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On the attractivity of a class of homogeneous dynamic economic systems☆

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Abstract

The attractivity properties of the set of equilibria of a special class of homogeneous dynamic economic systems are examined. The nonlinearity of the models and the presence of eigenvalues with zero real parts make the application of the classical theory impossible. Some principles of the modern theory of dynamical systems and invariant manifolds are applied, and the local attractivity of the set of equilibria is verified under mild conditions. As an application, special labor-managed oligopolies are investigated.

Keywords: Stability; Attractivity; Invariant manifold; Attractor; Labor-managed oligopoly

1. Introduction

Dynamic economic systems have been analyzed by many researchers during the last decade. Among the different model types the most attention has been given to dynamic oligopolies. Okuguchi [11] presented a comprehensive summary of single-product oligopolies without and with product differentiation and also gave a detailed analysis of earlier works on the subject. The existence and uniqueness of the equilibrium is examined and the stability of the equilibrium is analyzed with discrete and continuous

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time scales. The extensions of the models and results for multi-product oligopolies were presented in Okuguchi and Szidarovszky [12], where the different variants of the Cournot model are also discussed including labor-managed oligopolies, rent-seeking games, and models with production adjustment costs.

In this paper the asymptotic behavior of a special labor-managed oligopoly will be examined. The problem can be formulated as follows.

Let us consider an n-firm industry, where all firms are labor-managed. Let us assume the hyperbolic price function

$$p(s) = \frac{b}{s},$$

where s is the total output of the industry, and linear production functions l_i, and linear labor-independent cost functions c_i:

$$l_i(x_i) = a_i x_i \quad \text{and} \quad c_i(x_i) = \alpha_i x_i + \beta_i,$$

where x_i is the output of firm i $(i = 1, \ldots, n)$.

Economic interpretation requires that all parameters b, a_i, α_i, and β_i be positive. The surplus per unit of labor for firm i is given by

$$\begin{aligned} \phi_i(x_1, \ldots, x_n) &= \frac{x_i p(s) - w l_i(x_i) - c_i(x_i)}{l_i(x_i)} \\ &= \frac{b}{a_i(x_i + Q_i)} - w - \frac{\alpha_i}{a_i} - \frac{\beta_i}{a_i x_i}, \end{aligned} \tag{1.1}$$

where $Q_i = \sum_{l \neq i} x_l$ is the output of the rest of the industry, and w is the competitive wage rate. This economic situation can be modeled as an n-person game where the set of strategies for each firm is the interval $X_i = [0, \infty)$ and the payoff function of firm i is ϕ_i.

The existence of positive equilibria will be first examined and their asymptotic behavior will then be analyzed. We will show that there are infinitely many positive non-isolated equilibria under appropriate conditions. As the equilibrium set is connected, the classical Lyapunov theory cannot be used to analyze the asymptotic behavior of the equilibria. It will turn out that the modern theory of dynamical systems and invariant manifolds serves as a useful technique in our case.

This paper is developed as follows. In Section 2 we will examine the existence of equilibria and give a complete description of the equilibrium set. Then the dynamic extension of the model will be introduced with continuous time scales. The major attractivity properties of the equilibria will be formulated in Section 3. In Section 4 we will introduce and discuss the main theoretical issues and then apply these results in Section 5 to analyze the asymptotic behavior of the equilibrium set in a special class of homogeneous systems that includes our dynamic model as a special case. In Section 6 we will present an elementary proof of the strong attractivity part of our main result based on simple techniques in solving homogeneous systems.

2. Existence of positive equilibria

Let us consider the Nash–Cournot equilibrium of an n-person game. It is a vector of simultaneous strategies (in our case a vector of production levels) which can be considered as a steady state in the sense that no player can improve its payoff by unilaterally moving away from the equilibrium. In the dynamic extensions, the Nash–Cournot equilibria are usually the steady states of the dynamic systems [10,12].

For each firm i and $Q_i > 0$, the best response can be obtained as

$$x_i(Q_i) = \operatorname*{argmax}_{x_i \geqslant 0} \left\{ \frac{b}{a_i(x_i + Q_i)} - w - \frac{\alpha_i}{a_i} - \frac{\beta_i}{a_i x_i} \right\}.$$

Assuming an interior optimum, the first-order conditions are given in the following way:

$$-\frac{b}{a_i\,(x_i + Q_i)^2} + \frac{\beta_i}{a_i x_i^2} = 0,$$

which can be written as

$$x_i = \frac{\sqrt{\beta_i}}{\sqrt{b} - \sqrt{\beta_i}}\, Q_i \quad (i = 1, 2, \ldots, n). \tag{2.1}$$

In order to ensure that $x_i > 0$, we have to assume that $\beta_i < b$. The second-order conditions are always satisfied since at the optimum

$$\frac{2b}{a_i(x_i + Q_i)^3} - \frac{2\beta_i}{a_i x_i^3} = \frac{2\beta_i}{a_i x_i^3} \left(\sqrt{\frac{\beta_i}{b}} - 1 \right) < 0.$$

From Eq. (2.1) we have

$$s = x_i + Q_i = x_i \left(1 + \frac{\sqrt{b} - \sqrt{\beta_i}}{\sqrt{\beta_i}} \right) = x_i\, \frac{\sqrt{b}}{\sqrt{\beta_i}}$$

and finally,

$$1 = \frac{\sum_{i=1}^n x_i}{s} = \frac{\sum_{i=1}^n \sqrt{\beta_i}}{\sqrt{b}}.$$

The payoff of firm i at any equilibrium is

$$\phi_i(\hat{x}_1, \ldots, \hat{x}_n) = \frac{b}{a_i \hat{s}} - w - \frac{\alpha_i}{a_i} - \frac{\beta_i}{a_i \hat{x}_i} = \frac{b}{a_i \hat{s}} \left(1 - \frac{\sqrt{\beta_i}}{\sqrt{b}} \right) - w - \frac{\alpha_i}{a_i},$$

which is positive for all i if and only if $\hat{s}$ is sufficiently small:

$$\hat{s} < \min_i \left\{ \frac{\sqrt{b}(\sqrt{b} - \sqrt{\beta_i})}{a_i w + \alpha_i} \right\}. \tag{2.2}$$

Thus we have proved the following:

Proposition 2.1. *Assume that $\beta_i < b$ for all firms. Then positive equilibria of the labor-managed oligopoly exists, if and only if*

$$\sum_{i=1}^{n} \sqrt{\beta_i} = \sqrt{b}.$$

If this condition is satisfied then $\hat{x}_i = \sqrt{\beta_i}/\sqrt{b}\hat{s}$ $(i = 1, \ldots, n)$ is an equilibrium with any positive $\hat{s}$ satisfying relation (2.2), and all positive equilibria can be obtained in this way.

3. The dynamic model and attractivity analysis

Let us assume the continuous time scale and that at each time period each firm adjusts its output proportionally to its marginal profit. The resulting dynamic model is

$$\dot{x}_i = k_i \left(-\frac{b}{a_i s^2} + \frac{\beta_i}{a_i x_i^2} \right) \qquad (i = 1, \ldots, n), \tag{3.1}$$

where the speed of adjustment $k_i > 0$ is a given constant for each i. The first step in analyzing the asymptotic behavior of the system is to determine the Jacobian of the right-hand sides. Simple differentiation shows that at any equilibrium

$$J = \frac{2}{\hat{s}^3} \begin{pmatrix} k_1 \left(\frac{b}{a_1} - \frac{\beta_1 \hat{s}^3}{a_1 \hat{x}_1^3} \right) & k_1 \frac{b}{a_1} & \cdots & k_1 \frac{b}{a_1} \\ k_2 \frac{b}{a_2} & k_2 \left(\frac{b}{a_2} - \frac{\beta_2 \hat{s}^3}{a_2 \hat{x}_2^3} \right) & \cdots & k_2 \frac{b}{a_2} \\ \vdots & \vdots & \vdots & \vdots \\ k_n \frac{b}{a_n} & k_n \frac{b}{a_n} & \cdots & k_n \left(\frac{b}{a_n} - \frac{\beta_n \hat{s}^3}{a_n \hat{x}_n^3} \right) \end{pmatrix} = \frac{2}{\hat{s}^3} (\mathbf{D} + \mathbf{a}\mathbf{1}^{\mathrm{T}}),$$

where

$$\mathbf{D} = \mathrm{diag}\left(-k_1 \frac{\beta_1 \hat{s}^3}{a_1 \hat{x}_1^3}, \ldots, -k_n \frac{\beta_n \hat{s}^3}{a_n \hat{x}_n^3} \right), \qquad \mathbf{a} = \left(k_1 \frac{b}{a_1}, \ldots, k_n \frac{b}{a_n} \right)^{\mathrm{T}}$$

$$\text{and} \qquad \mathbf{1} = (1, \ldots, 1)^{\mathrm{T}}.$$

After neglecting the factor $2/\hat{s}^3$, the characteristic polynomial of matrix $\mathbf{D} + \mathbf{a}\mathbf{1}^{\mathrm{T}}$ can be written as

$$\begin{aligned} \det(\mathbf{D} - \lambda \mathbf{I} + \mathbf{a}\mathbf{1}^{\mathrm{T}}) &= \det(\mathbf{D} - \lambda \mathbf{I}) \cdot \det(\mathbf{I} + (\mathbf{D} - \lambda \mathbf{I})^{-1} \mathbf{a}\mathbf{1}^{\mathrm{T}}) \\ &= \prod_{i=1}^{n} \left(-k_i \frac{\beta_i \hat{s}^3}{a_i \hat{x}_i^3} - \lambda \right) \left(1 + \sum_{i=1}^{n} \frac{k_i b / a_i}{-k_i (\beta_i \hat{s}^3 / a_i \hat{x}_i^3) - \lambda} \right), \end{aligned} \tag{3.2}$$

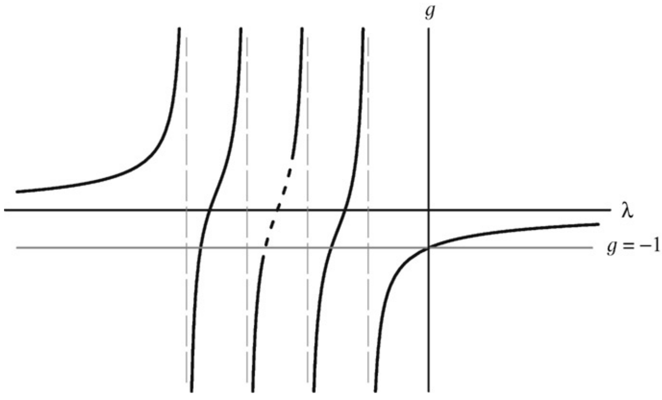

Fig. 1. The graph of function g.

where we used the Morrison's identity [5].

$$\det(\mathbf{I} + \mathbf{u}\mathbf{v}^{\mathrm{T}}) = 1 + \mathbf{v}^{\mathrm{T}}\mathbf{u}$$

with $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $\mathbf{I}$ being the $n \times n$ identity matrix. This relation can be easily proved by using the finite induction with respect to n. The roots of the first product are negative. In order to examine the locations of the roots of the second factor we introduce function (Fig. 1)

$$g(\lambda) = \sum_{i=1}^{n} \frac{k_i b/a_i}{-k_i(\beta_i \hat{s}^3/a_i \hat{x}_i^3) - \lambda}.$$

It is easy to see that it satisfies the following properties:

$$g(0) = -1, \qquad \lim_{\lambda \to \pm\infty} g(\lambda) = 0, \qquad \lim_{\lambda \to -k_i(\beta_i \hat{s}^3/a_i \hat{x}_i^3) - 0} g(\lambda) = +\infty,$$

$$\lim_{\lambda \to -k_i(\beta_i \hat{s}^3/a_i \, \hat{x}_i^3) + 0} g(\lambda) = -\infty$$

and

$$g'(\lambda) = \sum_{i=1}^{n} \frac{k_i b/a_i}{(-k_i(\beta_i \hat{s}^3/a_i \hat{x}_i^3) - \lambda)^2} > 0.$$

These relations and the fact that the equation $g(\lambda)+1=0$ is equivalent to a polynomial equation of degree n imply that all roots of the second factor of (3.2) are real: zero is a single root, and all other roots are negative by the Intermediate Value Theorem. The presence of a zero eigenvalue shows that based only on eigenvalue analysis, the asymptotic properties of any equilibrium are undetermined. Since the set of equilibria is an infinite connected set, Lyapunov theory cannot be applied either. As we will demonstrate in the next section, the modern theory of dynamical systems may serve as a useful tool to analyze the attractivity of the equilibrium set of this system and in general, of a certain class of homogeneous systems.

The main result can be summarized as

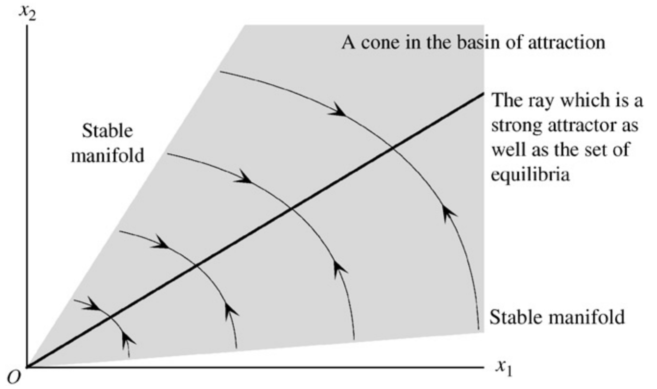

Fig. 2. The illustration of Theorem 3.1 in the case that $n = 2$, $k_1/a_1 = 1$, $k_2/a_2 = 0.5$, $\beta_1 = 25$, $\beta_2 = 9$, and $b = 64$.

Theorem 3.1. *Let us assume that the conditions of Proposition* 2.1 *are satisfied. Then in dynamical system* (3.1) (Fig. 2)

1. *The set of equilibria is an open ray starting from the origin, given by the parametric equation*
$$x_i = \frac{\sqrt{\beta_i}}{\sqrt{b}}\, s$$
where $s > 0$ *and* $i = 1, \ldots, n$.
2. *The ray in* 1 *is a strongly attracting set, i.e., any point near the ray is attracted to some particular point on the ray.*
3. *The basin of attraction contains a cone which is centered at the ray in* 1.
4. *At each point of the ray in* 1, *there is a stable manifold transversing the ray. These stable manifolds are mutually disjoint. They are* C^1*-continuous depending on the points on the ray. Furthermore, the cone described in* 3 *can be partitioned into the union of the local stable manifolds.*

4. Mathematical preliminaries

In this section, we are going to review some basic concepts and then introduce the relevant results of the modern theory of dynamical systems. For further details the reader can refer to the literature, e.g., see [1–4,6–9]. It is worthwhile to point out that most of the results remain valid for both maps and flows.

An n-dimensional topological manifold M is a set of points that locally looks like $\mathbb{R}^n$ via an atlas. Namely, for each $\mathbf{x} \in M$, there exists a neighborhood U of $\mathbf{x}$ and a homeomorphism h, such that h maps U to an open ball $B \subset \mathbb{R}^n$. A pair of (U, h) is called a chart or a system of local coordinates. A collection of charts is called an atlas. For any two charts (U_1, h_1) and (U_2, h_2) associated to the same point $\mathbf{x}$, the coordinate change $h_2 \circ h_1^{-1}$ is a homeomorphism on $h_1(U_1 \cap U_2) \subset B_1$. By gathering all charts compatible with the present ones, a unique maximal atlas is defined. If $h_2 \circ h_1^{-1}$ is r times continuously differentiable for every two charts then the resulting atlas is called a C^r-atlas and the resulting manifold is called a C^r-manifold. A k-dimensional ($k \leqslant n$) submanifold V of M is itself a differentiable manifold as well as a subset of M such that the maximal atlas of M contains a chart (U, h) for which the induced map $h|_{U\cap V}$ on $U \cap V$ maps to $\mathbb{R}^k \times \{\mathbf{0}\} \subset \mathbb{R}^n$, and defines charts for V compatible with the differentiable structure of V [4,9].

At each point $\mathbf{x} \in M$, there is an n-dimensional linear space $T_{\mathbf{x}}M$, called the tangent space of M at point $\mathbf{x}$, attached to M at $\mathbf{x}$. Each tangent vector $\mathbf{v}$ in $T_{\mathbf{x}}M$ represents an equivalence class of curves through $\mathbf{x}$ in M such that in any local coordinate system, $\mathbf{v}$ is tangent to all these curves at $\mathbf{x}$ in the Euclidean space. The disjoint union (or collection) of all these tangent spaces is called the tangent bundle of M, denoted as T_M [4,9].

A Riemannian manifold is a differentiable manifold with an inner product $g_{\mathbf{x}}(\cdot,\cdot)$ on each tangent space, which depends smoothly on the base point. The collection of the inner products along with the differentiable structure is called a Riemannian structure. If the phase "inner product" is replaced with "norm" in the above definition, the corresponding manifold is called a Finsler manifold and the collection of the norms along with the differentiable structure is called a Finsler structure [4,9].

A continuous dynamical system is a continuous time evolution process occurring in some phase space. Let the phase space M be a differentiable manifold of dimension n. A flow is the time evolution given by a differentiable function $\mathbf{F}(\mathbf{x}, t) = \varphi^t(\mathbf{x})$, $\mathbf{x} \in M, t \in \mathbb{R}$, which satisfies the group composition property $\varphi^t \circ \varphi^t = \varphi^{t+s}$. When we fix $\mathbf{x} \in M$ and vary t, we obtain a parameterized differentiable curve, which is called the flow line with initial condition $\mathbf{x}$. Let $\mathbf{f}(\mathbf{x})$ be the tangent vector to this curve at $t = 0$, i.e., at $\mathbf{x}$. The map $\mathbf{x} \to \mathbf{f}(\mathbf{x})$ forms a differentiable vector field on M in the tangent bundle. Note: $\mathbf{x} \in M$ and $\mathbf{f}(\mathbf{x}) \in T_{\mathbf{x}}M$. Thus we have constructed a differential equation system

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}), \tag{4.1}$$

to which the flow $\varphi^t(\mathbf{x})$ is a solution. The local existence and uniqueness of the solution can be guaranteed by some mild conditions on the vector field $\mathbf{f}(\mathbf{x})$, for instance, continuous differentiability. If the solution exists for all real values of t, then the vector field is called complete. Accordingly, if the solution exists for all positive (negative) values of t, then the vector field is called forward (backward) semi-complete. An example of a complete vector field is the one defined on a closed compact manifold. An example of a forward semi-complete vector field is the one in dynamical system (3.1). The solution to (3.1) with initial point near an axis reaches the axis within a finite backward time period.

An equilibrium $\hat{\mathbf{x}}$ of system (4.1) is the initial condition whose flow line is only a point, that is, $\varphi^t(\hat{\mathbf{x}}) = \hat{\mathbf{x}}$ for all t or, equivalently, $\mathbf{f}(\hat{\mathbf{x}}) = \mathbf{0}$. An equilibrium $\hat{\mathbf{x}}$ is asymptotically stable if for any $\varepsilon > 0$ there exists a neighborhood of $\hat{\mathbf{x}}$ such that for any initial condition $\mathbf{x}$ chosen in this neighborhood the flow line $\varphi^t(\mathbf{x})$ lies within the ball $B(\hat{\mathbf{x}}, \varepsilon)$ and $\lim_{t\to\infty} \varphi^t(\mathbf{x}) = \hat{\mathbf{x}}$. Similarly, one can define the asymptotic instability by using the backward time $t < 0$ and let $t \to -\infty$.

An invariant set, V, of a flow $\varphi^t(\mathbf{x})$ in M is the set where any flow initiated in it will stay in it in both forward and backward time, i.e., for any $\mathbf{x} \in V$, $\varphi^t(\mathbf{x}) \in V$ for all t. Especially, V is called invariant manifold if V is also a submanifold of M. It is easy to see an invariant manifold is tangent to the corresponding vector field at each of its points [3]. An invariant set V is called attractive (or repulsive) if there exists a neighborhood U of V, such that for any $\mathbf{x} \in U$, $\varphi^t(\mathbf{x})$ converges to V as $t \to \infty$ (or $t \to -\infty$). The set of all points which are attracted (or repulsed) by V is called the basin of attraction (or repulsion) of V [1]. V is called strongly attractive (or repulsive) if $\varphi^t(\mathbf{x})$ converges to some point in V for any initial condition $\mathbf{x}$ in the basin of V as $t \to \infty$ (or $t \to -\infty$).

A stable manifold $W^S(\hat{\mathbf{x}})$ of a flow $\varphi^t(\mathbf{x})$ at an equilibrium $\hat{\mathbf{x}}$ is a submanifold of M such that: (1) it is an invariant set of flow $\varphi^t(\mathbf{x})$, (2) it is classified as the set of points $\mathbf{x}$, where $\varphi^t(\mathbf{x}) \to \hat{\mathbf{x}}$ exponentially as $t \to \infty$. Replacing $t \to \infty$ with $t \to -\infty$, we get the definition of an unstable manifold $W^u(\hat{\mathbf{x}})$ of a flow $\varphi^t(\mathbf{x})$ at an equilibrium $\hat{\mathbf{x}}$.

As a well-known fact, the local stability and instability of a flow at an equilibrium is determined by the behavior of the local linearization of the flow at the equilibrium, by comparing the magnitudes of the eigenvalues of $D\varphi^t(\mathbf{x})$, the Jacobian or the derivative of φ^t with respect to $\boldsymbol{x}$, at $\hat{\mathbf{x}}$ to 1. This is equivalent to the comparison of the real parts of the eigenvalues of $D\mathbf{f}(\hat{\mathbf{x}})$, the Jacobian of $\mathbf{f}$ at $\hat{\mathbf{x}}$, to zero in system (4.1), since it is easy to verify that $D\varphi^t(\hat{\mathbf{x}}) = \exp(D\mathbf{f}(\hat{\mathbf{x}})t)$. A sufficient condition for asymptotic stability (instability) is that $\mathrm{Re}(\lambda_i) < 0$ (> 0) for every eigenvalue λ_i of $D\mathbf{f}(\hat{\mathbf{x}})$.

In stability theory, equivalence relations are a frequently used tool. We next introduce two special equivalence relations which are called flow equivalence (or conjugacy) and orbit equivalence.

Two C^r flows $\varphi^t : M \to M$, and $\psi^t : N \to N$ are said to be C^m ($m \leqslant r$) *flow equivalent or conjugate* if there exists a C^m diffeomorphism $\mathbf{h} : M \to N$ such that $\varphi^t = \mathbf{h}^{-1} \circ \psi^t \circ \mathbf{h}$ for all $t \in \mathbb{R}$. Usually, $\mathbf{h}$ is called a conjugacy. If $\mathbf{h}$ is only a C^m onto mapping and satisfies $\mathbf{h} \circ \varphi^t = \psi^t \circ \mathbf{h}$, then φ^t and ψ^t are said to be semi-conjugate, and $\mathbf{h}$ is called a semi-conjugacy.

Note that conjugacy or semi-conjugacy can be viewed as coordinate change, as it is illustrated in Fig. 3.

A flow ψ^t on M is a *time change* of another flow φ^t if for each $\mathbf{x} \in M$ the orbits $\mathcal{O}_\varphi(\mathbf{x}) = \{\varphi^t(\mathbf{x})\}_{t\in\mathbb{R}}$ and $\mathcal{O}_\psi(\mathbf{x}) = \{\psi^t(\mathbf{x})\}_{t\in\mathbb{R}}$ coincide and the orientations given by the change of t in the positive direction are the same.

Two C^r flows $\varphi^t : M \to M$, and $\psi^t : N \to N$ are said to be C^m ($m \leqslant r$) *orbit equivalent* if there exists a C^m diffeomorphism $h : M \to N$ such that $\chi^t = h^{-1} \circ \psi^t \circ h$ is a time change of φ^t.

A flow $\psi^t : N \to N$ is an *orbit factor* of $\varphi^t : M \to M$ if there exists an onto continuous map $h : M \to N$ that takes orbits of φ^t onto orbits of ψ^t.

Suppose φ^t and ψ^t, corresponding to vector fields $\mathbf{f}$ and $\mathbf{g}$, respectively, are semi-conjugate via $\mathbf{h}$. Then

$$\mathbf{h}(\varphi^t(\mathbf{x})) = \psi^t(\mathbf{h}(\mathbf{x})). \tag{4.2}$$

$$\begin{tikzcd} M \arrow[r, "\phi^t"] \arrow[d, "h"'] & M \arrow[d, "h"] \\ N \arrow[r, "\psi^t"'] & N \end{tikzcd}$$

Fig. 3. A diagram of semi-conjugacy.

Differentiating both sides of (4.2) with respect to t we obtain

$$\begin{aligned} D\mathbf{h}(\varphi^t(\mathbf{x}))\mathbf{f}(\varphi^t(\mathbf{x})) &= D\mathbf{h}(\varphi^t(\mathbf{x}))\,\frac{\mathrm{d}\varphi^t(\mathbf{x})}{\mathrm{d}t} \\ &= \frac{\mathrm{d}\psi^t(\mathbf{h}(\mathbf{x}))}{\mathrm{d}t} = \mathbf{g}(\psi^t(\mathbf{h}(\mathbf{x}))) = \mathbf{g}(\mathbf{h}(\varphi^t(\mathbf{x}))), \end{aligned}$$

that is,

$$D\mathbf{h}(\mathbf{x})\mathbf{f}(\mathbf{x}) = \mathbf{g}(\mathbf{h}(\mathbf{x})). \tag{4.3}$$

So the Jacobian of a semi-conjugacy, $D\mathbf{h}$, maps the vector field of one flow to the vector field of its semi-conjugate flow.

Suppose $\hat{\mathbf{x}}$ is an equilibrium of φ^t. Then $\psi^t(\mathbf{h}(\hat{\mathbf{x}})) = \mathbf{h}(\varphi^t(\hat{\mathbf{x}})) = \mathbf{h}(\hat{\mathbf{x}})$, so $\mathbf{h}(\hat{\mathbf{x}})$ is an equilibrium of ψ^t. Thus semi-conjugacy preserves equilibria.

Differentiate both sides of (4.2) with respect to $\mathbf{x}$ to get

$$D\mathbf{h}(\varphi^t(\mathbf{x}))\cdot D\varphi^t(\mathbf{x}) = D\psi^t(\mathbf{h}(\mathbf{x}))\cdot D\mathbf{h}(\mathbf{x}). \tag{4.4}$$

Rewrite (4.4) at equilibrium $\hat{\mathbf{x}}$ as

$$D\mathbf{h}(\hat{\mathbf{x}})\cdot D\varphi^t(\hat{\mathbf{x}}) = D\psi^t(\mathbf{h}(\hat{\mathbf{x}}))\cdot D\mathbf{h}(\hat{\mathbf{x}}). \tag{4.5}$$

Differentiating both sides of (4.5) with respect to t and interchanging the order of differentiation, which can be done since the partial derivatives are continuous, we get

$$D\mathbf{h}(\hat{\mathbf{x}})\cdot D\mathbf{f}(\hat{\mathbf{x}}) = D\mathbf{g}(\mathbf{h}(\hat{\mathbf{x}}))\cdot D\mathbf{h}(\hat{\mathbf{x}}). \tag{4.6}$$

If $\mathbf{h}$ is a conjugacy then $D\mathbf{h}(\hat{\mathbf{x}})$ is nonsingular, and $D\mathbf{f}(\hat{\mathbf{x}})$ and $D\mathbf{g}(\mathbf{h}(\hat{\mathbf{x}}))$, the Jacobians of $\mathbf{f}$ and $\mathbf{g}$ at the equilibria $\hat{\mathbf{x}}$ and $\mathbf{h}(\hat{\mathbf{x}})$, respectively, are thus similar. Hence conjugacy preserves the eigenvalues of the Jacobian of the vector field of any of the two conjugate flows at its equilibrium.

If $\mathbf{h}$ is only a semi-conjugacy then $D\mathbf{h}(\hat{\mathbf{x}})$ may be singular and $\mathbf{h}$ may not preserve all the eigenvalues. However, let λ be an eigenvalue of $D\mathbf{f}(\hat{\mathbf{x}})$ and let $\mathbf{a}$ be one of its eigenvectors. Then $D\mathbf{f}(\hat{\mathbf{x}})\mathbf{a} = \lambda\mathbf{a}$ and $D\mathbf{g}(\mathbf{h}(\hat{\mathbf{x}}))D\mathbf{h}(\hat{\mathbf{x}})\mathbf{a} = D\mathbf{h}(\hat{\mathbf{x}})D\mathbf{f}(\hat{\mathbf{x}})\mathbf{a} = D\mathbf{h}(\hat{\mathbf{x}})(\lambda\mathbf{a}) = \lambda D\mathbf{h}(\hat{\mathbf{x}})\mathbf{a}$. If $D\mathbf{h}(\hat{\mathbf{x}})\mathbf{a} \neq \mathbf{0}$, then λ is an eigenvalue of $D\mathbf{g}(\mathbf{h}(\hat{\mathbf{x}}))$. Therefore, only those eigenvalues of the Jacobian of the vector field at the equilibrium are preserved which have an eigenvector whose image mapped under the Jacobian of the semi conjugacy is nonzero. In other words, the eigenvalues whose generalized eigenspaces are mapped to the null space by the Jacobian of a semi-conjugacy will not be preserved under the semi-conjugacy. On the other hand, the following Lemma 4.1 guarantees that all eigenvalues of $D\mathbf{g}(\mathbf{h}(\hat{\mathbf{x}}))$ are also eigenvalues of $D\mathbf{f}(\hat{\mathbf{x}})$.

Lemma 4.1. *Let* $\mathbf{A}$, $\mathbf{B}$, *and* $\mathbf{P}$ *be matrices with sizes* $n \times n$, $m \times m$, *and* $m \times n$, *respectively, satisfying* $\mathbf{PA} = \mathbf{BP}$ *and rank*$(\mathbf{P}) = m$. *Then the set of all eigenvalues of* $\mathbf{B}$ *is contained in the set of all eigenvalues of* $\mathbf{A}$. *Moreover, the Jordan form of* $\mathbf{B}$ *can be obtained from the Jordan form of* $\mathbf{A}$ *by deleting* $n-m$ *rows and* $n-m$ *columns in the Jordan form of* $\mathbf{A}$ *corresponding to all generalized eigenvectors* $\mathbf{v}$ *for which* $\mathbf{Pv} = \mathbf{0}$.

Proof. See Appendix A.1. □

In summary, we have

Proposition 4.2. *A semi-conjugacy maps the vector field of a flow to the vector field of another flow by using Eq.* (4.3). *A semi-conjugacy preserves equilibrium and preserves those eigenvalues of the Jacobian of the vector field at the equilibrium, whose generalized eigenspaces are not mapped to the null space by the Jacobian of the semi-conjugacy. Especially, a conjugacy preserves equilibrium and all eigenvalues of the Jacobian of the vector field at any equilibrium. Thus a semi-conjugacy preserves the local stability properties of a dynamical system at an equilibrium.*

Let φ^t be a time change of ψ^t via $t = \alpha(s, \mathbf{x})$, i.e., $\varphi^t(\mathbf{x}) = \varphi^{\alpha(s,\mathbf{x})}(\mathbf{x}) = \psi^s(\mathbf{x})$. Let $\hat{\mathbf{x}}$ be an equilibrium of ψ^t. Then $\varphi^t(\hat{\mathbf{x}}) = \psi^s(\hat{\mathbf{x}}) \equiv \hat{\mathbf{x}}$. Hence, $\hat{\mathbf{x}}$ is also an equilibrium of φ^t.

Note that $\partial\alpha/\partial s > 0$ by the requirement of the same orientation of the time change. By the semi-group property $\psi^s(\mathbf{x}) = \psi^0(\psi^s(\mathbf{x}))$, we see $t = \alpha(s, \mathbf{x}) = \alpha(0, \psi^s(\mathbf{x}))$. Also $\mathrm{d}t = \partial\alpha(s, \mathbf{x})/\partial s \, \mathrm{d}s$ on the flow line with initial condition $\mathbf{x}$. Then we have

$$\frac{\mathrm{d}\psi^s(\mathbf{x})}{\mathrm{d}s} = \frac{\mathrm{d}\varphi^t(\mathbf{x})}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{\partial\alpha(s,\mathbf{x})}{\partial s}\mathbf{f}(\psi^s(\mathbf{x})) = \left.\frac{\partial\alpha(s,\psi^s(\mathbf{x}))}{\partial s}\right|_{s=0}\mathbf{f}(\psi^s(\boldsymbol{x})). \tag{4.7}$$

Let $\mathbf{g}$ be the vector field of ψ^t and $\tau(\mathbf{x}) = \partial\alpha(s, \mathbf{x})/\partial s|_{s=0}$. Relation (4.7) immediately derives $\mathbf{g}(\mathbf{x}) = \tau(\mathbf{x})\mathbf{f}(\mathbf{x})$. Then for any vector $\mathbf{v} \in \mathbb{R}^n$,

$$D\mathbf{g}(\mathbf{x})\mathbf{v} = (D\tau(\mathbf{x})\mathbf{v})\mathbf{f}(\mathbf{x}) + \tau(\mathbf{x})D\mathbf{f}(\mathbf{x})\mathbf{v}.$$

At equilibrium $\hat{\mathbf{x}}$, $\mathbf{f}(\hat{\mathbf{x}}) = \mathbf{0}$. So

$$D\mathbf{g}(\hat{\mathbf{x}})\mathbf{v} = \tau(\hat{\mathbf{x}})D\mathbf{f}(\hat{\mathbf{x}})\mathbf{v},$$

i.e.

$$D\mathbf{g}(\hat{\mathbf{x}}) = \tau(\hat{\mathbf{x}})D\mathbf{f}(\hat{\mathbf{x}}).$$

Hence the Jacobians of $\mathbf{g}$ and $\mathbf{f}$ are positive scalar multiples of each other. Therefore, we obtain the following proposition.

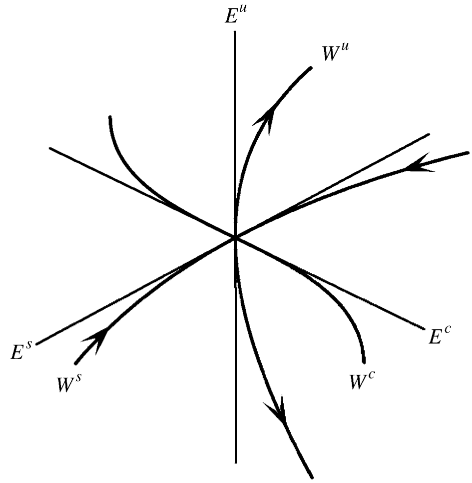

Fig. 4. The stable, unstable, and center manifolds.

Proposition 4.3. *The vector field of a time change ψ^t from flow φ^t is obtained by rescaling the length of each vector in the vector field of φ^t with a positive factor. Therefore, time change preserves equilibrium and preserves the signs of the real parts of the eigenvalues of the Jacobian of the vector field at any equilibrium. Thus a time change preserves the local stability properties of a dynamical system at any equilibrium.*

We will next introduce some known theorems which will be later used in establishing the main result of this paper.

Theorem 4.4 (Center manifold theorem for flows; Arnold and Il'yashenko [3], Guckenheimer and Holmes [6]). *Let $\mathbf{f}$ be a C^{r+1} $(0 \leqslant r \leqslant \infty)$ vector field with equilibrium $\hat{\mathbf{x}}$ and let $\mathbf{A} = D\mathbf{f}(\hat{\mathbf{x}})$. Divide the spectra (i.e. the set of eigenvalues) into three parts, σ_-, σ_+, σ_0 corresponding to the eigenvalues of $\mathbf{A}$ with negative, positive, and zero real parts, respectively. Let the eigenspaces of σ_-, σ_+, σ_0 be E^s, E^u, E^c, respectively. Then the differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ has invariant manifolds W^s, W^u, and W^c of class C^{r+1}, C^{r+1}, C^r, respectively, with dimensions coinciding with E^s, E^u, E^c, respectively, which go through $\hat{\mathbf{x}}$ and are tangent to E^s, E^u, and E^c, respectively, at $\hat{\mathbf{x}}$. Solutions with initial conditions on W^s (resp. W^u) tend exponentially to $\hat{\mathbf{x}}$ as $t \to \infty$ (resp. $-\infty$). W^s is called the stable manifold, W^u the unstable manifold, and W^c the center manifold of the equilibrium $\hat{\mathbf{x}}$. The stable and unstable manifolds are unique, but the center manifold need not be.*

The statement of this theorem is illustrated in Fig. 4.

The attractivity property of a center manifold is given as follows [8]:

Theorem 4.5. *In Theorem* 4.4, *if* σ_+ *is void, then there exists a neighborhood* U *of the equilibrium such that all solutions of* (4.1) *with initial conditions in* U *will maintain in this neighborhood for all* $t > 0$ *and tend exponentially to some solution of* (4.1) *on* W^c *as* $t \to \infty$.

The next result is about the stability of a normally hyperbolic invariant manifold. We first review some concepts before presenting the theorem.

- A hyperbolic equilibrium of a flow is a fixed point at which the Jacobian of the vector field has no eigenvalues with zero real parts, or equivalently, the Jacobian of the flow has no eigenvalues with absolute value equal to one.
- A flow φ^t is called r-normally hyperbolic at an invariant submanifold V where $1 \leqslant r \leqslant \infty$, if
  - φ^t is C^r.
  - $T_V M$, the tangent bundle of M restricted on V, has a $D\varphi^t$-invariant splitting into 3 continuous subbundles

    $$T_V M = N^u \oplus TV \oplus N^s$$

    where TV is the tangent bundle of V. Thus for any $\mathbf{x} \in M$,

    $$D\varphi^t(\mathbf{x}) = D\varphi^t|_{N^u_{\mathbf{x}}} \oplus D\varphi^t|_{T_{\mathbf{x}}V} \oplus D\varphi^t_{N^s_{\mathbf{x}}}.$$

  - There exists a Finsler structure $\|\cdot\|_{\mathbf{x}}$ on TM such that for all $\mathbf{x} \in M$, $0 \leqslant k \leqslant r$:

    $$m(D\varphi^t|_{N^u_{\mathbf{x}}}) > \|D\varphi^t|_{T_{\mathbf{x}}V}\|^k \quad \text{and} \quad \|D\varphi^t_{N^s_{\mathbf{x}}}\| < m(D\varphi^t|_{T_{\mathbf{x}}V})^k,$$

    where the derived norm $\|\mathbf{A}\|$ and the minimal norm $m(\mathbf{A})$ of a linear operator $\mathbf{A}$ is defined as

    $$\|\mathbf{A}\| = \sup\{\|\mathbf{A}\mathbf{x}\| : \|\mathbf{x}\| = 1\} \quad \text{and} \quad m(\mathbf{A}) = \inf\{\|\mathbf{A}\mathbf{x}\| : \|\mathbf{x}\| = 1\},$$

    respectively. Note: $\|\mathbf{A}^k\| \leqslant \|\mathbf{A}\|^k$, $m(\mathbf{A}^k) \geqslant m(\mathbf{A})^k$. When $\mathbf{A}$ is invertible, $m(\mathbf{A}) = \|\mathbf{A}^{-1}\|^{-1}$.
- A k-dimensional submanifold in an n-dimensional manifold can be locally viewed as the image of some mapping ϕ from some open set in $\mathbb{R}^k$ to $\mathbb{R}^n$. A parameterized collection of submanifolds is called continuous at a point $\mathbf{y} \in \mathbb{R}^n$ (where $\mathbf{y}$ is in some submanifold parameterized by a parameter θ_0 and locally represented by the image of a mapping ϕ_{θ_0}) on parameter θ in C^1-topology if there exists a neighborhood Θ_0 of θ_0 and an open neighborhood U of $\mathbf{x} = \phi_{\theta_0}^{-1}(\mathbf{x})$ in $\mathbb{R}^k$ such that the submanifolds near $\mathbf{y}$ are represented by $\phi_\theta(U)$ where $\theta \in \Theta_0$ and ϕ_θ converges to ϕ_{θ_0} in C_1-norm as $\theta \to \theta_0$.
- Suppose M is an n-dimensional differentiable manifold. A family of smoothly embedded manifolds $\{N_\alpha\}_{\alpha \in A}$ (called leaves or fibers) is called a foliation on M if $N_\alpha \cap N_\beta = \phi$ for $\alpha \neq \beta$, $M \subset \bigcup_\alpha N_\alpha$, and for each $\mathbf{x} \in M$, there exists a neighborhood U and a homeomorphism $h : U \to \mathbb{R}^n$ such that h maps every connected component of $\bigcup_\alpha N_\alpha \cap U$ to $h(U) \cap (\mathbb{R}^k \times \{\mathbf{y}\}) \subset \mathbb{R}^n$ for some $\mathbf{y} \in \mathbb{R}^{n-k}$.

Theorem 4.6 (Fundamental theorem of normally hyperbolic invariant manifolds). *Let $\varphi^t : M \to M$ be a C^r flow of a C^∞ manifold M with $r \geqslant 1$ leaving the C^1 submanifold $V \subset M$ invariant, where V is assumed either being compact or being a leaf or a union of leaves of a foliation of M. Assume that φ^t is r-normally hyperbolic at V respective to the tangent bundle splitting $T_V M = N^u \oplus TV \oplus N^s$, where $D\varphi^t$ exponentially expands and contracts the vectors in N^u and N^s, respectively. Then*

1. *Existence: There exist locally φ^t-invariant submanifolds $W^u(\varphi^t)$ and $W^s(\varphi^t)$, called a local unstable manifold and a local stable manifold at V, respectively, tangent at V to $N^u \oplus TV$, and $TV \oplus N^s$, respectively. (Remark: The local invariance of W^u, W^s means $\varphi^t(W^u) \supset W^u$, and $\varphi^t(W^s) \subset W^s$.)*
2. *Uniqueness: Any locally invariant set near V lies in $W^u \cap W^s$.*
3. *Characterization: W^s consists of all points whose forward φ-orbits never strays far from V, and W^u of all points whose reverse φ^t-orbits never stray far from V.*
4. *Smoothness: W^u, W^s and V are class C^r.*
5. *Foliation: W^u and W^s are invariantly fibered by C^r submanifolds $W^{uu}_{\mathbf{x}}$, $W^{ss}_{\mathbf{x}}$, $\mathbf{x} \in V$, tangent at $\mathbf{x}$ to $N^u_{\mathbf{x}}$ and $N^s_{\mathbf{x}}$ respectively. W^{uu} and W^{ss} are invariant in the sense that $\varphi^t(W^{uu}_{\mathbf{x}}) \subset W^{uu}_{\varphi^t(\mathbf{x})}$ for $t < 0$ and $\varphi^t(W^{ss}_{\mathbf{x}}) \subset W^{s}_{\varphi^t(\mathbf{x})}$ for $t > 0$. $W^{ss}_{\mathbf{x}}$ is characterized by $\|\varphi(\mathbf{y}) - \varphi(\mathbf{x})\| \to 0$ exponentially as $t \to \infty$ for any $\mathbf{y} \in W^{ss}_{\mathbf{x}}$, and $W^{uu}_{\mathbf{x}}$ is characterized by $\|\varphi(\mathbf{y}) - \varphi(\mathbf{x})\| \to 0$ exponentially as $t \to -\infty$ for any $\mathbf{y} \in W^{uu}_{\mathbf{x}}$.*
6. *Continuity: The leaves of foliation $W^{uu}_{\mathbf{x}}$ and $W^{ss}_{\mathbf{x}}$ are continuous on parameter $\mathbf{x} \in V$ in C^1-topology.*
7. *Permanence: If $\tilde{\varphi}^t$ is another C^r flow on M and is C^r close to φ^t (i.e., close in C^r-norm). Then $\tilde{\varphi}^t$ is r-normally hyperbolic at some unique submanifold $\tilde{V}$, which is C^r close to V. The invariant manifolds $W^u(\tilde{\varphi}^t)$, $W^s(\tilde{\varphi}^t)$, and the leaves $W^{uu}_{\mathbf{x}}(\tilde{\varphi}^t)$, $W^{ss}_{\mathbf{x}}(\tilde{\varphi}^t)$, are C^r close to those of φ^t.*
8. *Linearization: Near V, φ^t is topologically conjugate (i.e. C^0 conjugate) to $D\varphi^t|_{N^u \oplus N^s}$, the restriction of the Jacobian of the flow to the subspace $N^u \oplus N^s$.*

For more detail, refer to the fundamental theorem of normally hyperbolic invariant manifolds in Hirsch et al. [7] and the Hadamard–Perron Theorem in Katok and Hasselblatt [9].

5. The verification of the main result for a class of homogeneous systems

The main result of this paper, Theorem 3.1 in Section 3, is stated for a special class of homogeneous systems, where the vector field $\mathbf{f}(\mathbf{x})$ at the set of equilibria (which is a ray) has the local linearization that satisfies the property that one single eigenvalue is zero and the other eigenvalues have negative real parts. The reader may have noticed that the second half of the main result in fact contains the first half. However, the purpose of this paper is not only to present the relevant result, but also to demonstrate some useful ideas in working on differential equation systems. It is very beneficial to state our result in this way, since we are going to use different methods to prove each part of the result.

First we point out some straightforward properties of homogeneous systems by the use of the following lemma.

Lemma 5.1. *In system* (4.1), *assume that* M *is an* n*-dimensional manifold and* $\hat{\mathbf{x}}$ *is an equilibrium which is also the limit point of some other equilibria* $\hat{\mathbf{x}}_i$ $(i=1,2,\ldots)$. *Then there is at least one limiting position of the lines passing through* $\hat{\mathbf{x}}$ *and* $\hat{\mathbf{x}}_i$, *which is an eigen-direction corresponding to the zero eigenvalue of* $D\mathbf{f}(\hat{\mathbf{x}})$. *Furthermore, if there are* k $(k \leqslant n)$ *linearly independent limiting positions of these lines,* $D\mathbf{f}(\hat{\mathbf{x}})$ *has at least* k *linearly independent eigenvectors corresponding the zero eigenvalue.*

Proof. See Appendix A.2. □

Proposition 5.2. *Suppose* (4.1) *is a homogeneous system with a non-origin equilibrium. Then the set of equilibria of* (4.1) *is the union of some rays radiating from the origin, including or excluding the origin, depending whether the origin is in the domain of* (4.1) *or not. At any equilibrium,* $D\mathbf{f}$ *has at least one zero eigenvalue and the ray of equilibria on which the equilibrium lies is an eigen-direction of eigenvalue* 0. *Furthermore,* $D\mathbf{f}$ *are positively linearly dependent at the non-origin equilibria on the same ray of equilibria, and therefore their spectral sets are identical in terms of signs* $+$, $-$, *and* 0.

Proof. Assume that system (4.1) is homogeneous of degree α, i.e.,

$$\mathbf{f}(\lambda\mathbf{x}) = \lambda^{\alpha}\mathbf{f}(\mathbf{x}) \tag{5.1}$$

for all $\lambda > 0$. The origin $\mathbf{0}$ is an equilibrium when $\alpha > 0$, and is a singularity point when $\alpha < 0$. If $\hat{\mathbf{x}}$ is an equilibrium, then $\mathbf{f}(\lambda\hat{\mathbf{x}}) = \mathbf{0}$ either for $\forall\lambda > 0$, or for $\forall\lambda \neq 0 \in \mathbb{R}$ when α is an integer or the reciprocal of an odd number. This implies that the set of equilibria can be visualized as the union of rays radiating from the origin, including or excluding the origin, depending whether the origin is in the domain of (4.1) or not. Differentiating Eq. (5.1) with respect to t at $\lambda = 1$ yields identity

$$D\mathbf{f}(\mathbf{x}) \cdot \mathbf{x} = \alpha\mathbf{f}(\mathbf{x}).$$

At an equilibrium $\hat{\mathbf{x}}$ this yields $D\mathbf{f}(\hat{\mathbf{x}}) \cdot \hat{\mathbf{x}} = \mathbf{0}$, which implies that at any non-origin equilibrium, $D\mathbf{f}$ has a zero eigenvalue and the ray from the origin through the equilibrium is an eigen-direction of eigenvalue 0. By Lemma 5.1, the same conclusion is true to the case where the origin is an equilibrium. Using Eq. (5.1) and the chain rule, we have

$$\lambda D\mathbf{f}(\mathbf{y})|_{\mathbf{y}=\lambda\mathbf{x}} = D(\mathbf{f}(\lambda\mathbf{x})) = \lambda^{\alpha} D\mathbf{f}(\mathbf{x})$$

and

$$D\mathbf{f}(\mathbf{y})|_{\mathbf{y}=\lambda\mathbf{x}} = \lambda^{\alpha-1} D\mathbf{f}(\mathbf{x}), \tag{5.2}$$

where $\lambda \neq 0$. Thus $D\mathbf{f}$ are positively linearly dependent on the same open ray of equilibria. This implies that there is a one-to-one correspondence between the eigenvalues of $D\mathbf{f}$ at any two points on the same open ray of equilibria via some positive multiple scalar $\lambda^{\alpha-1}$. Therefore the spectral sets of $D\mathbf{f}$ on the same ray are identical in terms of signs $+$, $-$, and 0. □

Assume next that the Jacobian of the vector field at the equilibrium in the above homogeneous system has a zero eigenvalue with multiplicity one and all other eigenvalues with negative real parts. Then there is an open ray of equilibria starting from the origin such that each point on it satisfies the same eigenvalue assumption from Proposition 5.2. Such a ray is isolated from other equilibria, i.e., there exists a cone vertexed at the origin and centered at this ray which contains no other equilibrium. Otherwise, the ray is the limiting position of other rays of equilibria by Proposition 5.2. According to Lemma 5.1, there is an eigenvector corresponding to the zero eigenvalue transversing the original ray (i.e., they are not tangent to each other). This violates the assumption that the zero eigenvalue is single. Thus, we obtain the following result.

Proposition 5.3. *Assume that there is an equilibrium of a homogeneous system (4.1) such that the Jacobian of the vector field at the equilibrium has a single zero eigenvalue and all other eigenvalues with negative real parts. Then through this equilibrium there is an open ray of equilibria starting from the origin such that each point on the ray has the same eigenvalue property as the original equilibrium does. Such a ray of equilibria is isolated from any other equilibrium.*

We are going to verify the main result, Theorem 3.1. First we make use of the concepts and theorems introduced in Section 4. At each equilibrium on the ray, from Theorem 4.4, there is a center manifold which is the ray and a stable manifold with dimension $n-1$ which transverses the ray. From Theorem 4.5 we know that at each equilibrium, there exists a neighborhood such that any flow line starting in it will tend exponentially to a solution on the center manifold, which is some point on the ray. The union of these neighborhoods makes an attraction basin. This proves the strong attractivity of the ray. It is easy to verify that the ray of equilibria is a ∞-normally hyperbolic invariant manifold due to the property of the eigenvalues of the Jocobian of the vector field at each equilibrium. Besides, the ray is a leaf of a foliation of $\mathbb{R}^n\setminus\{\mathbf{0}\}$ by partitioning $\mathbb{R}^n\setminus\{\mathbf{0}\}$ with rays radiating from the origin. So we can use Theorem 4.6. The locally stable manifold of the ray is the basin of attraction. The leaves of the foliation of the basin are the stable manifolds at the equilibria with C^1-continuity on the equilibria on the ray. Using the homogeneity assumption (5.1), we may show that the basin of attraction contains a generalized cone vertexed at the origin and centered at the ray. Select any fixed stable manifold $W^s_{\hat{\mathbf{x}}}$ at $\hat{\mathbf{x}}$. We claim that $\lambda W^s_{\hat{\mathbf{x}}}$ is a stable manifold at the equilibrium $\lambda\hat{\mathbf{x}}$ for any $\lambda>0$. In fact, $W^s_{\hat{\mathbf{x}}}$ is the union of the flow lines with initial conditions in it. Let $\varphi^t(\mathbf{x})$ be any one of these flow lines. Then

$$\frac{\mathrm{d}(\lambda\varphi^t(\mathbf{x}))}{\mathrm{d}t}=\lambda\frac{\mathrm{d}\varphi^t(\mathbf{x})}{\mathrm{d}t}=\lambda\mathbf{f}(\varphi^t(\mathbf{x}))=\lambda^{1-\alpha}\mathbf{f}(\lambda\varphi^t(\mathbf{x})).$$

Changing the time by $\mathrm{d}s = \lambda^{1-\alpha}\,\mathrm{d}t$, we can see that $\lambda\varphi^t(\mathbf{x})$ is the time change of some flow lines of system (4.1), which tend exponentially to $\lambda\hat{\mathbf{x}}$ as $t \to \infty$. Besides, $\lambda W^s_{\hat{\mathbf{x}}}$ and $W^s_{\hat{\mathbf{x}}}$ are diffeomorphic. Then $\lambda W^s_{\hat{\mathbf{x}}}$ is a manifold and thus, a stable manifold at $\lambda\hat{\mathbf{x}}$. Varying λ from 0 to ∞, we obtain a generalized cone vertexed at the origin and centered at the ray.

Remark. Theorem 3.1 remains true without the homogeneity assumption, which is clear from the above proof. Of course, the terms of "ray" and "cone" may not be applicable in a general case without homogeneity. They ought to be replaced by "set of equilibria" and "basin of attraction", respectively.

6. An elementary method to prove the strong attractivity in the main result

In this section, we present an elementary method to prove the strong attractivity of the ray in Theorem 3.1 by applying conjugacy and the σ-process.

As we have seen in Section 4, a conjugacy preserves the local stability of a dynamical system. The homogeneity of a system guarantees that the system can be factored into a system in the projective space with lower dimension, which may be easier to analyze. This enables us to study the stability properties of the conjugate system and then to draw conclusion on the original system. Our main goal in this section is to demonstrate these useful techniques.

The projective space $\mathbb{P}^{n-1}$ is the collection of all lines passing through the origin in $\mathbb{R}^n \setminus \{\mathbf{0}\}$. It can also be obtained by identifying the diametrically opposite points on the $n-1$ dimensional sphere $\mathbb{S}^{n-1}$. $\mathbb{P}^{n-1}$ is a manifold with local coordinates given by the onto differentiable mappings $p_i : \mathbb{R}^n \setminus \{x_i = 0\} \to \mathbb{R}^{n-1}$ via $(x_1, \ldots, x_n) \to (x_1/x_i, \ldots, x_{i-1}/x_i, x_{i+1}/x_i, \ldots, x_n/x_i)$, where $i = 1, \ldots, n$ [4]. The passage from one coordinate system to another using the above p_i is called the σ-process [2].

A homogeneous differential system can be locally factored into the one-dimension-less projective space using the σ-process. Without loss of generality, we assume that $x_1 \neq 0$. Let p_1 be the mapping described as above, and let $u_i = x_i/x_1$ for $i = 2, \ldots, n$. p_1 is a coordinate change and thus, is a semi-conjugacy. From (4.3), Dp_1 maps the vector filed $\mathbf{f}$ in system (4.1) to a new vector field $\mathbf{g}_2$ in a semi-conjugate system.

Let α be the degree of homogeneity of vector field $\mathbf{f}$ defined by (5.1). Decompose $\mathbf{f}$ and $\mathbf{g}_2$ into their components, respectively, as $\mathbf{f} = (f_1, \ldots, f_n)^{\mathrm{T}}$, and $\mathbf{g}_2 = (g_2, \ldots, g_n)^{\mathrm{T}}$. Let $\tilde{f}_i(\mathrm{sgn}(x_1), x_2, \ldots, x_n) = f_i(\mathrm{sgn}(x_1), \mathrm{sgn}(x_1)x_2, \ldots, \mathrm{sgn}(x_1)x_n)$, for $i = 1, \ldots, n$, and furthermore let $\tilde{\mathbf{f}} = (\tilde{f}_1, \ldots, \tilde{f}_n)$ and $\mathbf{u}_2 = (u_2, \ldots, u_n)$. From (4.3), we see that

$$Dp_1 = \begin{pmatrix} -\dfrac{x_2}{x_1^2} & \dfrac{1}{x_1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -\dfrac{x_n}{x_1^2} & 0 & \cdots & \dfrac{1}{x_1} \end{pmatrix} = \frac{1}{x_1} \begin{pmatrix} -u_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ -u_n & 0 & \cdots & 1 \end{pmatrix}. \tag{6.1}$$

Note that

$$
\begin{aligned}
\mathbf{g}_2(x_1,\mathbf{u}) &= Dp_1 \cdot \mathbf{f}(x_1,x_2,\ldots,x_n) \\
&= Dp_1 \cdot \mathbf{f}(x_1,x_1u_2,\ldots,x_1u_n) = |x_1|^{\alpha} Dp_1 \cdot \tilde{\mathbf{f}}(\operatorname{sgn}(x_1),u_2,\ldots,u_n) \\
&= \operatorname{sgn}(x_1)|x_1|^{\alpha-1} \begin{pmatrix} -u_2\tilde{f}_1(\operatorname{sgn}(x_1),\mathbf{u}_2)+\tilde{f}_2(\operatorname{sgn}(x_1),\mathbf{u}_2) \\ \vdots \\ -u_n\tilde{f}_1(\operatorname{sgn}(u_1),\mathbf{u}_2)+\tilde{f}_n(\operatorname{sgn}(u_1),\mathbf{u}_2) \end{pmatrix} \\
&= |x_1|^{\alpha-1} \begin{pmatrix} \tilde{g}_2(\operatorname{sgn}(x_1),\mathbf{u}_2) \\ \vdots \\ \tilde{g}_n(\operatorname{sgn}(x_1),\mathbf{u}_2) \end{pmatrix} \\
&= |x_1|^{\alpha-1}\tilde{\mathbf{g}}_2(\operatorname{sgn}(x_1),\mathbf{u}_2), \qquad (6.2)
\end{aligned}
$$

where $\tilde{g}_i(\operatorname{sgn}(x_1),\mathbf{u}_2) = \operatorname{sgn}(x_1)\cdot(-u_i\tilde{f}_1(\operatorname{sgn}(x_1),\mathbf{u}_2)+\tilde{f}_i(\operatorname{sgn}(x_1),\mathbf{u}_2))$, $\forall i = 2,\ldots,n$; and $\tilde{\mathbf{g}}_2(\tilde{g}_2,\ldots,\tilde{g}_n)^{\mathrm{T}}$.

Note that x_1 have the same sign in either one of the two connected components of $\mathbb{R}^n \setminus \{x_1 = 0\}$. Thus, $\tilde{\mathbf{g}}_2$ can be viewed as a function of $\mathbf{u}_2$ only.

By rescaling the time t with a time s such that $\mathrm{d}s = |x_1|^{r-1}\,\mathrm{d}t$ the vector field $\mathbf{g}_2$ becomes $\tilde{\mathbf{g}}_2$. The original homogeneous system is factored into a subsystem

$$\frac{\mathrm{d}\mathbf{u}_2}{\mathrm{d}s} = \tilde{\mathbf{g}}_2(\mathbf{u}_2), \qquad (6.3)$$

which is an orbit factor of the original system (4.1).

If the eigenvalues of $\mathbf{f}$ at the equilibrium are assumed to be as in our main result Theorem 3.1, the eigenvectors corresponding to the zero eigenvalue are all along the ray of equilibria by Lemma 5.1 since the zero eigenvalue is single. It is easy to verify that $Dp_1(\mathbf{x})\cdot\mathbf{y} = \mathbf{0}$ if and only if $\mathbf{x}$ and $\mathbf{y}$ are parallel. Hence the null space of $Dp_1(\hat{\mathbf{x}})$ at an equilibrium $\hat{\mathbf{x}}$ is the ray of equilibria on which $\hat{\mathbf{x}}$ lies. By Proposition 4.2, semi-conjugacy p_1 preserves all the eigenvalues of $D\mathbf{f}(\hat{\mathbf{x}})$ at an equilibrium $\hat{\mathbf{x}}$ except the zero eigenvalue. By Proposition 4.3, the time change preserves the negative sign of the real parts of all the eigenvalues of $D\tilde{\mathbf{g}}_2(\hat{\mathbf{u}}_2)$, where $\hat{\mathbf{u}}_2 = p_1(\hat{\mathbf{x}})$ is an equilibrium, which is exactly the same point for any $\hat{\mathbf{x}}$ on the same ray of equilibria. Thus $\hat{\mathbf{u}}_2$ is an asymptotically stable equilibrium of system (6.3).

In order to receive the strong attractivity of the ray of equilibria in the original system, we construct a change of variable $p = (\mathrm{id}_{x_1}, p_1)$, where id_{x_1} is the identity map which maps the x_1-coordinate to itself. Then p is a conjugacy to the so-called canonical line bundle on $\mathbb{P}^{n-1}$. Dp maps vector field $\mathbf{f}$ to a new vector field $\mathbf{g}$, which determines a new flow conjugate to the original flow.

The Jacobian of p is

$$Dp = \begin{pmatrix} 1 & \mathbf{0}^{\mathrm{T}} \\ \mathbf{0} & Dp_1 \end{pmatrix},$$

furthermore,

$$\begin{aligned} \mathbf{g}(x_1,\mathbf{u}) &= Dp \cdot \mathbf{f}(x_1,x_2,\ldots,x_n) \\ &= \begin{pmatrix} f_1(x_1,x_2,\ldots,x_n) \\ \mathbf{g}_2(\mathrm{sgn}(x_1),\mathbf{u}_2) \end{pmatrix} \\ &= |x_1|^{\alpha-1} \begin{pmatrix} |x_1|\tilde{f}_1(\mathrm{sgn}(x_1),\mathbf{u}_2) \\ \tilde{\mathbf{g}}_2(\mathrm{sgn}(x_1),\mathbf{u}_2) \end{pmatrix} \\ &= |x_1|^{\alpha-1} \begin{pmatrix} x_1\tilde{g}_1(\mathrm{sgn}(x_1),\mathbf{u}_2) \\ \tilde{\mathbf{g}}_2(\mathrm{sgn}(x_1),\mathbf{u}_2) \end{pmatrix}, \end{aligned} \tag{6.4}$$

where $\tilde{g}_1(\mathrm{sgn}(x_1),\mathbf{u}_2) = \mathrm{sgn}(x_1)\tilde{f}_1(\mathrm{sgn}(x_1),\mathbf{u}_2)$.

Here g_1 depends only on $\mathbf{u}_2$ in either one of the connected components of $\{x_1 \neq 0\}$. We also rescale the time t to s by $\mathrm{d}s = |x_1|^{r-1}\,\mathrm{d}t$. Then an orbit equivalent system is obtained as follows:

$$\frac{\mathrm{d}}{\mathrm{d}s}\begin{pmatrix} x_1 \\ \mathbf{u}_2 \end{pmatrix} = \begin{pmatrix} x_1\tilde{g}_1(\mathbf{u}_2) \\ \tilde{\mathbf{g}}_2(\mathbf{u}_2) \end{pmatrix}. \tag{6.5}$$

From the first component in (6.5), we get $\mathrm{d}\ln|x_1|/\mathrm{d}s = \tilde{g}_1(\mathbf{u}_2)$, and $\ln|x_1(s)| = \ln|x_1(0)| + \int_0^s \tilde{g}_1(\mathbf{u}_2(\mu))\,\mathrm{d}\mu$. Since $\hat{\mathbf{u}}_2$ is an asymptotically stable equilibrium, $\mathbf{u}_2$ locally tends to $\hat{\mathbf{u}}_2$ exponentially as $s \to \infty$, i.e., there exist a $\theta_0 > 0$ and an $s_0 > 0$, such that $|\mathbf{u}_2(s) - \hat{\mathbf{u}}_2| < \mathrm{e}^{-\theta s}$ when $s > s_0$. Since $\mathbf{f}$ is C^1, continuously differentiable, so is $\tilde{g}_1$ in any one of the connected components of $\{x_1 \neq 0\}$. Note that $\tilde{g}_1$ vanishes at $\hat{\mathbf{u}}_2$ by the meaning of equilibrium. Hence there exist a $\delta > 0$ and a $\theta_1 > 0$ such that for all $\mathbf{u}_2$, $|\mathbf{u}_2 - \hat{\mathbf{u}}_2| < \delta$ implies $|\tilde{g}_1(\mathbf{u}_2)| = |\tilde{g}_1(\mathbf{u}_2) - \tilde{g}_1(\hat{\mathbf{u}}_2)| < \theta_1|\mathbf{u}_2 - \hat{\mathbf{u}}_2|$. (Just take a number larger than the absolute value of the derivative of $\tilde{g}_1$ at $\hat{\mathbf{u}}_2$ for the value of θ_1.) Choose $s^* > s_0$ such that $\mathrm{e}^{-\theta s^*} < \delta$. Then for all $s_1, s_2 > s^*$,

$$\begin{aligned} |\ln|x_1(s_2)| - \ln|x_1(s_1)|| &= \left| \int_0^{s_2} \tilde{g}_1(\mathbf{u}_2(\mu))\,\mathrm{d}\mu - \int_0^{s_1} \tilde{g}_1(\mathbf{u}_2(\mu))\,\mathrm{d}\mu \right| \\ &= \left| \int_{s_1}^{s_2} \tilde{g}_1(\mathbf{u}_2(\mu))\,\mathrm{d}\mu \right| \leqslant \int_{s_1}^{s_2} |\tilde{g}_1(\mathbf{u}_2(\mu))|\,\mathrm{d}\mu \\ &< \int_{s_1}^{s_2} \theta_1|\mathbf{u}_2(\mu) - \hat{\mathbf{u}}_2|\,\mathrm{d}\mu \end{aligned}$$

$$< \int_{s_1}^{s_2} \theta_1 \mathrm{e}^{-\theta_0 \mu}\, \mathrm{d}\mu$$

$$= \frac{\theta_1}{\theta_0} (\mathrm{e}^{-\theta_0 s_1} - \mathrm{e}^{-\theta_0 s_2}) \to 0$$

as $s_1, s_2 \to \infty$. So $\ln |x_1(s)|$ is Cauchy, $\lim_{s\to\infty} \ln |x_1(s)|$ exists and is finite. Therefore, $\lim_{s\to\infty} x_1(s)$ exists and is finite. On the other hand, $x_1(s) = x_1(0) \exp\{\int_0^s \tilde{g}_1(\mathbf{u}_2(\mu))\, \mathrm{d}\mu\}$. The sign of $x_1(s)$ remains the same as that of the initial condition, therefore the flow line stays in the same connected component of $\{x_1 \neq 0\}$ and thus the flow of system (6.5) is complete near the two rays $\{(x_1, \hat{\mathbf{u}}_2) \,|\, x_1 \neq 0\}$ in the canonical line bundle on $\mathbb{P}^{n-1}$. Hence we have proved the strong attractivity of the image of the ray of equilibria for the orbit equivalent system (6.5). So the strong attractivity of the same ray in the original system immediately follows from Propositions 4.2 and 4.3.

Appendix A.

A.1. *Proof of Lemma 4.1*

Without loss of generality, we assume that there is only one Jordan block associated with eigenvalue λ in the Jordan canonical form of matrix $\mathbf{A}$. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be a natural basis in this generalized eigenspace. Then $\mathbf{A}\mathbf{e}_i = \lambda \mathbf{e}_i + \mathbf{e}_{i+1}$, for $i = 1, \ldots, n-1$, and $\mathbf{A}\mathbf{e}_n = \lambda \mathbf{e}_n$. By $\mathbf{PA} = \mathbf{BP}$, we get

$$\begin{aligned} &\mathbf{BPe}_i = \lambda \mathbf{Pe}_i + \mathbf{Pe}_{i+1} \quad \text{for } i = 1, \ldots, n-1, \\ &\mathbf{BPe}_n = \lambda \mathbf{Pe}_n. \end{aligned} \tag{A.1}$$

It is easy to see that if $\mathbf{Pe}_r = \mathbf{0}$ for some $r < n$, then $\mathbf{Pe}_i = \mathbf{0}$ for all $i \geqslant r$. Since $\mathbf{P}$ is full-ranked, $\{\mathbf{Pe}_1, \mathbf{Pe}_2, \ldots, \mathbf{Pe}_n\}$ consists of m linearly independent vectors. Assume that $\mathbf{Pe}_k \neq \mathbf{0}$ and $\mathbf{Pe}_i = \mathbf{0}$ for all $i > k$. We claim that $\{\mathbf{Pe}_1, \mathbf{Pe}_2, \ldots, \mathbf{Pe}_k\}$ is a maximal linearly independent group. In fact, if there is an l, $1 \leqslant l < k$, such that $\mathbf{Pe}_{l+1}, \ldots, \mathbf{Pe}_k$ are linearly independent while $\mathbf{Pe}_l, \ldots, \mathbf{Pe}_k$ are linearly dependent, then there exist not-all-zero real numbers $c_l, \ldots, c_k$ such that $\sum_{i=l}^k c_i \mathbf{Pe}_i = \mathbf{0}$. Multiplying by $\mathbf{B}$ on both sides of this equation and using Eqs. (A.1) (note: n is replaced with k in the last equation of (A.1)), we obtain that $\sum_{i=l}^{k-1} c_i \mathbf{Pe}_{i+1} = \mathbf{0}$. We know $c_l, \ldots, c_{k-1}$ cannot be all zero, otherwise c_k will be zero since $\mathbf{Pe}_k \neq \mathbf{0}$ and this violates that $c_1, \ldots, c_k$ are not all zero. Thus $\mathbf{Pe}_{l+1}, \ldots, \mathbf{Pe}_k$ are linearly dependent, which is a contradiction. Hence, from Eqs. (A.1), we know that $\{\mathbf{Pe}_1, \mathbf{Pe}_2, \ldots, \mathbf{Pe}_k\}$ produces a Jordan block. Therefore, the Jordan blocks of $\mathbf{B}$ are actually obtained from the Jordan blocks of $\mathbf{A}$ by deleting some rows and columns if necessary.

A.2. *Proof of Lemma 5.1*

M is diffeomorphic to $\mathbb{R}^n$. We denote the unit ball of $\mathbb{R}^n$ centered at $\hat{\mathbf{x}}$ by $B(\hat{\mathbf{x}}, 1)$. The rays from $\hat{\mathbf{x}}$ to $\hat{\mathbf{x}}_i$ intersect $B(\hat{\mathbf{x}}, 1)$ at $\hat{\mathbf{y}}_i$. There is a one-to-one correspondence between $\hat{\mathbf{x}}_i$ and $\hat{\mathbf{y}}_i$. Since $B(\hat{\mathbf{x}},1)$ is compact, the set $\{\hat{\mathbf{y}}_i\}$ has at least one limit point. Each limit point determines one limiting position of the rays. Among all the limiting positions of the rays, there are not more than n linearly independent directions since $B(\hat{\mathbf{x}},1)$ is n-dimensional. At each limit point of $\hat{\mathbf{y}}_i$, say $\hat{\mathbf{y}}$, we will show that unit vector $\overrightarrow{\hat{\mathbf{x}}\hat{\mathbf{y}}}$ is a unit eigenvector of a zero eigenvalue.

Without loss of generality, we assume that $\hat{\mathbf{y}}_i \to \hat{\mathbf{y}}$.

From the Taylor's expansion,

$$\mathbf{0} = \mathbf{f}(\hat{\mathbf{x}}_i) - \mathbf{f}(\hat{\mathbf{x}}) = D\mathbf{f}(\hat{\mathbf{x}}) \cdot (\hat{\mathbf{x}}_i - \hat{\mathbf{x}}) + o(\|\hat{\mathbf{x}}_i - \hat{\mathbf{x}}\|).$$

Dividing by $\|\hat{\mathbf{x}}_i - \hat{\mathbf{x}}\|$, one gets

$$\mathbf{0} = D\mathbf{f}(\hat{\mathbf{x}}) \cdot \frac{\hat{\mathbf{x}}_i - \hat{\mathbf{x}}}{\|\hat{\mathbf{x}}_i - \hat{\mathbf{x}}\|} + o(1).$$

Let $i \to \infty$, then $(\hat{\mathbf{x}}_i - \hat{\mathbf{x}})/\|\hat{\mathbf{x}}_i - \hat{\mathbf{x}}\| \to \hat{\mathbf{y}} - \hat{\mathbf{x}}$. So

$$D\mathbf{f}(\hat{\mathbf{x}}) \cdot (\hat{\mathbf{y}} - \hat{\mathbf{x}}) = \mathbf{0}.$$

Therefore, $\hat{\mathbf{y}} - \hat{\mathbf{x}}$ is a unit eigenvector of the zero eigenvalue of $D\mathbf{f}(\hat{\mathbf{x}})$.

The above derivation also implies that if there are k $(k \leqslant n)$ linearly independent limiting positions of the rays, then there must be at least k linearly independent eigenvectors associated with the zero eigenvalue of $D\mathbf{f}(\hat{\mathbf{x}})$. □

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