Building blocks of self-organized criticality, part II: transition from very low drive to high drive

Ryan Woodard* and David E. Newman
University of Alaska Fairbanks
Fairbanks, Alaska 99775-5920, USA

Raúl Sánchez
Universidad Carlos III de Madrid
28911 Leganés, Madrid, Spain

Benjamin A. Carreras
Oak Ridge National Laboratory
Oak Ridge, Tennessee 37831-8070, USA
(Dated: July 25, 2004)

We analyze the transition of the self-organized criticality one dimensional directed running sandpile model of Hwa and Kardar [Phys. Rev. A **45**, 7002 (1992)] from very low external forcing to high forcing, showing how six distinct power law regions in the power spectrum at low drive become four regions at high drive. One of these regions is due to long time correlations among events in the system and scales as $\sim f^{-\beta}$ with $0 < \beta \leq 1$. The location in frequency space and the value of β both increase as the external forcing increases. β ranges from ≈ 0.4 for the weakest forcing studied here to a maximum value of 1 (i.e., a $1/f$ region) at stronger levels. The greatest rate of change in β is when the average quiet time between avalanche events is on the same order as the average duration of events. The correlations are quantified by a constant Hurst exponent $H \approx 0.8$ when estimated by R/S analysis for sandpile driving rates spanning over five orders of magnitude. The constant H and changing β in the same system as forcing changes suggests that the power spectrum does not consistently quantify long time dynamical correlations and that the relation $\beta = 2H - 1$ does not hold for the time series produced by this SOC model. Because of the constant rules of the model we show that the same physics that produces a $\beta = 1$ scaling region during strong forcing produces a $0 < \beta < 1$ region at weaker forcing.

I. INTRODUCTION

Simple models have been used to study the dynamics of some physical systems, such as confined fusion plasmas [1, 2], space plasmas [3, 4] and earthquakes [5], among others. These models comprise a connected network of local nonlinear gradients that can persist because of a critical threshold. Random external forcing of the system increases local gradients; when one of them exceeds the critical threshold a relaxation event is triggered that stabilizes the gradient. The gradient is reduced by transferring mass, heat or some other quantity specific to the system to neighboring regions which can make them unstable, creating a series of relaxations. This sequence of events, called an avalanche, occurs much faster than the external drive increases the gradients. These models and this type of dynamics are characteristic of self-organized criticality (SOC) [6–8].

One of the first SOC models was the sandpile [7, 9, 10]. A one dimensional variation of it was studied for strong external forcing by [11] and later for weak external forcing by [12]. Both studies show that even though the system is randomly driven, long time correlations exist in the dynamics on time scales much longer than the duration of any single avalanche. The question of whether long time correlations exist in a time series—a basis for predictability—is fundamental to many physical and geophysical fields.

One of the features of long time correlations in a system, including the SOC sandpile, is a region in the power spectrum that scales as a power law $f^{-\beta}$ with $\beta \neq 0$. [11] shows that $\beta = 1$ (i.e., $1/f$) in this region of correlations at high drive, where avalanches almost always overlap in time. [12] shows that $\beta \approx 0.4$ at very low drive, where avalanches essentially never overlap.

Values of $\beta = 1$ appear in the spectra of many physical processes [13, 14], where it is referred to as $1/f$ noise for historical reasons. This modern mystery still inspires much current work but no general theory explains the origin of $1/f$. One of the original motivations of SOC was to offer an explanation of $1/f$ noise but the conclusions have been mired in controversy since its introduction [15].

One way that correlations can be quantified is by the Hurst exponent, H [16], where a value of $0.5 < H < 1$ indicates positive correlations in a data series, $0 < H < 0.5$ indicates anticorrelations and $H = 0.5$ indicates lack of correlations [17, 18].

*Electronic address: `ryan@timehaven.org`

Algorithms exist to generate a data series with a power spectrum with any given value of H [13]. These artificial data sets, called fractional Gaussian noise (fGn), have Gaussian distributions and the relation $\beta = 2H - 1$ is derived based on this type of statistics [19]. β can then be interpreted directly through H as a measure of correlations in the time series. Note that if this equation were to hold for all data sets then any series with a $1/f$ spectrum would have $H = 1$ over the same time scales.

Using rescaled range (R/S) analysis, [12] found that $H \approx 0.8$ in the region discussed above for the low drive sandpile. Here we show that H maintains this value in the correlated region for over five orders of magnitude of driving rate, as the sandpile goes from low drive to high drive. H remains constant because the rules of the sandpile (the low level physics of the system) do not change as driving rate increases. The dynamics simply takes place on shorter time scales.

However, while H stays constant with driving rate β increases from ≈ 0.4 for the lowest drive studied here to a limiting value of 1 at the highest drive. The greatest rate of change in β occurs as driving rate increases and the average quiet time between avalanche events decreases until it is on the same order as the average duration of events. There are two main points to these results. First, the dynamics that produce a $1/f$ region at high drive is the same at low drive, even though $\beta < 1$. Second, the relation $\beta = 2H - 1$ that is often used to connect β and H does not hold for the SOC sandpile model.

The region with $H \approx 0.8$ (which we refer to as the SOC region) is just one of several in the spectral and R/S measures. We will show that six distinct power law regions in the low drive spectrum and five regions in the low drive R/S analysis both become four regions at high drive. [12] shows that the causes of the regions at low drive are, from shortest to longest time scales: low level physics, quiet times, memory stored in local gradients (the SOC region), system size effects and external drive. The reason that the number of regions changes at high drive is because events are triggered more frequently in time so that they almost constantly overlap and, therefore, virtually eliminate quiet times.

II. MODEL AND METHODS

We use the one dimensional directed running sandpile of [11], a cellular automaton. The model consists of a single column of L cells and each cell contains an integer number of ‘sand grains’, where the number is referred to as the height of the cell. At each time step for each cell, there is a probability $0 < P_0 < 1$ that U_0 grains of sand will fall on it from a ‘rain’ from above. The local gradient is the height difference between two cells. If a local gradient exceeds a defined critical gradient Z_{crit} then the gradient is stabilized by a transfer of N_{f} grains of sand from the higher cell to the lower. This action is a flip and it can make one or both of the neighboring cells unstable in the next time step so that the disturbance propagates. An uninterrupted sequence of one or more flips is called an avalanche. We used $U_0 = 1$, $N_{\mathrm{f}} = 3$ and $Z_{\mathrm{crit}} = 8$ in all results presented here; these are the same parameters used in [1].

Low and high drive are distinguished by the amount of sand falling on the system and how large the system is. Average input into the system is the driving rate, $J_{\mathrm{IN}} = P_0 L$. To compare systems of different size and driving rate, we use the effective driving rate $J_{\mathrm{E}} = P_0 L^2$. We discuss this distinction in Section IV.

The time series analyzed in this study are total flips at each time step. A flip is a single relaxation event, a transfer of N_{f} grains from one cell to the next. The total number of flips at each time step is, then, the total number of unstable cells in the system. This can be thought of as the instantaneous (potential) energy dissipation in the system.

We analyze the flips time series with the power spectrum and R/S analysis. For a data series $X(t)$, the power spectrum is defined as $S(f) = |F(f)|^2$, where $F(f) = N^{-1} \sum_{t=0}^{N-1} X(t) e^{-\imath 2\pi (f/N) t}$. The rescaled range is defined as $R'(\tau) \equiv R(\tau)/S(\tau)$, where $S(\tau)$ is the standard deviation and

$$R(\tau) = \max_{1 \le k \le \tau} W(k,\tau) - \min_{1 \le k \le \tau} W(k,\tau) \quad \text{(range)},$$

$$W(k,\tau) = \sum_{t=1}^{k} (X_t - \langle X \rangle_\tau) \ \text{(cumulative deviation) and}$$

$$\langle X \rangle_\tau = \frac{1}{\tau} \sum_{t=1}^{\tau} X_t \quad \text{(mean)}.$$

If the rescaled range of the time series scales as $R'(\tau) \sim \tau^H$, the slope of the plot of $R'(\tau)$ versus the time lag τ on a doubly logarithmic plot is the Hurst exponent, H.

III. RESULTS

Figure 1 shows the power spectra of the flips time series of the one dimensional directed running sandpile for over five orders of magnitude of effective driving rate, $P_0 L^2$, which increases from top to bottom in the figure. The sandpile size is L = 200, a size found in [1] to be adequate for studying SOC dynamics without corruption by edge effects. We also present results for sandpile sizes up to L = 2000. The lowest drive used is $P_0 L^2 = 0.002$ and the highest is $P_0 L^2 = 296$. The higher limit is chosen to stay below the normal overdrive limit of $P_0 L < N_{\mathrm{f}}/2$ (derived in Section IV).

Multiple distinct power law regions are seen in all of the spectra. The highest frequency regions share a common slope. At low drive a very prominent bump at low frequency moves to higher frequency as driving rate increases. This movement is due to more sand grains falling onto the system faster, thereby triggering avalanches more frequently so that the dynamics moves to shorter

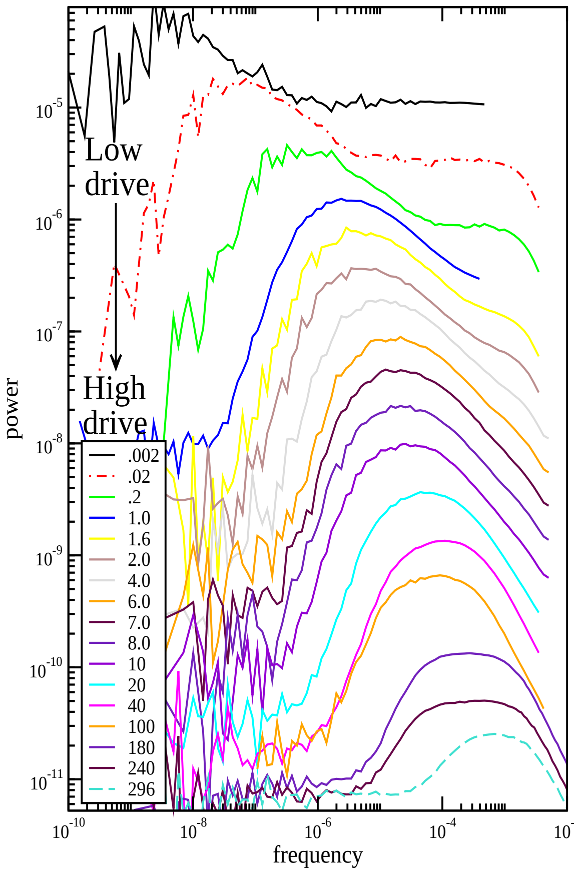

FIG. 1: Power spectra of flips time series of L = 200 sandpile for five orders of magnitude of effective driving rate in $P_0 L^2 \in [0.002, 296]$. Spectra have been shifted along y axis for easier viewing.

time scales. The avalanches overlap in time and thus eliminate quiet times. The loss of quiet times also accounts for the loss of two regions in the spectrum—six regions at low drive become four regions at high drive.

Three spectra from Figure 1 are shown in Figure 2(a), representing low, medium and high driving rates of the sandpile. The six regions of low drive and four regions of high drive are shown by the solid lines. The lines are power laws $f^{-\beta}$ and the numbers next to them are the values of β. The lowest frequency f^0 region of the low drive case is not seen because of the finite size of the time series. Its existence is assumed based on the f^0 regions seen in the spectra of higher drive cases.

The associated R/S analysis for the low, medium and high drive power spectra are shown in Figure 2(b). Five regions at low drive become four regions at high drive. Power law lines and their slopes are indicated in the figure. The slopes are the Hurst exponent H for each region. Again, the region for the longest time scales at the lowest drive is not seen because of the finite length of the time series but is inferred based on the higher drive cases.

To conveniently discuss the regions and breakpoints, we use the labelling convention of the cartoons in Figure 3. Figure 3(c) is taken from Figure 6 of [11] and the others are drawn in that spirit. Figures 3(a) and (b) are from [12] but are reproduced here for completeness. The sources of all of the regions are discussed in [11] and [12].

The breakpoints between regions in the two different measures, power spectrum and R/S, can be compared with each other. The results are shown in Figure 4. The breakpoints of the two measures, found independently, agree very closely with each other, though the R/S breakpoints appear at slightly longer time scales than those of the power spectrum. This effect is known from comparisons of R/S analysis with structure functions [20] and we conclude that both measures can distinguish the same dynamical regions through the identification of different power law regions.

The breakpoints between neighboring regions scale with driving rate as shown in Figures 5 and 6. T_{A} in the spectrum and T_{B} in both measures stay relatively constant, reflecting the unchanging rules of the system that produce discontinuous jumps in the gradient of the flips time series. The other breakpoints scale with driving rate as power laws. Region B of the spectrum and region C of both measures shrink and eventually disappear as drive increases and average quiet time shrinks.

At low drive, individual avalanches appear in the flips time series as trapezoidal pulses [12] and this is reflected in the slopes of regions A and B. $\beta \approx 3.4$ of region A and $H \gtrapprox 0.9$ of region A/B remain relatively constant as driving rate changes because of the fixed rules of the system. These values are consistent with those found for a random superposition of trapezoids. $\beta \approx 2$ in region B stays relatively constant until that region disappears, reflecting the distinct and separate trapezoids that eventually become extinct as avalanches overlap each other more and more.

$\beta = 0$ and $H = 0.5$ are signatures of an uncorrelated data series and these values seen in region C at the lowest drive reflect the uncorrelated triggering of avalanches on short time scales by the external drive. In the autocorrelation process on these time scales, the distinct avalanche pulses are shifted and multiplied by the relatively long periods of quiet times. That is, the avalanches correlate with the zeros of quiet times, producing an uncorrelated spectrum and rescaled range. β in regions C and D both increase with $P_0 L^2$ but at different rates until they reach the same value of $\beta \approx 1$ at high drive. This is a limiting value that does not change regardless of how high the drive becomes, up to the saturation limit of the model.

The Hurst exponent of region D remains a relatively constant $H \approx 0.8$ regardless of driving rate even as region D moves from long time scales at low drive to shorter time scales at high drive. H stays constant because the rules of the sandpile do not change as driving rate changes. Avalanches are still triggered by the random drive but on shorter and shorter time scales as the drive increases. The avalanches are still correlated at all driving rates because the same process triggers them. The constant H and changing β of region D is shown in the upper plot of

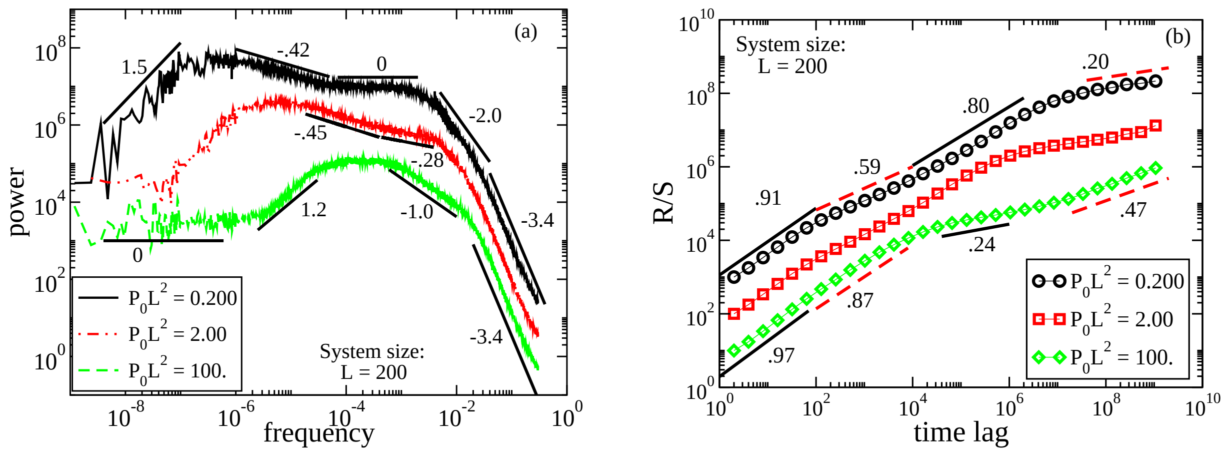

FIG. 2: (a) Power spectra and (b) R/S analysis of flips for different driving rates. The y values of both measures have been shifted for easier viewing. Numbers shown are the exponents of power law fits to regions, β for the spectra and H for R/S.

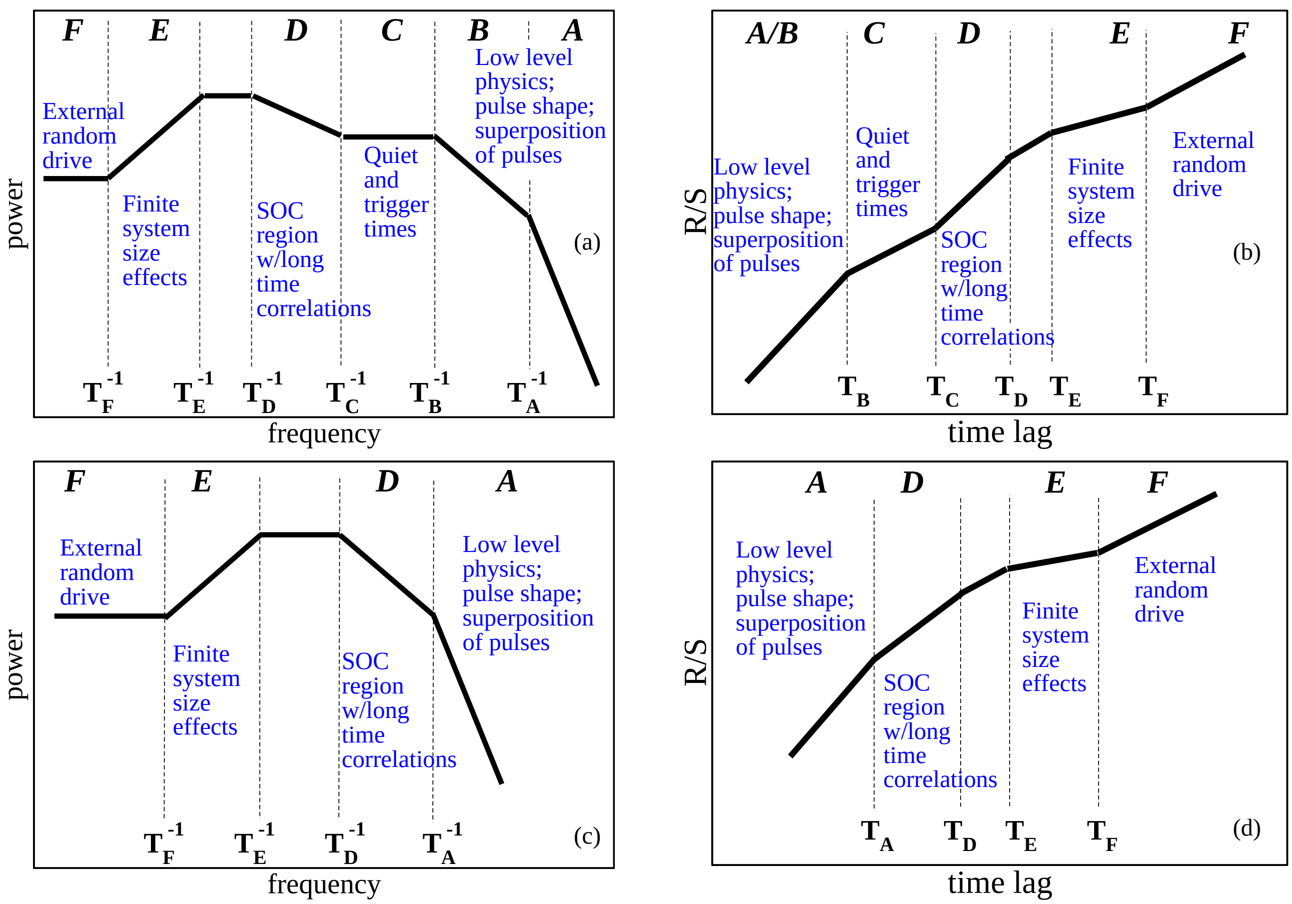

FIG. 3: Cartoons of distinct regions and their breakpoints and causes of power spectra and R/S analysis of sandpile flips. (a) Power spectrum of low drive, (b) R/S analysis of low drive, (c) power spectrum of high drive and (d) R/S analysis of high drive. (c) is taken from Figure 6 of [11] and the others are drawn in that spirit. (a) and (b) are from [12] but are reproduced here for completeness.

Figure 7. This will be discussed further in Section V.

β and H of region E both increase with driving rate, β towards 0 and H towards 0.5, Figure 8. At all drives, region E is due to anticorrelations among large discharge events that tend to reset the system and erase its memory. These events are anticorrelated because after a large event clears out the system another large event is unlikely since the majority of cells in the system are below critical. Discharge events can be either a single system-wide event or a rapid succession of smaller events; they are discussed in both [11] (high drive) and [12] (low drive). As drive increases, these large events, while still anticorrelated, occur closer together in time and reflect the random external drive of the system which itself is characterized by $\beta = 0$ and $H = 0.5$, the values to which region E approach.

Region F always has $\beta = 0$ and $H = 0.5$. We conjecture that this region reflects the random system drive and that β and H remain constant at any driving rate for all time scales beyond T_{F}.

IV. DISTINGUISHING BETWEEN LOW AND HIGH DRIVE

High drive means that avalanches overlap with each other in time but not necessarily in space (though for very high drive both types of overlap usually occur). This is relevant because some systems that have been discussed as possibly SOC, such as earthquake fault systems, have distinct events that do not overlap in time and, therefore, may be relatively weakly driven. In the sandpile, high drive is not simply defined by the ratio of input current to maximum output current, $\overline{J_{\mathrm{IN}}}/N_{\mathrm{f}}$. This just defines whether the system is overdriven or not, since flux in must never exceed the maximum possible flux out (N_{f}) of the bottom cell.

There are two overdrive limits, N_{f} and $N_{\mathrm{f}}/2$. Consider a cell that is unstable in the midst of a large avalanche. Being unstable, it dumps N_{f} grains to its downhill neighbor. In the next time step, for what we call a normal avalanche, this first cell is stable. But since it has lost N_{f} grains, its uphill neighbor is now unstable and will send N_{f} grains into the original cell. In this way, during a normal avalanche, a cell alternates between stable and unstable until the avalanche ends or washes past the cell. The time average flux through the cell is $N_{\mathrm{f}}/2$ and the maximum steady state input current is then $\overline{J_{\mathrm{IN}}} = N_{\mathrm{f}}/2$.

We define $\overline{J_{\mathrm{IN}}} = N_{\mathrm{f}}/2$ as the normal overdrive limit and $\overline{J_{\mathrm{IN}}} = N_{\mathrm{f}}$ as the super overdrive limit. A sandpile can still be in steady state between the two limits but the dynamics of the system changes since many cells are unstable for successive time steps to transport the increased flux in. Steady state is not possible for $\overline{J_{\mathrm{IN}}} > N_{\mathrm{f}}$ since flux into the system will always exceed flux out. We will only discuss driving rates below the normal overdrive limit.

The limit for spatial overlap has been previously given as $P_0 < \mathrm{L}^{-1}$ [11]. This comes from the condition of $P_0\,\overline{s} < 1$, where $\overline{s}$ is the average size of an avalanche and $\overline{s} \sim \mathrm{L}$ (in nonoverlapping regime). But since we analyze the flips *time* series, we need to quantify overlapping in time, not space.

To find the condition for overlap in time we must consider quiet times. If the average avalanche duration $\overline{s}$ is greater than the average trigger time T_t then avalanches will overlap in time. Using $T_t = N_{\mathrm{f}}/P_0 L U_0$ [21] (flux out divided by flux in), the condition for overlapping in time is $\overline{s} > T_t$ or $\mathrm{L} > N_{\mathrm{f}}/P_0 \mathrm{L} U_0$, giving $P_0 L > N_{\mathrm{f}}/\mathrm{L}U_0$. The most general quantity, then, to measure for comparing sandpiles in the same drive regime but where all dynamical parameters are different is $V_{\mathrm{g}} = P_0 L^2 U_0/N_{\mathrm{f}}$, where $V_{\mathrm{g}} > 1$ indicates high drive.

For our model, $U_0 = 1$ and $N_{\mathrm{f}} = 3 \ll \mathrm{L}$ so the high drive condition reduces to $P_0 L > \mathrm{L}^{-1}$. It is convenient to remember this condition as (driving rate) > (system size)$^{-1}$. In practice, we find the condition for low drive (when region C has $\beta = 0$ and $H = 0.5$) to be $P_0 L \ll \mathrm{L}^{-1}$.

The average quiet time $\overline{q}$ decreases as system size increases even as input current $\overline{J_{\mathrm{IN}}} = P_0 L$ remains constant. This effect is due to the increase in the average size of an avalanche with system size, $\overline{s} \sim \mathrm{L}$, and larger avalanches lasting longer in time. That $\overline{s}$ is independent of driving rate (for low, nonoverlapping drive) reflects the critical nature of the steady state of the sandpile. A critical cell is the food of an avalanche. A larger sandpile has more cells and therefore can have more neighboring cells that are close to critical. Any avalanche, then, has more food to eat and can live longer. Since avalanches propagate at the same speed (1 cell per time step) regardless of system size, the larger systems will have larger avalanches regardless of driving rate.

With the low drive condition $P_0 L \ll \mathrm{L}^{-1}$ we see that there are two routes to high drive: 1) increasing driving rate P_0 while keeping the system size L fixed and 2)

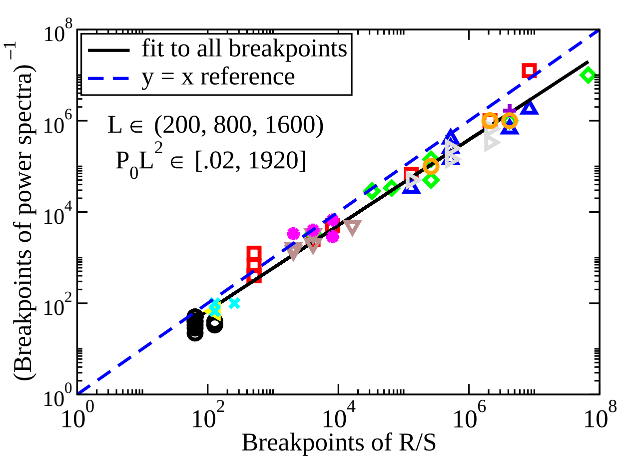

FIG. 4: Inverse breakpoints of the power spectra versus breakpoints of R/S analysis for flips time series of different size sandpiles and driving rate.

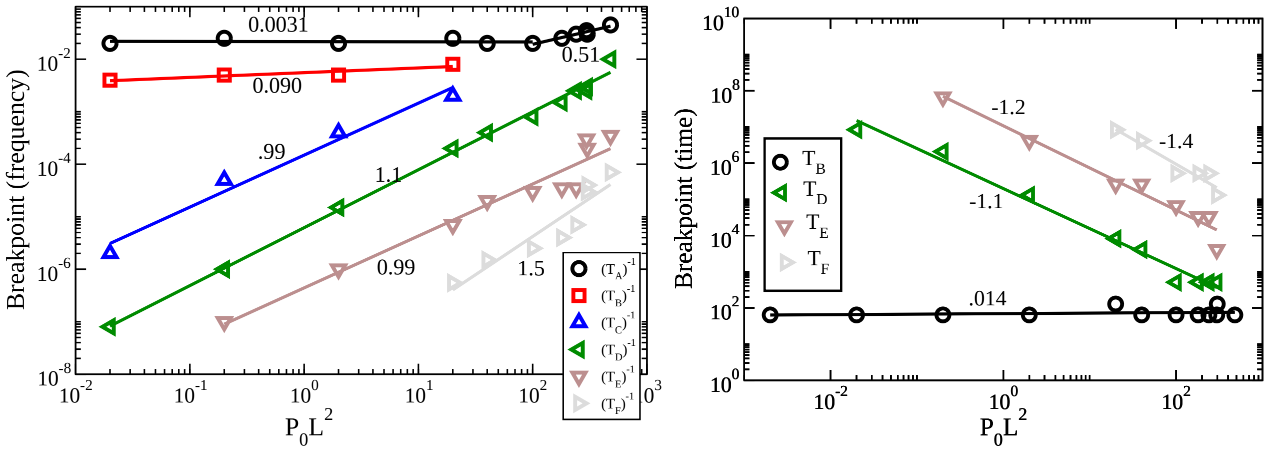

FIG. 5: Breakpoints of power spectra and R/S analysis versus driving rate for L = 200 sandpile. Numbers shown are exponents of power law fits to the data.

increasing L while keeping P_0L fixed. The first method is easy to visualize and is the more 'traditional' way of increasing drive. Consider a sandpile being driven by an input current of $\overline{J_{\mathrm{IN}}} = P_0L$ grains per time step over the entire system. Increasing P_0 will obviously increase $\overline{J_{\mathrm{IN}}}$ and decrease $\overline{q}$. More grains of sand fall in fewer time steps, avalanches initiate much more frequently and therefore begin to overlap in time.

The other way to increase drive is not as intuitive: consider a fixed $\overline{J_{\mathrm{IN}}} = P_0L$ and increase the system size L. To keep $\overline{J_{\mathrm{IN}}}$ constant, P_0 must decrease. But, because of the increase in L, the average avalanche size increases and avalanches begin to overlap in time. Since P_0L is the same as before, the same number of avalanches per time are triggered but they last longer on average and thus overlap. This is seen in the power spectrum. When L is increased and P_0L is kept constant, the breakpoint T_{B}^{-1} moves to lower frequencies and T_{C}^{-1} stays fixed. Region C, then, has shrunk in width because quiet times have decreased due to the larger system. This is a restatement of the trapezoid analysis of Part I [12], where we show that breakpoint T_{B}^{-1} moves to lower frequencies for larger trapezoids (that represent larger avalanches due to larger systems). Therefore, increasing system size while keeping constant the average flux in (P_0L) has the effect of increasing the driving rate.

This has implications, for example, in investigations of finite size and/or multifractal scaling. Figure 9 shows justification for comparing systems with the same effective driving rate, $P_0L^2 = 2.0$. The spectra of systems with different driving rates P_0L and system sizes L but with the same P_0L^2 can be rescaled to lie on top of each other. The spectra of systems with the same driving rate P_0L but with different system size L cannot be rescaled to lie on top of each other. We used a rescaling function

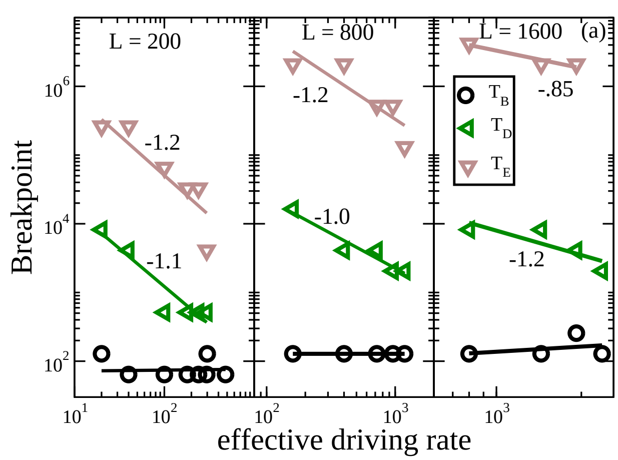

FIG. 6: Breakpoints of power spectra and R/S analysis versus driving rate for L = 800 and 1600 sandpile. Numbers shown are exponents of power law fits to the data.

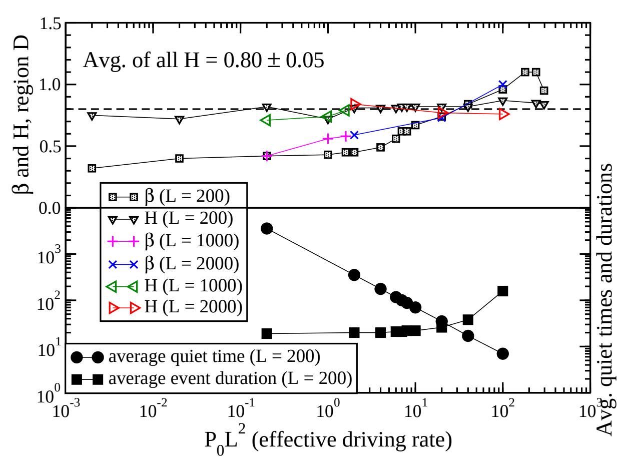

FIG. 7: H, β, average quiet time and average duration versus five orders of magnitude of effective driving rate.

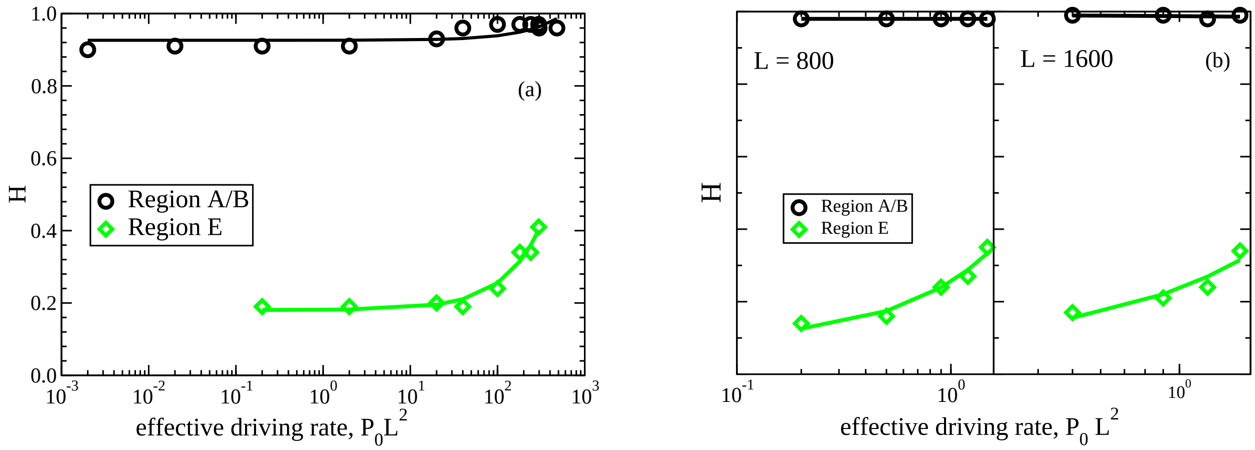

FIG. 8: Hurst exponent versus driving rate for (a) L = 200 and (b) L = 800, 1600. Solid lines are linear fits on a linear y axis and a logarithmic x axis.

of the form

$$y_0 \log_{10} S(f) = g\left(x_0 log_{10}(f/x_1)\right),$$

where $S(f)$ is the power spectrum and $g(f)$ is a scaling function. Different sets of parameters $[x_0, x_1, y_0]$ are needed for each spectrum so this is not the same as, for instance, the multifractal rescaling of avalanche size PDFs of [9], where the same value of each parameter is used for different system sizes.

We mention two notes about the effective driving rate. First, P_0L^2 is used because our measure, flips, is a global quantity where information about the entire system must be known. Hence, P_0L, total input into the system, and L system size must be known. But other local measures, such as flux through a single cell, may only need to know P_0L. The other note is that the appropriate rescaling quantity may instead by $P_0^aL^b$ where a and b may be values near to but different from 1 and 2, respectively. We have not investigated this and have found P_0L^2 to be a reasonable value for comparison.

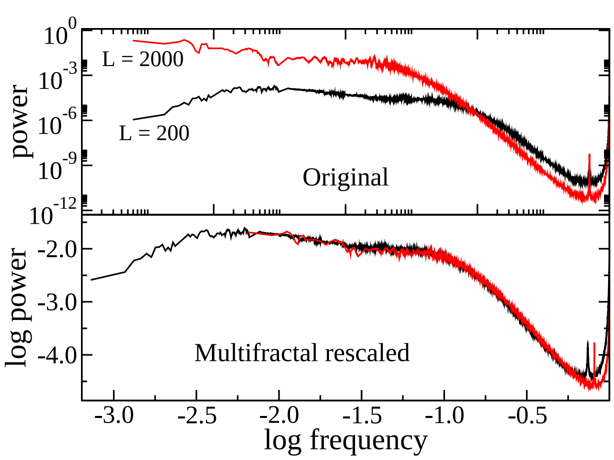

FIG. 9: Rescaling of power spectra of two different systems with $P_0L^2 = 2.0$, one with L = 200 and one with L = 2000.

V. DISCUSSION

The cause of the change in the spectra and R/S analysis is that the average size and frequency of quiet times decrease as effective driving rate increases. Sand falls on the system more frequently and triggers avalanches more quickly so that all dynamics moves to shorter time scales. Avalanches overlap in space and time and the correlations among them move to shorter time scales.

Region C is the quiet times indicator. When it exists and $\beta = 0$ and $H = 0.5$ the system is in a low drive regime and events are distinct and well-separated. The presence of correlations among events on long time scales is seen in values of $\beta > 0$ and $H \approx 0.8$ in region D. As events are triggered more closely together in time, the average quiet time decreases to below the average avalanche duration and the separation between regions C and D is lost. This decrease of quiet times also causes β in region D to change most drastically. The simultaneous effects are seen in Figure 7.

There are two main points that we would like to particularly emphasize and discuss. The first is that β in region D increases with driving rate until it reaches a limiting value of 1, a $1/f$ region. The second is that H in region D stays constant while β changes and, thus, $\beta = 2H - 1$ is not satisfied.

At high drive, region D is known to have a $1/f$ scaling [11] and the reason was given that overlapping of events produces this special spectral region. Our results show that at low drive, when events do not overlap and $0 < \beta < 1$, the system still has the same physics (by definition) and the same correlations exist at all driving rates, as shown by the constant H. Given the import attached to the specific $1/f$ scaling over the years, we point out that perhaps $1/f$ is not always $1/f$. By this we mean that the physics that produces $1/f$ may still exist in a system that does not actually reveal a $1/f$ spectrum simply because the system is more slowly driven and the 'signal' is broken up by large periods of quiet times. It implies that a $f^{-\beta}$ scaling with $0 < \beta < 1$ can be just as 'special' as $\beta = 1$.

We can test for this effect by inserting Poisson-distributed quiet times into a series of fractional Gaussian noise (fGn) [13]. FGn is a time series that has a Gaussian probability distribution and an arbitrary Hurst exponent. We inserted quiet times into a fGn series of nonnegative integers created with $H = 0.8$ such that the average of the quiet times was approximately the same as the average of the original fGn series (≈ 20). The results are shown in Figure 10. The spectrum and R/S analysis are shown for the original and modified series. The original series follows $\beta = 2H - 1$, with $H \approx 0.8$ and $\beta \approx 0.6$. When quiet times are added between each point of the original series a region of $H \approx 0.5$ appears up to a certain time lag. Beyond this time lag, $H \approx 0.8$ as expected since the correlations among the data have not changed. In the spectrum, $\beta \approx 0$ down to the inverse of the same time lag and then $\beta \approx 0.43$ (note that the R/S shows the breakpoint at a longer time than in the spectrum, the same effect seen in Figure 4). This shows that the introduction of quiet times into a correlated series affects β but not H. This effect accounts for the change seen in the sandpile flips data, when β decreases with decreasing drive and increasing average quiet time.

A difference between the sandpile time series and other physical systems that have a $1/f$ spectrum is that in the sandpile series, there are definite lower and upper limits to its values. These limits are 0 when there is no activity in the sandpile and L when all sites are unstable. In other systems that exhibit $1/f$ noise, such as resistance fluctuations [22], there are no such limits though the probability of extremely large deviations from the mean are very small.

There is still the issue, though, that $\beta = 1$ appears to be a limiting value as driving rate increases. Is this simply due to (overlapping of events? [12] shows that removing all quiet times from between separate events of a low drive flips time series produces a spectrum with a region D where $\beta = 1$. There is no overlapping of events by design. But what this altered series and an unaltered high drive series have in common is an almost complete lack of quiet times. It appears that $1/f$ is due more to a lack of quiet times and distinct pulse shapes than to overlapping.

There are some quiet times in the high drive sandpile but they are very small, on average. To test if they are important to the dynamics, we shuffle superpulses. A superpulse is defined as the structure between successive quiet times in a high drive flips time series (Figure 11). A superpulse comprises many overlapping avalanches. By randomly shuffling the superpulses of a high drive time series, we see that the correlations that lead to $1/f$ and $H \approx 0.8$ are on time scales shorter than the average superpulse. Beyond these time scales, $\beta \approx 0$ and $H \approx 0.5$, signatures of uncorrelated data (Figure 12). We conclude that quiet times are unimportant in the dyanamics of the high drive sandpile.

Our second point is that the relation $\beta = 2H - 1$ does not hold for the sandpile, a SOC system. This relationship was derived based on Gaussian statistics [19]. But R/S analysis itself has no requirement for Gaussian statistics in order for it to measure correlations. Therefore, the relation $\beta = 2H - 1$ that is often associated with the Hurst exponent and power spectrum does not always hold for all systems, such as SOC and the running sandpile. So a physical system cannot be modeled by both SOC and fGn.

VI. CONCLUSIONS

We have analyzed the one dimensional directed running sandpile SOC model for five orders of magnitude of effective driving rate and for different system sizes and shown how the power spectrum and R/S analysis change from low drive to high drive. The most noticeable feature of the change in signatures is the loss of the power law region C at low drive with $\beta = 0$ and $H = 0.5$. This region is due to uncorrelated quiet times between distinct individual avalanches. The region disappears because events are triggered more frequently in the sandpile as driving rate increases; this causes a virtual extinction of quiet times. β and H of this uncorrelated region increase with driving rate until they reach limiting values of $\beta \approx 1$ and $H \approx 0.8$, both being signs of long time correlations. The greatest change in β with increasing driving rate is when the average quiet time is on the order of the average avalanche duration.

At low drive, the power law region D exists on time scales longer than region C. Here, $\beta \approx 0.4$ at very low drive and β increases with increasing driving rate, reaching the limiting value of 1 as region D merges with region C. The Hurst exponent of this region stays approximately constant, $H \approx 0.8$, as driving rate changes, reflecting the same types of correlations among separate events and the same underlying rules of the system at any level of external forcing.

The changing β and constant H of the correlated region D imply that $1/f$ is not always $1/f$ and that $\beta = 2H - 1$ does not hold for the SOC sandpile model. This first part means that the dynamics that produces a $1/f$ signature at high drive is still present at low drive (since H is constant) but that the spectrum will instead scale as $f^{-\beta}$ with $0 < \beta < 1$. This is taken to mean that such values of β are not necessarily any less 'special' than $\beta = 1$ and that the search for a general underlying process that leads to $1/f$ may be found in systems with different values of β.

$\beta = 2H - 1$ is often accepted as a true statement that relates β and H, regardless of the system. It was derived based upon an artificial data set that is created by an algorithm that is designed to make the relation

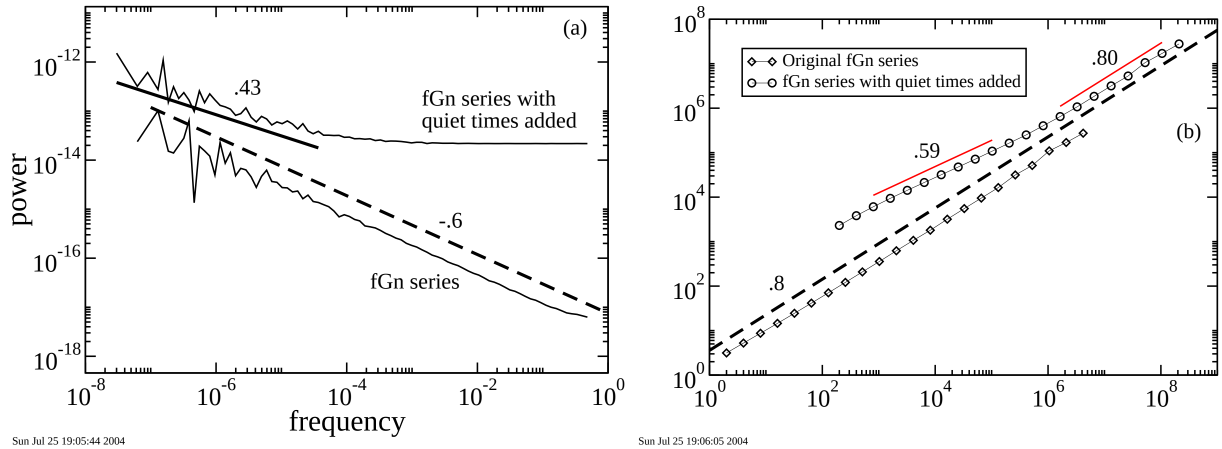

FIG. 10: (a) Power spectra and (b) R/S analysis for fractional Gaussian noise of $H = 0.8$ with and without quiet times added. Numbers shown are β for spectra and H for R/S.

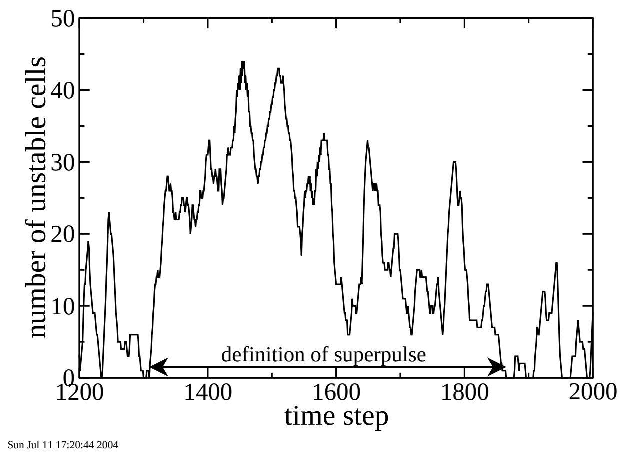

FIG. 11: Section of flips time series for high drive sandpile and definition of a superpulse.

true. We see that this relation does not hold for a system that creates time series that are, perhaps, more ‘natural’. When looking at a physical system, then, both the power spectrum and the R/S analysis can be calculated and compared to see if the system under study can be modeled as a simple fractional Gaussian noise process where $\beta = 2H - 1$ does hold or by some other process where it does not hold, such as SOC.

[1] D. E. Newman, B. A. Carreras, P. H. Diamond, and T. S. Hahm, Phys. Plasmas **3**, 1858 (1996).

[2] D. E. Newman, B. A. Carreras, and P. H. Diamond, Phys. Let. A pp. 58–63 (1996).

[3] A. T. Y. Lui, S. C. Chapman, K. Liou, P. T. Newell, C. I. Meng, M. Brittnacher, and G. K. Parks, Geophys. Res. Let. **27**, 911 (2000).

[4] S. C. Chapman and N. W. Watkins, Space Sci. Rev. **95** (2001).

[5] D. L. Turcotte, *Fractals and Chaos in Geology and Geophysics* (Cambridge University Press, 1997), 2nd ed.

[6] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Let. **59**, 381 (1987).

[7] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988).

[8] P. Bak, *How Nature Works* (Springer-Verlag, New York, 1996).

[9] L. P. Kadanoff, S. R. Nagel, L. Wu, and S. min Zhou, Phys. Rev. A **39**, 6524 (1989).

[10] S. S. Manna, J. Phys. A **24** (1991).

[11] T. Hwa and M. Kardar, Phys. Rev. A **45**, 7002 (1992).

[12] R. Woodard, D. E. Newman, R. Sánchez, and B. A. Carreras (2004), submitted to PRE.

[13] B. B. Mandelbrot, *Multifractals and* $1/f$ *noise* (Springer-Verlag, 1999).

[14] URL `http://www.nslij-genetics.org/wli/1fnoise/`.

[15] H. J. Jensen, *Self-organized criticality: emergent complex behaviour in physical and biological systems* (Cambridge University Press, Cambridge, 1998).

[16] H. E. Hurst, Trans. Am. Soc. Civ. Eng. **116**, 770 (1951).

[17] B. B. Mandelbrot, *Gaussian self-affinity and fractals* (Springer-Verlag, 2002).

[18] B. B. Mandelbrot and J. R. Wallis, Water Resources Research **5**, 321 (1969).

[19] B. D. Malamud and D. L. Turcotte, Adv. Geophys. **40**, 1 (1999).

[20] M. Gilmore, C. X. Yu, T. Rhodes, and W. Peebles, Phys. Plasmas **9**, 1312 (2002).

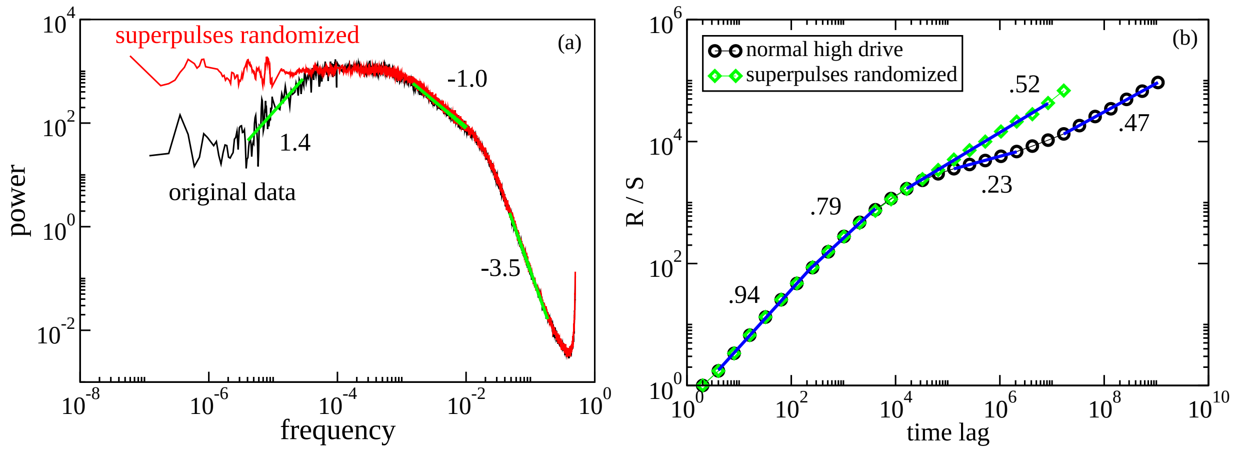

FIG. 12: Power spectra and R/S for high drive with superpulses changed. Numbers shown are β for spectra and H for R/S.

[21] R. Sánchez, D. E. Newman, and B. A. Carreras, Phys. Rev. Let. **88**, 068302 (2002).

[22] R. F. Voss and J. Clarke, Phys. Rev. B **13**, 556 (1976).