THE KOORNWINDER POLYNOMIALS ARE THE EIGENFUNCTIONS OF A STURM-LIOUVILLE PROBLEM IN THE SIMPLEX

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Abstract. This paper extends work first published in a 1998 technical report [20] and presented at the International Conference for Spectral and Higher Order Methods (ICOSAHOM) in June of 1998 in Herzliya, Israel.

Motivated by spectral element methods on triangles and tetrahedra we derive a self-adjoint singular Sturm-Liouville problem for the simplex and explicitly compute the eigenfunctions and eigenvalues. The eigenfunctions are the N-dimensional extension of the Koornwinder orthogonal polynomials for the triangle and reduce, in one-dimension, to the Legendre polynomials on the line. This results gives justification for using the Koornwinder polynomials as a basis for triangular spectral element methods – they are the Legendre (uniform weight) polynomials for the simplex. Knowing the singular Sturm-Liouville equation we give an estimate for the convergence rate of eigenfunction expansions of smooth functions in the simplex. The conventional polynomial truncation of degree K used in triangular spectral element methods is given by the set of all Koornwinder polynomials with eigenvalues less than a constant.

Key words. finite element, spectral element, orthogonal polynomials, Sturm-Liouville, triangle, tetrahedra, simplex

AMS subject classifications. 34B24 65M60 65M70 65N35 41A10 41A25 05E35

1. Introduction.

The theory of numerical computation using spectral methods for one-dimension and tensor-product domains is well developed. This is also true for methods on triangles if the maximum degree of the polynomial on each subdomain is small, say 2 or 3. On the quadrilateral, the preferred space is the span of a tensor product basis of Legendre polynomials,

$$\mathcal{P}^2_{\square} = \{\text{span}(x^n y^m) | 0 \le m \le K; 0 \le n \le K\}, \tag{1.1}$$

where K is the maximum degree of the polynomial, or the truncation. On the triangle the preferred space is,

$$\mathcal{P}^2_{\triangle} = \{\text{span}(x^n y^m) | 0 \le m, n \le K; m + n \le K\}. \tag{1.2}$$

Unfortunately, for $K > 7$ 1.2 is nearly dependent if used as a basis for the space and leads to ill-conditioned approximations.

On the quadrilateral, with the diamond truncation, $\mathcal{P}^2_{\square}$, the basis functions are eigenfunctions of singular a Sturm-Liouville problem. Subsequently, the convergence rate for the approximation of smooth functions depends on the regularity of the function being approximated and no special conditions at element boundaries. We would like something similar for the triangle and tetrahedron. In this paper we construct a singular Sturm-Liouville problem on a simplex and show that the eigenfunctions are Legendre-like polynomials. It turns out that our equation in two-dimensions will be a special case of the work of Krall and Sheffer [12], who looked at all classes of Sturm-Liouville problems on the triangle which have polynomial solutions. Similar work was done earlier by Appel and Feriet [2] who used Sturm-Liouville problems to

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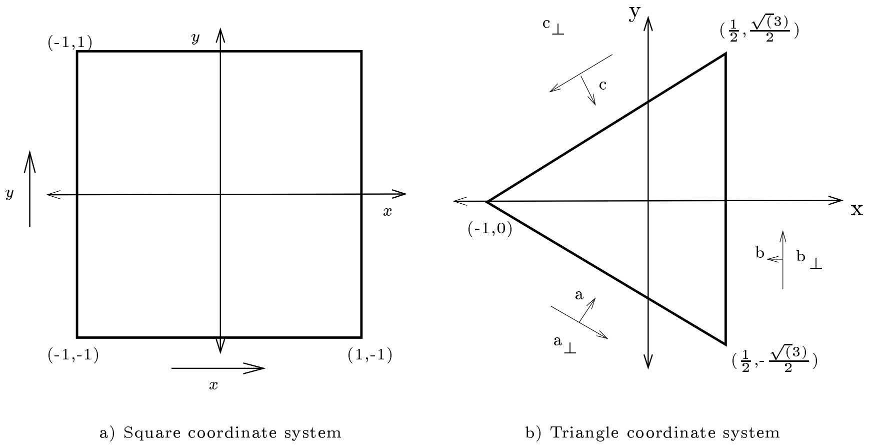

FIG. 2.1. *Coordinate systems in the square and triangle.*

construct the semi-orthogonal Appel polynomials in the triangle. Our results extend this these ideas to the simplex.

In what follows we start with the Sturm-Liouville equation for the square which has a tensor-product of Legendre polynomials as eigenfunctions. In section 3 construct a similar equation for the triangle. The solution to this equation is shown to be the Koornwinder orthogonal polynomials[11] with eigenvalues $\lambda_{mn} = (m+n)*(m+n+2)$. In section 4 we discus the generalization of these results to the simplex. And finally, we give an estimate for the convergence rate of eigenfunction expansions of smooth functions in the simplex in section 5.

This result justifies the practice of using the Koornwinder polynomials, which span $\mathcal{P}^2_\triangle$, as basis functions in triangular spectral element methods [7, 3, 16, 17], and allows classic Sturm-Liouville theory to be used in evaluating the convergence properties of these basis functions.

2. Quadrilateral Eigenvalue Problems. In this section we state the eigenvalue problem on the quadrilateral. The features of this equation will be used to generate a similar eigenvalue problem for the triangle.

Classical one-dimensional theory on the line directly extends to the quadrilateral,

$$\frac{\partial}{\partial x}\big[(1-x^2)\frac{\partial u}{\partial x}\big] + \frac{\partial}{\partial y}\big[(1-y^2)\frac{\partial u}{\partial y}\big] + \lambda u = 0, \tag{2.1}$$

on Ω where $\Omega = \{(x,y)| -1 \le x \le 1, -1 \le y \le 1\}$. This is a singular Sturm-Liouville problem. It is invariant under all rotations and reflections of the square, since these transformations are just permutations and sign changes of x and y and the equations is clearly invariant under such operations. The equations is a sum of terms, each involving a derivative tangential to one boundary (see figure 2.1), and each term has a coefficient which vanishes on boundaries not tangent to the direction of differentiation. These two properties will be used to construct self-adjoint operators in the simplex.

The eigenfunctions of equation 2.1 are tensor products of the form,

$$u = u(x,y) = P^{0,0}_m(x)P^{0,0}_n(y), \tag{2.2}$$

where $P_m^{0,0}(x)$ and $P_n^{0,0}(y)$, the Legendre polynomials of degree m and n respectively. The eigenvalues are,

$$\lambda_{mn} = m(m+1) + n(n+1) \tag{2.3}$$

The set of eigenfunctions described by 2.2 is the basis most commonly used for the approximate solution of partial differential equations on quadrilaterals and combinations of quadrilaterals found in the spectral element method. The truncation normally used is given by $\mathcal{P}^2_\square$ which is a subset of the eigenfunctions with eigenvalue less than $2K(K+1)$. For this truncation, standard arguments from Sturm-Liouville theory are used to show that the convergence rate of an eigenfunction expansion depends only on the regularity of the function being expanded, without imposing boundary conditions on the function. See for example [4] and [6].

3. Sturm-Liouville problem on Triangles. We now construct a Sturm-Liouville equation for the triangle which has similar symmetry and self-adjoint properties as the equation in the square.

To obtain an equation which is invariant under all rotations and reflections of the triangle, we use the barycentric coordinate systems, depicted in figure 2.1b. In the equilateral triangle,

$$T_e^2 = \{(x,y)|x \le \frac{1}{2}; \sqrt{3}y - x \le 1; -\sqrt{3}y - x \le 1\}, \tag{3.1}$$

The group of rotations and reflections (D_3) of the triangle is simply all permutations of a, b and c.

Just as in the case of the square, we take a differential operator which consists of the sum of second order operators, each of which involves a derivative only in the direction tangent to one of the boundaries of the triangle. Also as in the case of the square, we look for quadratic coefficients in each of these terms which vanish along the boundaries which are not tangent to the direction of differentiation, so that the resulting Sturm-Liouville equation will be self-adjoint. Once a term associated with one edge of the triangle is chosen, the remaining terms are generated by permuting the coordinates. The simplest equation based on these principals is

$$\frac{\partial}{\partial a}\big[(a_\perp^2 - 3a^2)\frac{\partial u}{\partial a}\big] + \frac{\partial}{\partial b}\big[(b_\perp^2 - 3b^2)\frac{\partial u}{\partial b}\big] + \frac{\partial}{\partial c}\big[(c_\perp^2 - 3c^2)\frac{\partial u}{\partial c}\big] + 3\lambda u = 0. \tag{3.2}$$

It is easy to check that this equation is self-adjoint and invariant under D_3.

This equation is easier to solve in a more conventional coordinate system, so we map it to the right triangle

$$T_r^2 = \{(x,y)| -1 \le x, y; x + y + 1 \le 1\}. \tag{3.3}$$

In this coordinate system the Sturm-Liouville equation has the form

$$\begin{aligned} &\frac{\partial}{\partial x}\left((1+x)\big[(1-x)\frac{\partial u}{\partial x} - (1+y)\frac{\partial u}{\partial y}\big]\right) \\ &+\frac{\partial}{\partial y}\left((1+y)\big[(1-y)\frac{\partial u}{\partial y} - (1+x)\frac{\partial u}{\partial x}\big]\right) + \lambda u = 0. \end{aligned} \tag{3.4}$$

and we find the Koornwinder polynomials satisfy this equation exactly, which leads us to the following theorem.

THEOREM 3.1. *The Koornwinder polynomials, equations 3.9 - 3.11 are the eigenfunctions to equation 3.4 and the eigenvalues are $(m+n)(m+n+2)$ in T_r^2 for all m, n.*

Proof. The outline of the proof of part one of the theorem is as follows:

1. Simplify equation 3.4 using the coordinate transformation to *warped* space,

$$x = -1 + (1-Y)(1+X)/2, \quad \text{and} \quad y = Y. \tag{3.5}$$

Where the new coordinates, (X, Y) are defined on the square

$$S^2 = \{(X,Y) | -1 \le X \le 1, -1 \le Y \le 1\}. \tag{3.6}$$

2. We derive an equation with derivatives in X using the Rodrigues formula for Legendre polynomials.
3. We derive an equation with derivatives in Y using the Rodrigues formula for Jacobi polynomials of degree n, weights $\alpha = 2m+1$, and $\beta = 0$.
4. We add the two previous equations, to get the equation in step 1, which completes the proof.

Step 1 This step is the well known proof that the Rodrigues formula for Legendre polynomials satisfies the Sturm-Liouville equation. We write down the details here so the reader will see how the equations are constructed for the tetrahedra and the simplex. Using the chain rule we obtain the derivatives in the square coordinate system. They are

$$\frac{\partial}{\partial x} = \frac{2}{1-Y}\frac{\partial}{\partial X} \qquad \frac{\partial}{\partial y} = \frac{1+X}{1-Y}\frac{\partial}{\partial X} + \frac{\partial}{\partial Y}. \tag{3.7}$$

Substituting equations 3.5 and 3.7 into the Sturm-Liouville equation on the triangle, equation 3.4, we obtain,

$$\begin{aligned}\frac{\partial}{\partial X}[(1-X^2)\frac{\partial u(X,Y)}{\partial X}] + \frac{\partial}{\partial Y}[(1-Y^2)\frac{1-Y}{2}\frac{\partial u(X,Y)}{\partial Y}] \\ + \lambda_{mn}\frac{1-Y}{2}u(X,Y) = 0.\end{aligned} \tag{3.8}$$

The Koornwinder functions are written in this coordinate system as

$$u(X,Y) = \overset{1-2}{g}_{mn}(X,Y) = \overset{1}{g}_m(X)\overset{2}{g}_{mn}(Y), \tag{3.9}$$

where

$$\overset{1}{g}_m(X) = P_m^{0,0}(X) \tag{3.10}$$

and.

$$\overset{2}{g}_{nm}(Y) = P_n^{2m+1,0}(Y)\Big(\frac{1-Y}{2}\Big)^m. \tag{3.11}$$

The polynomials, $P_k^{\alpha,\beta}(x)$ are the Jacobi polynomials. They are orthogonal in the inner product

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_n^{\alpha,\beta}(x)P_m^{\alpha,\beta}(x)dx = h_n\delta_{mn}, \tag{3.12}$$

where δ_{mn} is the Dirac delta function. See Abramowitz and Stegan [1] for further details. The Jacobi polynomials with $\alpha = 0$ and $\beta = 0$ are the familiar Legendre polynomials.
The well-known Rodrigues formulas are,

$$\overset{1}{g}_m(X) = \frac{1}{2^m m!}\frac{d^m}{dX^m}(X^2-1)^m, \tag{3.13}$$

and

$$\overset{2}{g}_{mn}(Y) = \frac{1}{2^{n+m}n!}\frac{1}{(1-Y)^{m+1}}\frac{d^n}{dY^n}(Y^2-1)^n(1-Y)^{2m+1}. \tag{3.14}$$

Step 2 We derive an equation that is satisfied by 3.13, and show that it is satisfied equally by $\overset{1-2}{g}_{mn}(X,Y)$. First say $v(X) = (X^2-1)^m$. Differentiate this equation to obtain,

$$v' = 2mX(X^2-1)^{m-1}.$$

Multiplying through by X^2-1 gives,

$$(1-X^2)v' + 2mXv = 0.$$

Here the primed variables indicate differentiation with respect to X. Differentiate repeatedly,

$$(1-X^2)v'' + 2(m-1)Xv' + 2mv = 0$$

$$(1-X^2)v''' + 2(m-2)Xv'' + 2(2m-1)v' = 0.$$

After $m+1$ differentiations,

$$(1-X^2)v^{m+2} - 2Xv^{m+1} + m(m+1)v^m = 0.$$

But

$$\overset{1}{g}_m(X) = P_m^{0,0}(X) = \frac{d^m}{dX^m}v,$$

and the equation becomes,

$$(1-X^2)\frac{d^2P_n^{0,0}}{dX^2} - 2X\frac{dP_n^{0,0}}{dX} + m(m+1)u = 0, \tag{3.15}$$

or

$$\frac{d}{dX}(1-X^2)\frac{dP_n^{0,0}}{dX} + m(m+1)P_n^{0,0} = 0, \tag{3.16}$$

Equation 3.16 is an equation that is satisfied by the Legendre polynomials. Without any difficulty we can multiply this equation by through by $\overset{2}{g}_{mn}(Y)$ to obtain,

$$\frac{\partial}{\partial X}(1-X^2)\frac{\partial u}{\partial X} + m(m+1)u = 0. \tag{3.17}$$

where we have used equation 3.9.

Step 3 We derive an equation that is satisfied by 3.14, and show that it is also satisfied by $\overset{1-2}{g}_{mn}(X,Y)$. We follow the same procedure as for the Legendre polynomials. First say $v(Y) = (Y^2-1)^n(1-Y)^{2m+1}$. Just as before we differentiate this equation to obtain,

$$(1-Y^2)(1-Y)v' + ((2nY(1-Y)-(2m+1)(Y^2-1))v = 0.$$

Here the primed variables indicate differentiation with respect to Y. Differentiating repeatedly,

$$(1-Y^2)v^{n+2} + ((2m-1)Y+2m+1)v^{n+1} + (n+1)(n+2m+1)v^n = 0.$$

And if

$$w = \frac{d^n v}{dY^n},$$

we get

$$(1-Y^2)\frac{d^2w}{dY^2} + ((2m-1)Y+2m+1))\frac{dw}{dY} + (n+1)(n+2m+1)w = 0. \tag{3.18}$$

Since

$$\overset{2}{g}_n(Y) = \frac{1}{2^{n+m}n!}\frac{1}{(1-Y)^{m+1}}\frac{d^n}{d^n}(Y^2-1)^n(1-Y)^{2m+1},$$

we make the change of variables,

$$w = \frac{1}{(1-Y)^{m+1}}q. \tag{3.19}$$

Differentiate equation 3.19 to get,

$$\frac{dw}{dY} = (1-Y)^{m+1}\frac{dq}{dY},$$

and

$$\frac{d^2w}{dY^2} = m(m+1)(1-Y)^{m-1}q - 2(m+1)(1-Y)^m\frac{dq}{dY} + (1-Y)^{m+1}\frac{d^2q}{dY^2}. \tag{3.20}$$

Substitute these into equation 3.18, and rearrange,

$$(1-Y^2)(1-Y)\frac{d^2q}{dY^2} - (1-Y)(3y+1)\frac{dq}{dY} + \big[(n+m)(n+m+2)(1-Y) - 2m(m+1)\big] = 0. \tag{3.21}$$

Which can also be written,

$$\frac{d}{dY}\left[(1-Y^2)(1-Y)\frac{dq}{dY}\right] + \big[(n+m)(n+m+2)(1-Y) - 2m(m+1)\big]q = 0. \tag{3.22}$$

Without any difficulty we can multiply this equation by through by $\overset{1}{g}_m(X)$ to obtain,

$$\frac{\partial}{\partial Y}\left[(1-Y^2)(1-Y)\frac{du}{dY}\right] + \left[(n+m)(n+m+2)(1-Y) - 2m(m+1)u\right] = 0, \tag{3.23}$$

where we have used equation 3.9.

Step 4 Add equations 3.23 and 3.17, we get identically equation 3.8, where the eigenvalues are,

$$\lambda_{mn} = (m+n)(m+n+2). \tag{3.24}$$

Unlike in the one-dimensional case, where each eigenvalue is unique and corresponds to a specific eigenfuncion, the simplex gives families of eigenspaces. The commonly used triangle truncation is a union of these eigenspaces for all eigenvalues less than a constant.

4. Sturm-Liouville on an N-dimensional simplex. The extension of 3.1 to the N-simplex is slightly more complicated. The differential equation is again constructed in a barycentric coordinate system out of terms involving directional derivatives along each face of the simplex. But within each face, we have $N-2$ degrees of freedom for the tangent vector. The tangent vector can thus be chosen so that is also tangent to $N-2$ other faces of the simplex. Then a quadratic polynomial can be constructed which vanishes on the remaining 2 faces of the simplex. Once such a term in constructed for one face of the simplex, the remaining terms are formed by permuting the coordinates. Just as for the triangle, this procedure can be used to produce an invariant, self-adjoint Sturm-Liouville equation for the simplex, with Koornwinder polynomials as eigenvectors. For completeness, we state the result below. Our proof is given in [20]

The Sturm-Liouville equation, in Cartesian coordinates for the N-dimensional right simplex is given by

$$\sum_{k=1}^{N}\frac{\partial}{\partial x_k}\left[(1+x_k)\left((1-x_k)\frac{\partial f(\tilde{x})}{\partial x_k} - \sum_{p=1,p\neq k}^{N}(1+x_p)\frac{\partial f(\tilde{x})}{\partial x_p}\right)\right] + \lambda_{\tilde{m}} f(\tilde{x}) = 0. \tag{4.1}$$

Where $\tilde{x} = (x_1, x_2, ..., x_N)$, $\tilde{m} = (m_1, m_2, ..., m_N)$. and the eigenfunctions are most easly given in a warped coordinate system,

$$f(\tilde{X}) = P_{m_1}^{0,0}(X_1)\prod_{k=2}^{N} P_{m_k}^{\gamma(k),0}(X_k)\left(\frac{1-X_k}{2}\right)^{\sum_{i=1}^{k} m_i}, \tag{4.2}$$

where

$$\gamma(k) = k - 1 + 2\sum_{p=1}^{k-1} m_p, \tag{4.3}$$

with eigenvalues,

$$\lambda_{\tilde{m}} = (\sum_{i=1}^{N} m_i)(N + \sum_{i=1}^{N} m_i). \tag{4.4}$$

The mappings are, for $i = 1, N$,

$$x_i = -1 + (1 + X_i) \prod_{j=i+1}^{N} \left(\frac{1 - X_j}{2}\right), \tag{4.5}$$

and

$$X_i = \frac{2(1 + x_i)}{-\sum_{j=i}^{N}(1 + r_{ij})} - 1, \tag{4.6}$$

where

$$\begin{aligned} r_{ij} &= x_i \quad \text{for} \quad i \neq j \\ &= -3 \quad \text{for} \quad i = j. \end{aligned} \tag{4.7}$$

THEOREM 4.1. *The singular Sturm-Liouville equation 4.1 is self-adjoint in the simplex. Furthermore, the eigenfunctions are the N-dimensional set of products of Jacobi polynomials, equation 4.2 and the eigenvalues are give by equation 4.4 for for all m_i.*

5. Convergence Proof in a Simplex. In this section we give an estimate for the convergence rate for eigenfunction expansions of smooth functions in the simplex. Since we now know the Sturm-Liouville equation, its eigenfunctions, and eigenvalues we apply the standard repeated integration-by-parts argument.

We first define Ω to be a bounded simplex in $\mathbf{R}^N$ with $\mathbf{x} = (x_1, x_2, ..., x_N)$. Let $d\mathbf{x}$ be the usual area measure on $\mathbf{R}^N$.

$$L^2(\Omega) = \left\{ \phi : \Omega \to \mathbf{R} \quad \text{measurable} : \int_\Omega \phi^2(\mathbf{x}) d\mathbf{x} < \infty \right\}. \tag{5.1}$$

The space $L^2(\Omega)$ is a Hilbert space with the inner product

$$(u, v) = \int_\Omega u(\mathbf{x}) v(\mathbf{x}) d\mathbf{x}, \tag{5.2}$$

which induces the norm

$$||u||_{L^2(\Omega)} = \left(\int_\Omega |u(\mathbf{x})|^2 d\mathbf{x} \right)^{\frac{1}{2}}. \tag{5.3}$$

Then as usual we define $H^m(\Omega)$ to be the vector space of functions $u \in L^2(\Omega)$ such that all the distributional derivatives of u of order up to m can be represented by functions in $L^2(\Omega)$. Let $\alpha = (\alpha_1, \alpha_2, ...\alpha_N)$ be a multi-index of non-negative integers. We set $|\alpha| = \alpha_1 + \alpha_2 + \cdots \alpha_N$ and

$$D^\alpha u = \frac{\partial^{\alpha_1 + \cdots + \alpha_2} u}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}}. \tag{5.4}$$

Then we define,

$$H^m(\Omega) = \{u \in L^2(\Omega) : \text{for each non-negative multi-index } \alpha \text{ with } |\alpha| \le m, \tag{5.5}$$
$$\text{the distributional derivative } D^\alpha u \text{ belongs to } L^2(\Omega)\}.$$

This is a Hilbert space with inner product

$$(u,v)_m = \sum_{|\alpha|\le m} \int_\Omega D^\alpha u(\mathbf{x}) D^\alpha v(\mathbf{x}) d\mathbf{x}, \tag{5.6}$$

which induces the norm

$$||u||_{H^m(\Omega)} = \left(\sum_{|\alpha|\le m} ||D^\alpha u||^2_{L^2(\Omega)} \right)^{\frac{1}{2}}. \tag{5.7}$$

Finally, we label the eigenfunctions in 4.2 by $P_{\tilde{m}}$, where $\tilde{m}$ is an N-dimensional multi-index $\tilde{m} = (m_1, m_2, \cdots, m_N)$,

THEOREM 5.1. *For positive integers p and M, there exists a constant C_p such that for $f \in H^{2p}(\Omega)$*

$$||f - \sum_{|\tilde{m}|\le M} c_{\tilde{m}} P_{\tilde{m}}||_{H^0} \le \frac{1}{M^{2p}} C_p ||f||_{H^{2p}}. \tag{5.8}$$

Proof of Theorem 5.1 Let M and p be positive integers. The eigenfunctions, 3.9 are an orthogonal basis for $P^M(\Omega)$, the space of all polynomials up to and including degree M in $L^2(\Omega)$. Polynomials are known to be complete in $L^2(\Omega)$ for Ω, and we assume the eigenfunctions, $P_{\tilde{m}}$, are orthonormalized. Therefore, for any $f \in L^2(\Omega)$, and for some multi-index, $\tilde{m}$, defined the same way as α in equation 5.4, we can write its Fourier series

$$f = \sum_{|\tilde{m}|=0}^{\infty} c_{\tilde{m}} P_{\tilde{m}} \tag{5.9}$$

where the Fourier coefficients are given by

$$c_{\tilde{m}} = (f, P_{\tilde{m}}) \tag{5.10}$$

and the series will converge in the L^2 norm. Next we write the usual error estimate in the L^2 norm as

$$||f - \sum_{|\tilde{m}|\le M} c_{\tilde{m}} P_{\tilde{m}}||_{H^0} = \left(\sum_{|\tilde{m}|>M}^{\infty} c^2_{\tilde{m}} \right)^{\frac{1}{2}}. \tag{5.11}$$

Using the properties of the self-adjoint Sturm-Liouville operator 4.1,

$$\begin{aligned} T(f) = & \sum_{j=1}^{N} \frac{\partial}{\partial x_j} \Bigg[(1+x_j) \Bigg((1-x_j) \frac{\partial f}{\partial x_j} \\ & - \sum_{p=1, p\ne j}^{N} (1+x_p) \frac{\partial f}{\partial x_p} \Bigg) \Bigg] \end{aligned} \tag{5.12}$$

Since

$$T(P_{\tilde{m}}) = -\lambda_{\tilde{m}} P_{\tilde{m}}, \tag{5.13}$$

we have for $|\tilde{m}| > 0$

$$c_{\tilde{m}} = (-1)^p \frac{(f, T^p P_{\tilde{m}})}{\lambda_{\tilde{m}}^p}, \tag{5.14}$$

Using the self-adjoint property of the operator and integrating by parts p times,

$$c_{\tilde{m}} = (-1)^p \frac{(T^p f, P_{\tilde{m}})}{\lambda_{\tilde{m}}^p}. \tag{5.15}$$

Substituting into (5.11) ,

$$||f - \sum_{|\tilde{m}| \le M} c_{\tilde{m}} P_{\tilde{m}}||_{H^0} = \left(\sum_{|\tilde{m}|>M}^{\infty} \frac{(T^p f, P_{\tilde{m}})^2}{\lambda_{\tilde{m}}^{2p}} \right)^{\frac{1}{2}}. \tag{5.16}$$

Using $\lambda_{\tilde{m}} \ge |\tilde{m}|^2$ from 4.4,

$$||f - \sum_{|\tilde{m}| \le M} c_{\tilde{m}} P_{\tilde{m}}||_{H^0} \le \frac{1}{M^{2p}} \left(\sum_{|\tilde{m}|>M}^{\infty} (T^p f, P_{\tilde{m}})^2 \right)^{\frac{1}{2}} \tag{5.17}$$

$$\le \frac{1}{M^{2p}} ||T^p f||_{H^0} \tag{5.18}$$

$$\le \frac{1}{M^{2p}} C_p ||f||_{H^{2p}}. \tag{5.19}$$

The last step comes from the fact that T is a second order operator with coefficients bounded by one so that $||T^p f||_{H_0} \le C_p ||f||_{H^{2p}}$.

6. Summary. In this work we have constructed and solved a self-adjoint Sturm-Liouville equation for the N dimensional simplex. This equation resembles as closely as possible the classic Sturm-Liouville equation in the N-dimensional cube whose eigenfunctions are tensor products of Legendre polynomials. In particular, the equation is second order with quadratic coefficients and is invariant under the group of rotations and reflections of the simplex. The eigenfunctions of this equation are given by the N-dimensional Koornwinder polynomials, with eigenvalues $(\sum_{i=1}^N m_i)(N + \sum_{i=1}^N m_i)$. In one dimension these eigenfunctions reduce to the Legendre polynomials, implying the N-dimensional Koornwinder polynomials are the Legendre polynomials (uniform weight) for the simplex.

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