blocks, $F^{(a)}_{i,\,\mathrm{osp}}(z)$. The X_{ab} are, as yet, undetermined coefficients which we shall determine by imposing single-valuedness in the vicinity of $z=0$ and $z=1$. In order to impose the condition of single-valuedness on the embedding blocks, we consider the monodromy transformations of its constituent chiral blocks as was done in §3.2.2. Using the analytic continuation formulae for the hypergeometric functions it is straightforward to see that the monodromy matrices $g_0^{\mathcal{A}}$ and $g_1^{\mathcal{A}}$, in the notation of equations (3.27), are given by

$$g_0^{\mathrm{su}} = \begin{pmatrix} i & 0 \\ -2 & i \end{pmatrix}, \quad g_1^{\mathrm{su}} = \begin{pmatrix} i & -2 \\ 0 & i \end{pmatrix}, \quad g_0^{\mathrm{osp}} = \begin{pmatrix} -i & 0 \\ 2 & -i \end{pmatrix}, \quad g_1^{\mathrm{osp}} = \begin{pmatrix} -i & 2 \\ 0 & -i \end{pmatrix}.$$

Demanding that the embedding functions be invariant under both $\mathcal{C}_0$ and $\mathcal{C}_1$, requires that $X_{11} = X_{22} = 0$ and $X_{12} = -X_{21}$. In other words,

$$E_{ij}(z) = \mathcal{N}' \left[F^{(1)}_{i,\,\mathrm{su}}(z) F^{(2)}_{j,\,\mathrm{osp}}(z) - F^{(2)}_{i,\,\mathrm{su}}(z) F^{(1)}_{j,\,\mathrm{osp}}(z) \right], \tag{3.36}$$

where the overall normalization $\mathcal{N}' = 2\epsilon\gamma/\pi$ is chosen for later convenience. Since there are two invariant tensors in the su(2) sector ($i = 1, 2$), and three in the osp(2|2) sector ($j = 1, 2, 3$), this defines a total of six functions $E_{ij}(z)$. In our explicit evaluation of these embedding blocks, we are once again able to utilize the Legendre's relation (3.31). Straightforward substitution of the $\widehat{su}(2)_0$ and $\widehat{osp}(2|2)_{-2}$ chiral blocks into (3.36) reveals that the elliptic integrals always appear in the manner prescribed in Legendre's relation (3.31) — not only are the embedding blocks single-valued, but they do not contain any elliptic integrals. The embedding blocks may be summarized:

$$E_{11} = -1, \quad E_{12} = \frac{1}{z}, \quad E_{13} = -\frac{4\epsilon\gamma}{z}, \quad E_{21} = \frac{z}{z-1}, \quad E_{22} = \frac{1}{1-z}, \quad E_{23} = 0.$$

Indeed, one may verify by direct substitution, that these embedding blocks satisfy the Knizhnik–Zamolodchikov equations pertaining to $\widehat{su}(2)_0$ and $\widehat{osp}(2|2)_{-2}$ — see equations (2.71) and (2.104):

$$\frac{dE_{\alpha\beta}}{dz} = \left[\frac{P^{\alpha\gamma}_{\text{su}}}{z} + \frac{Q^{\alpha\gamma}_{\text{su}}}{z-1}\right] E_{\gamma\beta} + \left[\frac{P^{\beta\delta}_{\text{osp}}}{z} + \frac{Q^{\beta\delta}_{\text{osp}}}{z-1}\right] E_{\alpha\delta}, \tag{3.37}$$

where summation over repeated indices is implied. The matrices P_{su} and Q_{su} are given by equation (2.72) with $N = 2$ and $k = 0$, and the matrices P_{osp} and Q_{osp} are given by equation (2.105) with $k = -2$. In addition, it is also necessary to check for the crossing covariance of the correlation function

$$\mathcal{F}^{\alpha\beta\bar{\alpha}\bar{\beta}}(z_i, \bar{z}_i) = |z_{14}z_{23}|^{-2} F^{\alpha\beta}(z) F^{\bar{\alpha}\bar{\beta}}(\bar{z}) \tag{3.38}$$

under the interchange of the fields 2 and 3 where,

$$F^{\alpha\beta}(z) \equiv \sum_{ij} I^{\alpha}_{i,\text{su}} I^{\beta}_{j,\text{osp}}\, E_{ij}(z). \tag{3.39}$$

Under this transformation. the $su(2)$ invariant tensors $I^{\alpha}_{1,\text{su}} = \delta_{\alpha_1\alpha_2}\delta_{\alpha_3\alpha_4}$, and $I^{\alpha}_{2,\text{su}} = \delta_{\alpha_1\alpha_3}\delta_{\alpha_2\alpha_4}$, are interchanged:

$$I^{\alpha}_{1,\text{su}} \to I^{\alpha}_{2,\text{su}}, \quad I^{\alpha}_{2,\text{su}} \to I^{\alpha}_{1,\text{su}}.$$

The $osp(2|2)$ tensors given in equation (2.103) transform according to equations (84) and (85) of reference [53]:

$$I^{\beta}_{1,\text{osp}} \to -\tilde{\mathcal{P}} I^{\beta}_{1,\text{osp}}, \quad I^{\beta}_{2,\text{osp}} \to -\tilde{\mathcal{P}} \left[I^{\beta}_{1,\text{osp}} - I^{\beta}_{2,\text{osp}} + 4\epsilon\gamma I^{\beta}_{3,\text{osp}}\right], \quad I^{\beta}_{3,\text{osp}} \to -\tilde{\mathcal{P}} I^{\beta}_{3,\text{osp}}.$$

where $\tilde{\mathcal{P}}$ is the parity of the permutation. Combining these transformation laws with the explicit transformations of the embedding blocks under $z \to 1 - z$, one finds

$$F^{\alpha\beta}(z) \to \tilde{\mathcal{P}} F^{\alpha\beta}(z). \tag{3.40}$$

That is to say, in equation (3.38) we have succeeded in constructing a single-valued and crossing-symmetric solution of the $\widehat{su}(2)_0 + \widehat{osp}(2|2)_{-2}$ Knizhnik–Zamolodchikov equation which is free from logarithms.

§3.4 Discussion

In this chapter we have studied a number of examples of non-Abelian bosonization at the level of correlation functions. In particular, we have highlighted that the Knizhnik–Zamolodchikov equations pertaining to such non-simple groups may possess two independent solutions [171]. In general, one of these solutions corresponds to the 'ordinary' product of the appropriate non-chiral WZNW theories, whereas the other involves a subtle chiral interplay between them. In condensed matter applications of the non-Abelian bosonization rules for Dirac fermions (and their supersymmetric extensions) it is essential to embrace these latter solutions. The presence of additional selection rules on the WZNW models may be interpreted as the requirement of single-valuedness on the fermionic chiral blocks [171]. Moreover, we have provided explicit evidence for the recovery of Dirac fermion correlation functions from logarithmic CFTs. We have indicated that the logarithmic construction employed by the author in reference [54] has a natural non-logarithmic analogue in the work of Fuchs [171]. We anticipate that such considerations apply to other models which are equivalent to Dirac fermions [184], and their supersymmetric extensions.

Appendices to Chapter 3

3.A Level–Rank Relation

In section §3.2.1 we made use of the 'level–rank' relation:

$$\begin{aligned}&{}_2F_1[\alpha,-\alpha;\beta;z]\ {}_2F_1[\alpha,-\alpha;1-\beta;z]+(Nk)^{-1}z(1-z)\times\\ &\qquad {}_2F_1[1+\alpha,1-\alpha;1+\beta;z]\ {}_2F_1[1+\alpha,1-\alpha;2-\beta;z]=1,\end{aligned} \tag{3.41}$$

where $\alpha = 1/(N+k)$ and $\beta = N/(N+k)$. For completeness, we give a non-perturbative proof of this relation below. Following Fuchs [171], we combine the hypergeometric functions using identity (3) appearing on page 88 of Bailey [191]:

$${}_2F_1\left[\begin{matrix}a,b;z\\ c\end{matrix}\right]\ {}_2F_1\left[\begin{matrix}a,b;z\\ a+b-c+1\end{matrix}\right] = {}_4F_3\left[\begin{matrix}a,b,\frac{1}{2}(a+b+1),\frac{1}{2}(a+b);w\\ a+b,c,a+b-c+1\end{matrix}\right], \tag{3.42}$$

where $w = 4z(1-z)$. In the limit $b = -a$, there is a compensation between the upper argument $(a+b)/2$ and the lower argument $a+b$, and it follows that,

$${}_2F_1\left[\begin{matrix}\alpha,-\alpha;z\\ \beta\end{matrix}\right]\ {}_2F_1\left[\begin{matrix}\alpha,-\alpha;z\\ 1-\beta\end{matrix}\right] = \frac{1}{2}+\frac{1}{2}\ {}_3F_2\left[\begin{matrix}\alpha,-\alpha,\frac{1}{2};w\\ \beta,1-\beta\end{matrix}\right]. \tag{3.43}$$

It also follows from equation (3.42) that,

$${}_2F_1\left[\begin{matrix}1+\alpha,1-\alpha;z\\ 1+\beta\end{matrix}\right]\ {}_2F_1\left[\begin{matrix}1+\alpha,1-\alpha;z\\ 2-\beta\end{matrix}\right] = {}_4F_3\left[\begin{matrix}1+\alpha,1-\alpha,\frac{3}{2},1;w\\ 1+\beta,2-\beta,2\end{matrix}\right]. \tag{3.44}$$

We may now use identity 17 on page 439 of Prudnikov, Brychov and Marichev [189],

$${}_pF_q\left[\begin{matrix}(a_{p-1}),1;z\\ (b_{q-1}),2\end{matrix}\right] = z^{-1}\prod_{j=1}^{q-1}(b_j-1)\prod_{k=1}^{p-1}(a_k-1)^{-1}\left\{\ {}_{p-1}F_{q-1}\left[\begin{matrix}(a_{p-1})-1;z\\ (b_{q-1})-1\end{matrix}\right]-1\right\}$$

in order to reduce the order of the hypergeometric function appearing in (3.44):

$${}_4F_3\left[\begin{matrix}1+\alpha,1-\alpha,\frac{3}{2},1;w\\ 1+\beta,2-\beta,2\end{matrix}\right] = w^{-1}\frac{\beta(1-\beta)}{\alpha(-\alpha)(1/2)}\left\{\ {}_3F_2\left[\begin{matrix}\alpha,-\alpha,\frac{1}{2};w\\ \beta,1-\beta\end{matrix}\right]-1\right\}. \tag{3.45}$$

By combining the results (3.43), (3.44) and (3.45), and inserting the explicit forms of α and β, the relation (3.41) follows.

3.B Generalized Legendre Relation

Theorem

$$E_N(z)K_N(1-z) + E_N(1-z)K_N(z) - K_N(z)K_N(1-z) = \frac{\pi}{N}\sin\left(\frac{\pi}{N}\right). \tag{3.46}$$

Proof *Theorem (3.46) is commonly referred to as Legendre's relation [70, 187] (in the case $N = 2$) and is straightforwardly proven in the following manner. One may first differentiate the left hand side of equation (3.46) with respect to z, by means of the rather simple differentiation formulae for generalized complete elliptic integrals (A.21), in order to demonstrate that the combination of functions appearing is in fact a constant. In order to identify this constant one may develop the left hand side of equation (3.46) in a Laurent series expansion about $z = 0$ by means of the analytic continuation formulae for hypergeometric functions — see Appendix A. The first and last terms combine to give a contribution involving powers of z greater than or equal to unity, and thus the constant term arises from the expansion of $E(1-z)K(z)$ alone. Combining the definitions (A.20) with the analytic continuation formula (A.16) — with z replaced by $1-z$, and $a = -1/N$, $b = 1/N$, $m = 1$ — this constant is seen to be*

$$\left(\frac{\pi}{N}\right)^2 \frac{\Gamma(1)\Gamma(1)}{\Gamma(1-1/N)\Gamma(1+1/N)}.$$

Employing the facts that $\Gamma(1) = 1$ and $\Gamma(1-1/N)\Gamma(1+1/N) = (\pi/N)\sin(\pi/N)$ the result (3.46) follows.

Chapter 4

Conclusions

In this thesis we have employed the powerful tools of non-Abelian bosonization and logarithmic conformal field theory to study a model introduced by Nersesyan, Tsvelik and Wenger in the context of two-dimensional disordered d-wave superconductivity [27–29]. This model takes the form of a two-dimensional relativistic field theory of Dirac fermions interacting with a random non-Abelian $su(N)$ gauge field. We have calculated the disorder averaged four-point correlation functions of the local density of states (LDoS) by means of three non-perturbative approaches. These are based on the commonly employed replica and supersymmetry methods, together with a third (model specific) approach valid in the limit of strong disorder. In the replica and strong disorder approaches the logarithmic features are encoded in the $\widehat{u}(0)_N$ and the $\widehat{su}(N)_{-2N}$ WZNW models respectively, whereas in the supersymmetric approach (with $N = 2$) they are described by the $\widehat{osp}(2|2)_{-2}$ WZNW model. In contradistinction to earlier treatments, we have demonstrated that this *ménage à trois* of theoretical approaches leads to identical results.

§4.1 Simplifications

Considerable simplifications are observed to occur in the supersymmetric approach. In particular, the level $k = -2$ of the underlying $\widehat{osp}(2|2)_k$ Kač–Moody algebra plays a crucial rôle: two generalized hypergeometric solutions of the third-order Knizhnik–Zamolodchikov equations (pertaining to the $[0, 1/2]$ representation) reduce to (ordinary hypergeometric) complete elliptic integrals. Indeed, the Knizhnik–Zamolodchikov equations factorize, and it is possible to satisfy the stringent demands of the conformal bootstrap (single-valuedness and crossing symmetry) on a reduced subspace of functions. The decoupling of the third solution (or conformal block) is essential for the marriage of approaches presented in this thesis.

We have demonstrated that the disorder averaged correlation functions of the non-Abelian Dirac fermion problem inherit their non-trivial logarithmic structure from the non-unitary minimal model with central charge $c = -2$. This theory describes the geometrical critical behaviour of dense polymers and has served as an illuminating example of a LCFT. In particular, a remarkably simple free field representation of the $\widehat{osp}(2|2)_{-2}$ WZNW model is obtained by augmenting this $c = -2$ theory by compact and non-compact free bosons, without the use of screening operators. We have established a chiral correspondence between the primary fields of the supersymmetric model and the $\mathbb{Z}_N$ twist operators of the polymer theory, 'dressed' by exponentials (or vertex operators) of the bosonic fields. Whilst this prescription has been used successfully to reproduce the chiral blocks of the $\widehat{osp}(2|2)_{-2}$ WZNW model, we have thus far been unable to establish a general non-chiral operator correspondence. These non-chiral (or braiding) aspects deserve further investigation.

§4.2 Legendre's Relation

The study of the non-Abelian Dirac fermion problem leads to a separation between the 'active' and 'passive' theories; the active degrees of freedom encode the non-trivial behaviour of the LDoS, whereas the 'passive' degrees of freedom represent the massive or decoupled Jacobian degrees of freedom. Considerations of the internal consistency of the various approaches has lead to natural relations for the mutual cancellation of logarithms between the 'active' and 'passive' LCFTs. In this pursuit, we have been aided by the Legendre relation in the theory of elliptic integrals. Moreover, our study of the general $su(N)$ problem leads us to introduce generalizations of the complete elliptic integrals which furnish a corolloray to the Legendre relation. We have drawn attention to the fact that the Knizhnik–Zamolodchikov equations on non-simple groups may yield two independent solutions [171].

Whilst in the non-Abelian Dirac fermion problem, there is a clear separation between the 'active' and the 'passive' degrees of freedom, the requirement of the cancellation of logarithms suggest that there exist relations between them. In particular, the presence of the $c = -2$ theory may also be seen in all of them. It may be interesting to investigate whether these are connected to the notions of 'level-rank' duality and 'conjugation' duality: $\widehat{su}(0)_N \leftrightarrow \widehat{su}(N)_0 \leftrightarrow \widehat{su}(N)_{-2N}$ [193]. Investigations of the supersymmetric 'passive' theory — the $\widehat{su}(2)_0$ WZNW model — and its relations to both the $\widehat{osp}(2|2)_{-2}$ WZNW model and the $c = -2$ minimal model have been continued in references [178, 194]. It is a subject of considerable interest to investigate the extent to which these correspondences hold at the level of (logarithmic) currents [194].

§4.3 Outlook

In addition to the questions outlined above, there are quite naturally, a number of open questions and directions for further research.

We have applied the supersymmetry approach to the general $su(N)$ non-Abelian Dirac fermion problem only in the case $N = 2$. It would be interesting to generalize the decomposition (2.36) further; the recent progress in this direction is more suited to the study of Dirac fermions in a random $sp(N)$ potential [168]. In the light of our results obtained in the replica and strong disorder approaches, one anticipates that the generalized elliptic integrals will play a rôle here. In particular, they should provide a basis for an infinite series of logarithmic solutions to some (presently unknown) supersymmetric Knizhnik–Zamolodchikov equations. Indeed, a detailed and systematic construction of the invariant tensors and the Knizhnik–Zamolodchikov equations for Lie superalgebras is highly desirable.

In this thesis we have confined ourselves to the study of the LDoS. In particular we have not addressed the subject of conductivity [67]. In the light of recent connections between conductivity in the spin quantum Hall effect [18], and twist operators in percolation [19], one might endeavor to link dense polymer twist operators and conductivity in the Dirac fermion problem. Indeed, it is desirable to develop a simple picture which accounts for the emergence of twist operators, even at the level of the LDoS. In a somewhat more formal context, we note that twist operators arise in the conformal field theory of orbifolds, and it may be fruitful to investigate the rôle of logarithmic operators here — see §4.3 of reference [195].

An interesting (but potentially involved) project would be to investigate the structure of the multipoint correlation functions in the $c = -2$ minimal model and in the WZNW models studied here. Do the elliptic integrals indicate some underlying simplicity? Does the twist operator correspondence extend beyond the four-point functions? The detailed knowledge of such correlation functions is expected to shed light on the exact multifractal spectrum of the random non-Abelian Dirac fermion problem [48], and on the (lower) correlation functions of logarithmic currents in the WZNW models. We note that such non-trival logarithmic currents may play an important rôle in describing the plateau transitions in the integer quantum Hall effect [196]. It may also be illuminating to study the full operator content and character decompositions of many of the models discussed in this thesis.

We hope that we have convinced the reader that the non-Abelian Dirac fermion problem is an instructive example of a disordered critical point, linking apparently disparate approaches and theories.

Appendix A

Hypergeometric Series and Elliptic Integrals

"I do not wish to imply that the hypergeometric function is the only function about which mathematicians know anything. That is far from being true. There are other fertile valleys with which the wooden ploughs of the twentieth century can cope; but the valley inhabited by schoolboys, by engineers, by physicists, and by students of elementary mathematics, is the valley of the Hypergeometric Function, and it's boundaries are (but for one or two small clefts explored by pioneers) virgin rock." W. W. Sawyer, 'Prelude to Mathematics'.

Hypergeometric Series

We gather here some useful properties of the hypergeometric series and their connections to the complete elliptic integrals. A good general reference is [70]. More details may be found in the monographs [191, 192] or in the various books on special functions [69, 188–190]. A generalized hypergeometric series is of the form (see [69] 9.14(1))

$$ {}_pF_q\left[a_1, \ldots, a_p; b_1, \ldots, b_q; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)\ldots(a_p)_n}{(b_1)_n \ldots (b_q)_n} \frac{z^n}{n!} \tag{A.1} $$

where the so-called Pochammer symbol is defined as

$$ (a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{A.2} $$

When $p = 2$ and $q = 1$ we obtain the ordinary hypergeometric series. It is well known in the theory of elliptic integrals that (see [69] 8.113(1) and 8.114(1) and note our different conventions)

$$K(z) = \frac{\pi}{2}\ {}_2F_1\left[\tfrac{1}{2},\tfrac{1}{2};1;z\right] \qquad E(z) = \frac{\pi}{2}\ {}_2F_1\left[-\tfrac{1}{2},\tfrac{1}{2};1;z\right] \tag{A.3}$$

where $K(z)$ is the complete elliptic integral of the first kind and $E(z)$ is the complete elliptic integral of the second kind[1] (see [69] 8.110(1) and the subsequent discussion)

$$K(z) = \int_0^1 \frac{1}{\sqrt{(1-x^2)(1-zx^2)}}dx \qquad E(z) = \int_0^1 \frac{\sqrt{1-zx^2}}{\sqrt{1-x^2}}dx \tag{A.4}$$

The elliptic integrals have rather simple properties under differentiation with respect to the parameter z (see [69] 8.123(2) and 8.132(4) and note our different conventions):

$$\frac{dK(z)}{dz} = \frac{E(z)-(1-z)K(z)}{2z(1-z)} \qquad \frac{dE(z)}{dz} = \frac{E(z)-K(z)}{2z} \tag{A.5}$$

Given the relations (A.3) it is possible to express many other hypergeometric functions in terms of the elliptic integrals. In particular, the definition (A.1) implies that

$${}_2F_1'\left[a_1,a_2;b_1;z\right] = \frac{a_1 a_2}{b_1}\ {}_2F_1\left[a_1+1,a_2+1;b_1+1;z\right] \tag{A.6}$$

and thus from (A.3) and (A.5) one obtains

$${}_2F_1\left[\tfrac{1}{2},\tfrac{3}{2};2;z\right] = -\frac{4}{\pi z}\left[E(z)-K(z)\right] \tag{A.7}$$

Further, using the Gauss recursion formula (see [69] 9.137(3))

$$\begin{aligned}(2\beta-\gamma-\beta z+\alpha z)\ {}_2F_1\left[\alpha,\beta;\gamma;z\right] + (\gamma-\beta)\ {}_2F_1\left[\alpha,\beta-1;\gamma;z\right]&\\ +\beta(z-1)\ {}_2F_1\left[\alpha,\beta+1;\gamma;z\right] = 0&\end{aligned} \tag{A.8}$$

with $\alpha=\beta=1/2$, $\gamma=1$, one finds that

$${}_2F_1\left[\tfrac{1}{2},\tfrac{3}{2};1;z\right] = (1-z)^{-1}\ {}_2F_1\left[\tfrac{1}{2},-\tfrac{1}{2};1;z\right] \tag{A.9}$$

Recalling (A.3) we may recast this in terms of elliptic integrals

$${}_2F_1\left[\tfrac{1}{2},\tfrac{3}{2};1;z\right] = \frac{2E(z)}{\pi(1-z)} \tag{A.10}$$

[1]Note that many texts on the theory of elliptic integrals (including Gradshteyn and Ryzhik [69]) denote the parameter z by k^2 - the so-called modulus. Although this is purely a mater of convention, we find it preferable to deal with functions of z, rather than k^2, or indeed $\sqrt{k}$.

Quadratic Transformations and osp$(2|2)_{-2}$

Amongst the many relations satisfied by the ordinary hypergeometric functions are the quadratic transformations

$$ {}_2F_1[\alpha,\beta;\alpha+\beta+\tfrac{1}{2};4z(1-z)] = \begin{cases} {}_2F_1[2\alpha,2\beta;\alpha+\beta+\tfrac{1}{2};z] & \text{(a)} \\ {}_2F_1[2\alpha,2\beta;\alpha+\beta+\tfrac{1}{2};1-z] & \text{(b)} \end{cases} \tag{A.11} $$

where form (a) is valid inside the loop of the lemniscate $|4z(1-z)| = 1$ surrounding $z = 0$, and form (b) is valid inside the loop surrounding $z = 1$ — see for example 4. (iii) and 4. (v) on page 97 of [191]. In particular, when applied to the ancillary function appearing in (2.163) with $x = 4$ (and $\alpha = -1/4$, $\beta = 3/4$, $\gamma = 1$) one obtains

$$ F_0^- = \begin{cases} [z(1-z)]^{-1/4}\ {}_2F_1\left[-\tfrac{1}{2},\tfrac{3}{2};1;z\right] & \text{(a)} \\ [z(1-z)]^{-1/4}\ {}_2F_1\left[-\tfrac{1}{2},\tfrac{3}{2};1;1-z\right] & \text{(b)} \end{cases} \tag{A.12} $$

Using the Gauss recursion formula (see [69] 9.137(4))

$$ \gamma\ {}_2F_1\left[\alpha,\beta-1;\gamma;z\right] - \gamma\ {}_2F_1\left[\alpha-1,\beta;\gamma;z\right] + (\alpha-\beta)z\ {}_2F_1\left[\alpha,\beta;\gamma+1;z\right] = 0 \tag{A.13} $$

with $\alpha = 1/2$, $\beta = 3/2$, $\gamma = 1$, together with the results (A.3) and (A.7) one finds that

$$ {}_2F_1\left[-\tfrac{1}{2},\tfrac{3}{2};1;z\right] = \frac{2}{\pi}\left[2E(z) - K(z)\right] \tag{A.14} $$

Substituting this result into (A.12) and recalling that the solutions to the osp$(2|2)_{-2}$ Knizhnik–Zamolodchikov equation (2.106a) are obtained through the differential relation $F_3 \propto (1-4D)F_0^-$, one may utilize (A.5) and arrive at the solutions (2.118) stated in the text.

Analytic Continuation Formulae

We gather here some useful analytic properties of the hypergeometric function.

• $c = a + b$

$$ \begin{gathered} {}_2F_1[a,b;a{+}b;z] = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(n!)^2}[\psi_n - \ln(1-z)](1-z)^n \\ \text{where} \quad \psi_n = 2\psi(n+1) - \psi(n+a) - \psi(n+b) \end{gathered} \tag{A.15} $$

• $c = a + b + m$ for $m = 1, 2, 3 \cdots$

$$
\begin{aligned}
{}_2F_1[a, b; a+b+m; z] &= \frac{\Gamma(m)\Gamma(a+b+m)}{\Gamma(a+m)\Gamma(b+m)} \sum_{n=0}^{m-1} \frac{(a)_n (b)_n}{n!(1-m)_n}(1-z)^n \\
&+ \frac{\Gamma(a+b+m)}{\Gamma(a)\Gamma(b)}(-1)^m (1-z)^m \sum_{n=0}^{\infty} \frac{(a+m)_n (b+m)_n}{n!(n+m)!}[\psi_n - \ln(1-z)](1-z)^n
\end{aligned} \tag{A.16}
$$

where $\quad \psi_n = \psi(n+1) + \psi(n+1+m) - \psi(n+a+m) - \psi(n+b+m)$

Continuation of the Complete Elliptic Integrals

In this section we apply the previous formulae to the complete elliptic integrals. The complete elliptic integral of the first kind has the analytic continuation

$$
\begin{aligned}
K(z) &\equiv \frac{\pi}{2}\ {}_2F_1[\tfrac{1}{2}, \tfrac{1}{2}; 1; z] \\
&= \frac{\pi}{2} \frac{\Gamma(1)}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{2})_n}{(n!)^2}[\psi_n - \ln(1-z)](1-z)^n \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{2})_n}{(1)_n n!}[\psi_n - \ln(1-z)](1-z)^n \\
&= \frac{1}{2}\left[\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{2})_n}{(1)_n n!}\psi_n (1-z)^n - \ln(1-z)\ {}_2F_1[\tfrac{1}{2}, \tfrac{1}{2}; 1; 1-z]\right] \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{1}{2})_n}{(1)_n n!}\psi_n (1-z)^n - \frac{1}{\pi}\ln(1-z)K(1-z)
\end{aligned} \tag{A.17}
$$

where $\psi_n = 2[\psi(n+1) - \psi(n+\frac{1}{2})]$. The complete elliptic integral of the second kind has the analytic continuation

$$
\begin{aligned}
E(z) &\equiv \frac{\pi}{2}\ {}_2F_1[-\tfrac{1}{2}, \tfrac{1}{2}; 1; z] \\
&= \frac{\pi}{2}\left[\frac{\Gamma(1)\Gamma(1)}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2})} + \frac{\Gamma(1)}{\Gamma(-\frac{1}{2})\Gamma(\frac{1}{2})}(1-z)\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{3}{2})_n}{n!(2)_n}[\ln(1-z) - \psi_n](1-z)^n\right] \\
&= 1 - \frac{1}{4}(1-z)\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{3}{2})_n}{n!(2)_n}[\ln(1-z) - \psi_n](1-z)^n \\
&= 1 - \frac{1}{4}(1-z)\left[\ln(1-z)\ {}_2F_1[\tfrac{1}{2}, \tfrac{3}{2}; 2; 1-z] - \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{3}{2})_n}{n!(2)_n}\psi_n (1-z)^n\right] \\
&= 1 + \frac{\ln(1-z)}{\pi}[\tilde{E}(z) - \tilde{K}(z)] + \frac{(1-z)}{4}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n (\frac{3}{2})_n}{n!(2)_n}\psi_n (1-z)^n
\end{aligned} \tag{A.18}
$$

where $\psi_n = \psi(n+1) + \psi(n+2) - \psi(n+\frac{1}{2}) - \psi(n+\frac{3}{2})$, and we have used the fact that $(n+1)! = (2)_n$ together with the result that

$$\begin{aligned} {}_2F_1[\tfrac{1}{2},\tfrac{3}{2};2;1-z] &= 4\frac{d}{dz}\,{}_2F_1[-\tfrac{1}{2},\tfrac{1}{2};1;1-z] \\ &= \frac{8}{\pi}\frac{d}{dz}E(1-z) \\ &= -\frac{4}{\pi}\frac{[E(1-z)-K(1-z)]}{1-z}. \end{aligned} \tag{A.19}$$

Generalized Elliptic Integrals

In the course of our study of the random $su(N)$ Dirac fermion problem, we have found it convenient to define the following generalizations of the complete elliptic integrals:

$$K_N(z) \equiv \frac{\pi}{N}\,{}_2F_1\left[1-\tfrac{1}{N},\tfrac{1}{N};1;z\right] \qquad E_N(z) \equiv \frac{\pi}{N}\,{}_2F_1\left[-\tfrac{1}{N},\tfrac{1}{N};1;z\right]. \tag{A.20}$$

The generalized elliptic integrals have rather simple properties under differentiation with respect to the parameter z:

$$\frac{dK_N(z)}{dz} = \frac{E_N(z)-(1-z)K_N(z)}{Nz(1-z)} \qquad \frac{dE_N(z)}{dz} = \frac{E_N(z)-K_N(z)}{Nz}. \tag{A.21}$$

Using the analytic continuation formulae for the hypergeometric series, it is easily seen that the generalized elliptic integrals have the following nontrivial monodromy properties:

$$\mathcal{C}_0\tilde{K}_N(z) = \tilde{K}_N(z) - a_N K_N(z) \tag{A.22}$$

$$\mathcal{C}_0\tilde{E}_N(z) = \tilde{E}_N(z) + a_N[E_N(z) - K_N(z)] \tag{A.23}$$

$$\mathcal{C}_1 K_N(z) = K_N(z) - a_N\tilde{K}_N(z) \tag{A.24}$$

$$\mathcal{C}_1 E_N(z) = E_N(z) + a_N[\tilde{E}_N(z) - \tilde{K}_N(z)] \tag{A.25}$$

where $a_N = 2i\sin(\pi/N)$, together with the trivial transformations $\mathcal{C}_0 K_N(z) = K_N(z)$, $\mathcal{C}_1\tilde{K}_N(z) = \tilde{K}_N(z)$, and similarly for E_N. The reader may find the identities $\Gamma(1+x) = x\Gamma(x)$, and $\Gamma(1-x)\Gamma(x) = \pi/\sin(\pi/x)$ helpful.

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