TECHNICAL REPORT R-96

ANALYSIS OF MULTIPOINT-MULTITIME CORRELATIONS AND DIFFUSION IN DECAYING HOMOGENEOUS TURBULENCE

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SUMMARY

Two-point, two-time correlation equations are obtained by considering the Navier-Stokes equations for two points in a turbulent fluid at two different times. By neglecting the triple correlations in the equations, a solution is obtained for the space-time velocity correlation in the final period of decay. The analysis is extended to earlier times by considering the Navier-Stokes equations at three points in the fluid at three different times. The resulting set of equations is made determinate by neglecting the quadruple correlations in comparison with the triple correlations, as in a previous paper by the author which considered correlations involving only one time.

The diffusion of particles from a source in a decaying turbulent field is calculated approximately by assuming that the velocity fluctuations are small. The theoretical results are compared with experiments for diffusion from a line source in a decaying turbulent stream.

INTRODUCTION

Most of the theoretical work on homogeneous turbulence has been based on correlations between fluctuating quantities at several points in a fluid at a single time (e.g., ref. 1). Correlations involving several different times as well as several points in the fluid are also of considerable interest and have been studied by several authors (refs. 2 to 8). These studies were concerned mostly with the kinematics of space-time correlations, although some aspects of the dynamical problems were also considered. In connection with the dynamical problem, Bass (ref. 5) set up the space-time equivalents of the Kármán-Howarth equation (ref. 9), but no solutions were obtained.

This paper is concerned primarily with the dynamical problem. First, a solution for the final period is obtained by neglecting the triple correlations in the two-point, two-time equation. A similar solution was obtained by Batchelor and Townsend (ref. 3) by use of a method that considered unaveraged velocities rather than the two-point, two-time equations considered here. However, the method used in this report is more convenient for extension to earlier times. The extension to earlier decay times, or to higher Reynolds numbers, is made by retaining the triple correlations. An expression for these correlations is obtained by neglecting the quadruple correlations in a three-point, three-time equation. Solutions for still earlier times could be obtained by considering the turbulent fluid at a larger number of points and times. This procedure is analogous to that used previously by the author for multipoint correlations at a single time (refs. 10 and 11).

By assuming that the turbulent fluctuations are sufficiently small for squares and products of the fluctuations to be negligible, turbulent diffusion from a source is calculated approximately; it can be shown that the Lagrangian correlation and the Eulerian time correlation are essentially equal for this case. The possibility of replacing the Lagrangian by the time correlation at a point has been suggested by Burgers (ref. 4). Recently Baldwin (ref. 12) obtained an experimental indication that this is a reasonable approximation.

In the next section the space-time correlations for the final period are considered; a higher order approximation for earlier times is taken up in later sections of the paper.

SYMBOLS

D	indicates substantial derivative
E	energy spectrum function
f,g,h	arbitrary functions
i	$\sqrt{-1}$
J_0	constant that depends on initial conditions
n	integer
P,P',P''	points
p	instantaneous pressure
R_{11}	longitudinal space-time correlation coefficient, defined by eq. (21)
$\boldsymbol{r},\boldsymbol{r}'$	distance vectors
T,T'	dimensionless time, defined by eq. (43)
ΔT	dimensionless time increment $T'-T$
T_m	dimensionless time halfway between T and T'
t,t',t''	times
t_m	time halfway between t and t'
t_0	reference time
$\Delta t,\Delta t'$	time increments $t'-t$ and $t''-t$, respectively
u_i,u_j	instantaneous velocity components
v	component of velocity in y-direction
W	given by eq. (38)
x_i,x_j,x_k	space coordinates
Y	distance in y-direction that a fluid particle originally at $y=0$ travels during time interval $t'-t_1$
y,z	space coordinates
α,β	Fourier transforms defined by eqs. (27), (28), and (29)
β_0	constant that depends on initial conditions
θ	angle between $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}'$
$\boldsymbol{\kappa},\boldsymbol{\kappa}'$	wave number vectors
$d\boldsymbol{\kappa}$	equals $d\kappa_1 d\kappa_2 d\kappa_3$
λ	microscale based on space interval
λ_t	microscale based on time interval
ν	kinematic viscosity
ρ	density
φ	Fourier transform defined by eq. (7) or (8)

Subscripts:

L	Lagrangian
i,j,k,l	tensor subscripts that have values 1, 2, or 3 and designate coordinate directions

Superscripts:

$','' $	referring to points P' and P''

TWO-POINT, TWO-TIME CORRELATION AND SPECTRAL EQUATIONS AND APPLICATION TO FINAL PERIOD OF DECAY

For obtaining the two-point, two-time correlation equations, first write the Navier-Stokes equations for the points P and P' separated by the distance vector $\boldsymbol{r}$ and the time increment Δt:

$$\frac{\partial u_i}{\partial t}+\frac{\partial(u_iu_k)}{\partial x_k}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}+\nu\frac{\partial^2 u_i}{\partial x_k\partial x_k} \tag{1}$$

$$\frac{\partial u_j'}{\partial t'}+\frac{\partial(u_j'u_k')}{\partial x_k'}=-\frac{1}{\rho}\frac{\partial p'}{\partial x_j'}+\nu\frac{\partial^2 u_j'}{\partial x_k'\partial x_k'} \tag{2}$$

where the subscripts can take on the values 1, 2, 3 and a repeated subscript in a term indicates a summation. The quantities u_i and u_j' are instantaneous velocity components, x_i is a space coordinate, t is the time, ρ is the density, ν is the kinematic viscosity, and p is the instantaneous pressure. Multiplying the first equation by u_j', the second by u_i, and taking space averages result in

$$\frac{\partial\overline{u_iu_j'}}{\partial t}+\frac{\partial\overline{u_iu_j'u_k}}{\partial x_k}=-\frac{1}{\rho}\frac{\partial\overline{pu_j'}}{\partial x_i}+\nu\frac{\partial^2\overline{u_iu_j'}}{\partial x_k\partial x_k} \tag{3}$$

$$\frac{\partial\overline{u_iu_j'}}{\partial t'}+\frac{\partial\overline{u_iu_j'u_k'}}{\partial x_k'}=-\frac{1}{\rho}\frac{\partial\overline{p'u_i}}{\partial x_j'}+\nu\frac{\partial^2\overline{u_iu_j'}}{\partial x_k'\partial x_k'} \tag{4}$$

where the fact that quantities at x_i and t are independent of x_i' and t' was used. By introducing the transformations $\partial/\partial x_i=-\partial/\partial r_i$, $\partial/\partial x_i'=\partial/\partial r_i$, $(\partial/\partial t)_{t'}=(\partial/\partial t)_{\Delta t}-\partial/\partial\Delta t$, and $\partial/\partial t'=\partial/\partial\Delta t$, which are obtained by writing a correlation as a function of r_i, t, and Δt and differentiating, the following equations are obtained from (3) and (4):

$$\frac{\partial\overline{u_iu_j'}}{\partial t}+\frac{\partial}{\partial r_k}\overline{u_ju_ku_i'(-\boldsymbol{r},-\Delta t,t+\Delta t)}-\frac{\partial}{\partial r_k}\overline{u_iu_ku_j'(\boldsymbol{r},\Delta t,t)}$$
$$=\frac{1}{\rho}\frac{\partial}{\partial r_i}\overline{pu_j'}-\frac{1}{\rho}\frac{\partial}{\partial r_j}\overline{pu_i'(-\boldsymbol{r},-\Delta t,t+\Delta t)}+2\nu\frac{\partial^2\overline{u_iu_j'}}{\partial r_k\partial r_k} \tag{5}$$

$$\frac{\partial\overline{u_iu_j'}}{\partial\Delta t}+\frac{\partial}{\partial r_k}\overline{u_ju_ku_i'(-\boldsymbol{r},-\Delta t,t+\Delta t)}$$
$$=-\frac{1}{\rho}\frac{\partial}{\partial r_j}\overline{pu_i'(-\boldsymbol{r},-\Delta t,t+\Delta t)}+\nu\frac{\partial^2\overline{u_iu_j'}}{\partial r_k\partial r_k} \tag{6}$$

Equations (5) and (6) are the space-time equivalents of the Kármán-Howarth equation. They were obtained in a slightly different form, for the case of isotropic turbulence, by Bass (ref. 5).

In order to convert equations (5) and (6) to spectral form, the following three-dimensional Fourier transforms are introduced:

$$\overline{u_i u_j'(\boldsymbol{r},\Delta t,t)}=\int_{-\infty}^{\infty}\varphi_{ij}(\boldsymbol{\kappa},\Delta t,t)e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}}d\boldsymbol{\kappa} \tag{7}$$

$$\overline{u_j u_k u_i'(\boldsymbol{r},\Delta t,t)}=\int_{-\infty}^{\infty}\varphi_{jki}(\boldsymbol{\kappa},\Delta t,t)e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}}d\boldsymbol{\kappa} \tag{8}$$

$$\overline{p u_j'(\boldsymbol{r},\Delta t,t)}=\int_{-\infty}^{\infty}\lambda_j(\boldsymbol{\kappa},\Delta t,t)e^{i\boldsymbol{\kappa}\cdot\boldsymbol{r}}d\boldsymbol{\kappa} \tag{9}$$

where $\boldsymbol{\kappa}$ is a wave number vector and $d\kappa=d\kappa_1 d\kappa_2 d\kappa_3$. By introducing these transforms, equations (5) and (6) become

$$\frac{\partial\varphi_{ij}}{\partial t}+i\kappa_k\varphi_{jki}(-\boldsymbol{\kappa},-\Delta t,t+\Delta t)-i\kappa_k\varphi_{ikj}(\boldsymbol{\kappa},\Delta t,t)$$
$$=\frac{1}{\rho}i\kappa_i\lambda_j-\frac{1}{\rho}i\kappa_j\lambda_i(-\boldsymbol{\kappa},-\Delta t,t+\Delta t)-2\nu\kappa^2\varphi_{ij} \tag{10}$$

$$\frac{\partial\varphi_{ij}}{\partial\Delta t}+i\kappa_k\varphi_{jki}(-\boldsymbol{\kappa},-\Delta t,t+\Delta t)$$
$$=-\frac{1}{\rho}\kappa_j\lambda_i(-\boldsymbol{\kappa},-\Delta t,t+\Delta t)-\nu\kappa^2\varphi_{ij} \tag{11}$$

In order to convert the tensor equations (10) and (11) to scalar equations, contract the indices i and j:

$$\frac{\partial\varphi_{ii}}{\partial t}+2\nu\kappa^2\varphi_{ii}=i\kappa_k\varphi_{iki}(\boldsymbol{\kappa},\Delta t,t)$$
$$+i(-\kappa_k)\varphi_{iki}(-\boldsymbol{\kappa},-\Delta t,t+\Delta t) \tag{12}$$

$$\frac{\partial\varphi_{ii}}{\partial\Delta t}+\nu\kappa^2\varphi_{ii}=i(-\kappa_k)\varphi_{iki}(-\boldsymbol{\kappa},-\Delta t,t+\Delta t) \tag{13}$$

The pressure terms drop out of these scalar equations because of the continuity relation $\partial u_i/\partial x_i=\partial u_i'/\partial x_i'=0$ and the relation $\partial/\partial r_i=-\partial/\partial r_i'$ (see eqs. (3) and (4)).

FINAL PERIOD

Equations (12) and (13), as they stand, contain too many unknowns for solutions to be obtained. For the final period of decay, however, the triple correlation or inertia terms should be negligible compared with the double correlation terms. Thus, the terms on the right sides of equations (12) and (13) are neglected, and the following solutions are obtained:

$$\varphi_{ii}=f_1(\boldsymbol{\kappa},\Delta t)e^{-2\nu\kappa^2(t-t_0)} \tag{14}$$

$$\varphi_{ii}=f_2(\boldsymbol{\kappa},t)e^{-\nu\kappa^2\Delta t} \tag{15}$$

In order for these equations to be consistent,

$$E=f(\kappa)e^{-\nu\kappa^2\Delta t}e^{-2\nu\kappa^2(t-t_0)} \tag{16}$$

where the energy spectrum function $E=2\pi\kappa^2\varphi_{ii}$ has been introduced. Evaluate $f(\kappa)$ by letting $E=J_0\kappa^4/3\pi$ when κ is small (Lin, ref. 13). This gives

$$E=\frac{J_0\kappa^4}{3\pi}e^{-2\nu\kappa^2\left(t-t_0+\frac{1}{2}\Delta t\right)} \tag{17}$$

where J_0 is a constant that depends on initial conditions. For $\Delta t=0$, equation (17) reduces to the usual expression for the energy spectrum function in the final period, which involves only one time. By integrating equation (17) with respect to κ, the time correlation is obtained as

$$\frac{\overline{u_i u_i'}}{2}=\frac{J_0}{32(2\pi)^{1/2}}\nu^{-5/2}\left(t-t_0+\frac{1}{2}\Delta t\right)^{-5/2} \tag{18}$$

and, for isotropic turbulence, the longitudinal space-time correlation is

$$\overline{u_1 u_1'(\boldsymbol{r},\Delta t,t)}=\frac{J_0}{48(2\pi)^{1/2}}\nu^{-5/2}\left(t-t_0+\frac{1}{2}\Delta t\right)^{-5/2}$$
$$exp\left[-\frac{r^2}{8\nu\left(t-t_0+\frac{1}{2}\Delta t\right)}\right] \tag{19}$$

Equations (18) and (19) again reduce to the usual expressions involving only one time if $\Delta t=0$ (e.g., ref. 1, p. 94).

If a new time t_m, which lies halfway between t and $t'(t_m=t+\Delta t/2)$ is defined, then Δt does not appear explicitly in equations (17), (18), and (19), and $\overline{u_1u_1'}$ is a function only of t_m and r. For instance, the longitudinal space-time correlation becomes

$$\overline{u_1 u_1'(\boldsymbol{r},t_m)}=\frac{J_0}{48(2\pi)^{1/2}}\nu^{-5/2}(t_m-t_0)^{-5/2}$$
$$exp\left[-\frac{r^2}{8\nu(t_m-t_0)}\right] \tag{20}$$

However, the correlation *coefficient* is not independent of Δt. The longitudinal space-time correlation coefficient R_{11} is defined as

$$R_{11}\equiv\frac{\overline{u_1u_1'(r,t_m)}}{\left[\overline{u_1^2\left(t_m+\frac{1}{2}\Delta t\right)}\;\overline{u_1^2\left(t_m-\frac{1}{2}\Delta t\right)}\right]^{1/2}} \tag{21}$$

and, for the final period, becomes

$$R_{11}(r,\Delta t,t_m)=\left[1-\frac{1}{4}\left(\frac{\Delta t}{t_m-t_0}\right)^2\right]^{5/4} exp\left\{-\frac{1}{2}\left[\frac{r}{\sqrt{4\nu(t_m-t_0)}}\right]^2\right\} \tag{22}$$

Batchelor and Townsend (ref. 3) previously obtained this equation but by a different method, which considered unaveraged velocities. A dimensionless plot of R_{11} is presented in figure 1. The values of R_{11} decrease as time interval Δt increases. This is similar to the variation of correlation coefficient with distance and would be expected physically. The curves go to zero at a finite value of $\Delta t(\Delta t/2=t_m-t_0)$ whereas, as r increases, they go to zero only at $r=\infty$. The point where the value of R_{11} is zero corresponds to the point where one of the velocity fluctuations becomes infinite. These curves for the final period would not, of course, be expected to be accurate in the vicinity of that point. The singular behavior could be avoided, for positive values of Δt, by evaluating the correlation coefficient at t rather than at t_m, as in equation (19). However, this coefficient would not be symmetric with respect to Δt.

A microscale λ_t, which is based on time interval, can be defined by analogy with the usual microscale λ, which is based on space interval. Thus, λ_t might be defined as

$$\frac{1}{\lambda_t^2}=\frac{1}{\overline{u^2}}\left(\frac{\partial^2 R_{11}}{\partial\Delta t^2}\right)_0$$

where $\overline{u^2}=\frac{1}{3}\overline{u_iu_i}$ for isotropic turbulence. The ratio of λ_t^2 to λ^2 is then

$$\left(\frac{\lambda_t}{\lambda}\right)^2=\frac{\overline{u^2}(\partial^2 R_{11}/\partial r^2)_0}{(\partial^2 R_{11}/\partial\Delta t^2)_0}$$

For the final period this becomes

$$\left(\frac{\lambda_t}{\lambda}\right)^2=\frac{J_0\,(t_m-t_0)^{-3/2}}{120\,(2\pi)^{1/2}\,\nu^{7/2}}$$

Thus, λ_t/λ in the final period is a function of decay time as well as of J_0 and ν. Calculation of λ_t/λ

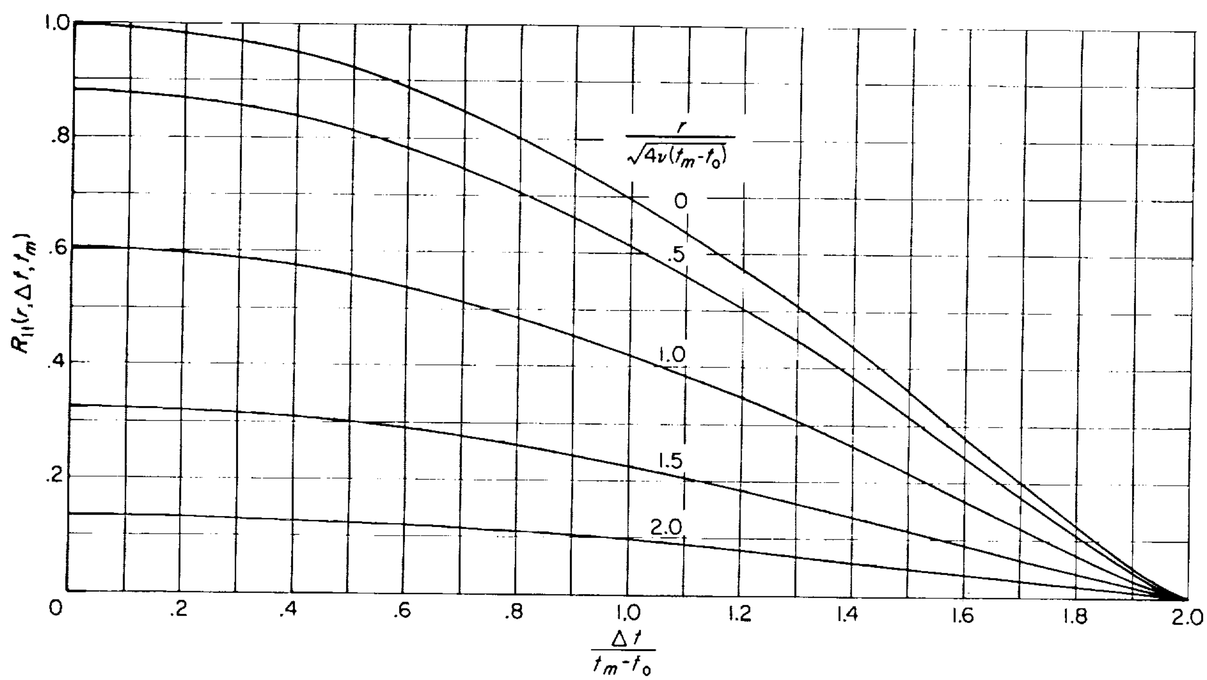

FIGURE 1.—Variation of longitudinal space-time double-velocity correlation coefficient in final period (eq. (21)) with space and time intervals. Correlation coefficient evaluated at $t_m=t+\Delta t/2$.

from this equation and the experimental data of Batchelor and Townsend in the final period (ref. 3) indicated values of that ratio on the order of unity.

THREE-POINT, THREE-TIME CORRELATION AND SPECTRAL EQUATIONS

To obtain the three-point, three-time correlation equations, write the Navier-Stokes equation at the points P, P', and P'' separated by the distance vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$ and the time increments Δt and $\Delta t'$. The vector configuration is shown in figure 2.

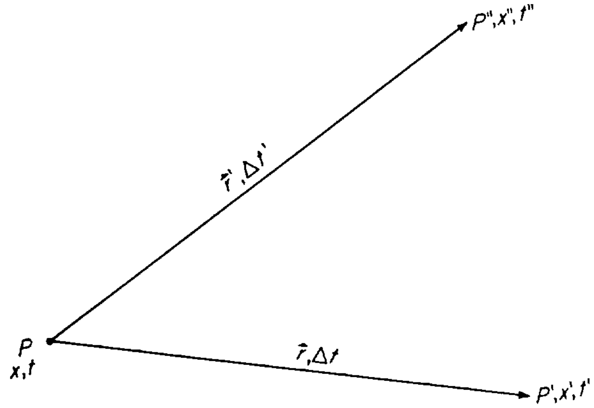

FIGURE 2.—Vector configuration for deriving three-point, three-time correlation equations.

The first two equations are the same as equations (1) and (2) with the subscripts k replaced by l. The third equation is

$$\frac{\partial u_k''}{\partial t''}+\frac{\partial}{\partial x_l''}(u_k''u_l'')=-\frac{1}{\rho}\frac{\partial p''}{\partial x_k''}+\nu\frac{\partial^2 u_k''}{\partial x_l''\partial x_l''} \quad (23)$$

By multiplying the first equation by $u_j'u_k''$, the second by u_iu_k'', the third by u_iu_j' and by taking space averages, the following correlation equations can be constructed:

$$\begin{aligned}
&\frac{\partial}{\partial t}\overline{u_iu_j'u_k''}-\frac{\partial}{\partial r_l}\overline{u_iu_j'u_k''u_l}-\frac{\partial}{\partial r_l'}\overline{u_iu_j'u_k''u_l}\\
&\qquad+\frac{\partial}{\partial r_l}\overline{u_iu_j'u_k''u_l'}+\frac{\partial}{\partial r_l'}\overline{u_iu_j'u_k''u_l''}\\
&=-\frac{1}{\rho}\left(-\frac{\partial}{\partial r_i}\overline{pu_j'u_k''}-\frac{\partial}{\partial r_i'}\overline{pu_j'u_k''}\right.\\
&\qquad\left.+\frac{\partial}{\partial r_j}\overline{p'u_iu_k''}+\frac{\partial}{\partial r_k'}\overline{p''u_iu_j'}\right)\\
&\quad+2\nu\left(\frac{\partial^2\overline{u_iu_j'u_k''}}{\partial r_l\partial r_l}+\frac{\partial^2\overline{u_iu_j'u_k''}}{\partial r_l\partial r_l'}+\frac{\partial^2\overline{u_iu_j'u_k''}}{\partial r_l'\partial r_l'}\right)
\end{aligned} \quad (24)$$

$$\frac{\partial}{\partial\Delta t}\overline{u_iu_j'u_k''}+\frac{\partial}{\partial r_l}\overline{u_iu_j'u_k''u_l'}=-\frac{1}{\rho}\frac{\partial}{\partial r_j}\overline{p'u_iu_k''}+\nu\frac{\partial^2\overline{u_iu_j'u_k''}}{\partial r_l\partial r_l} \quad (25)$$

$$\frac{\partial}{\partial\Delta t'}\overline{u_iu_j'u_k''}+\frac{\partial}{\partial r_l'}\overline{u_iu_j'u_k''u_l''}=-\frac{1}{\rho}\frac{\partial\overline{p''u_iu_j'}}{\partial r_k'}+\nu\frac{\partial^2\overline{u_iu_j'u_k''}}{\partial r_l'\partial r_l'} \quad (26)$$

where the following transformations were used:

$$\frac{\partial}{\partial x_l}=-\frac{\partial}{\partial r_l}-\frac{\partial}{\partial r_l'},\quad \frac{\partial}{\partial x_l'}=\frac{\partial}{\partial r_l},\quad \frac{\partial}{\partial x_l''}=\frac{\partial}{\partial r_l'},$$

$$\left(\frac{\partial}{\partial t}\right)_{t',t''}=\left(\frac{\partial}{\partial t}\right)_{\Delta t,\Delta t'}-\frac{\partial}{\partial\Delta t}-\frac{\partial}{\partial\Delta t'},$$

$$\frac{\partial}{\partial t'}=\frac{\partial}{\partial\Delta t},\quad \frac{\partial}{\partial t''}=\frac{\partial}{\partial\Delta t'}$$

Equations (24), (25), and (26) are the three-point, three-time correlation equations. In order to convert these equations to spectral form, the following six-dimensional Fourier transforms can be defined:

$$\overline{u_iu_j'u_k''}(\boldsymbol{r},\Delta t,\boldsymbol{r}',\Delta t',t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\beta_{ijk}(\boldsymbol{\kappa},\Delta t,\boldsymbol{\kappa}',\Delta t',t)e^{i(\boldsymbol{\kappa}\cdot\boldsymbol{r}+\boldsymbol{\kappa}'\cdot\boldsymbol{r}')}d\boldsymbol{\kappa}d\boldsymbol{\kappa}' \quad (27)$$

$$\overline{u_iu_lu_j'u_k''}(\boldsymbol{r},\Delta t,\boldsymbol{r}',\Delta t',t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\beta_{iljk}(\boldsymbol{\kappa},\Delta t,\boldsymbol{\kappa}',\Delta t',t)e^{i(\boldsymbol{\kappa}\cdot\boldsymbol{r}+\boldsymbol{\kappa}'\cdot\boldsymbol{r}')}d\boldsymbol{\kappa}d\boldsymbol{\kappa}' \quad (28)$$

$$\overline{pu_j'u_k''}=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\alpha_{jk}e^{i(\boldsymbol{\kappa}\cdot\boldsymbol{r}+\boldsymbol{\kappa}'\cdot\boldsymbol{r}')}d\boldsymbol{\kappa}d\boldsymbol{\kappa}' \quad (29)$$

Also,

$$\begin{aligned}
\overline{u_iu_j'u_l'u_k''}(\boldsymbol{r},\Delta t,\boldsymbol{r}',\Delta t')&=\overline{u_ju_lu_i'u_k''}(-\boldsymbol{r},-\Delta t,\boldsymbol{r}'-\boldsymbol{r},\Delta t'-\Delta t,t+\Delta t)\\
&=\int_{-\infty}^{\infty}\beta_{jlik}(-\boldsymbol{\kappa}-\boldsymbol{\kappa}',-\Delta t,\boldsymbol{\kappa}',\Delta t'-\Delta t,\\
&\qquad t+\Delta t)e^{i(\boldsymbol{\kappa}\cdot\boldsymbol{r}+\boldsymbol{\kappa}'\cdot\boldsymbol{r}')}d\boldsymbol{\kappa}d\boldsymbol{\kappa}'
\end{aligned} \quad (30)$$

and

$$\begin{aligned}
\overline{u_iu_j'u_k''u_l''}(\boldsymbol{r},\Delta t,\boldsymbol{r}',\Delta t')&=\overline{u_ku_lu_i'u_j''}(-\boldsymbol{r}',-\Delta t',\boldsymbol{r}-\boldsymbol{r}',\Delta t-\Delta t',t+\Delta t')\\
&=\int_{-\infty}^{\infty}\beta_{klij}(-\boldsymbol{\kappa}-\boldsymbol{\kappa}',-\Delta t',\boldsymbol{\kappa},\Delta t-\Delta t',\\
&\qquad t+\Delta t')e^{i(\boldsymbol{\kappa}\cdot\boldsymbol{r}+\boldsymbol{\kappa}'\cdot\boldsymbol{r}')}d\boldsymbol{\kappa}d\boldsymbol{\kappa}'
\end{aligned} \quad (31)$$

Similar expressions can be obtained for pressure correlations.

Substituting the preceding relations into equations (24), (25), and (26) gives, for the three-point, three-time spectral equations:

$$\frac{\partial}{\partial t}\beta_{ijk}+2\nu(\kappa^2+\kappa_l\kappa_l'+\kappa'^2)\beta_{ijk}=i(\kappa_l+\kappa_l')\beta_{iljk}$$
$$-i\kappa_l\beta_{jlik}(-\boldsymbol{\kappa}-\boldsymbol{\kappa}',-\Delta t,\boldsymbol{\kappa}',\Delta t'-\Delta t,t+\Delta t)$$
$$-i\kappa_l'\beta_{klij}(-\boldsymbol{\kappa}-\boldsymbol{\kappa}',-\Delta t',\boldsymbol{\kappa},\Delta t-\Delta t',t+\Delta t')$$
$$-\frac{1}{\rho}[-i(\kappa_i+\kappa_i')\alpha_{jk}+i\kappa_j\alpha_{ik}(-\boldsymbol{\kappa}-\boldsymbol{\kappa}',-\Delta t,\boldsymbol{\kappa}',$$
$$\Delta t'-\Delta t,t+\Delta t)+i\kappa_k'\alpha_{ij}(-\boldsymbol{\kappa}-\boldsymbol{\kappa}',-\Delta t',\boldsymbol{\kappa},$$
$$\Delta t-\Delta t',t+\Delta t')] \tag{32}$$

$$\frac{\partial}{\partial\Delta t}\beta_{ijk}+\nu\kappa^2\beta_{ijk}=-i\kappa_l\beta_{jlik}(-\boldsymbol{\kappa}-\boldsymbol{\kappa}',-\Delta t,\boldsymbol{\kappa}',$$
$$\Delta t'-\Delta t,t+\Delta t)-\frac{1}{\rho}\,i\kappa_j\alpha_{ik}(-\boldsymbol{\kappa}-\boldsymbol{\kappa}',$$
$$-\Delta t,\boldsymbol{\kappa}',\Delta t'-\Delta t,t+\Delta t) \tag{33}$$

$$\frac{\partial\beta_{ijk}}{\partial\Delta t'}+\nu\kappa'^2\beta_{ijk}=-i\kappa_l'\beta_{klij}(-\boldsymbol{\kappa}-\boldsymbol{\kappa}',-\Delta t',\boldsymbol{\kappa},$$
$$\Delta t-\Delta t',t+\Delta t')-\frac{1}{\rho}i\kappa_k'\alpha_{ij}(-\boldsymbol{\kappa}-\boldsymbol{\kappa}',-\Delta t',\boldsymbol{\kappa},$$
$$\Delta t-\Delta t',t+\Delta t') \tag{34}$$

SOLUTION FOR TIMES BEFORE FINAL PERIOD

For the final period of decay, a solution was obtained by neglecting the triple correlations in the two-point, two-time equations. Similarly, a solution applicable to times before the final period could be obtained by retaining the triple correlations and neglecting the quadruple correlations in the three-point, three-time equations. A fuller discussion of this procedure is given in references 10 and 11. In reference 10 it is shown that, if terms corresponding to quadruple correlations are neglected, the terms corresponding to pressure correlations must also be neglected. Thus, if all the terms on the right sides of equations (32), (33), and (34) are neglected, the equations can be integrated between t_0 and t to give

$$\beta_{ijk}=f_{ijk}(\boldsymbol{\kappa},\boldsymbol{\kappa}',\Delta t,\Delta t')\,exp\,[-2\nu(\kappa^2+\kappa_l\kappa_l'+\kappa'^2)(t-t_0)]$$

$$\beta_{ijk}=g_{ijk}(\boldsymbol{\kappa},\boldsymbol{\kappa}',t,\Delta t')\,exp\,(-\nu\kappa^2\Delta t)$$

$$\beta_{ijk}=h_{ijk}(\boldsymbol{\kappa},\boldsymbol{\kappa}',t,\Delta t)\,exp\,(-\nu\kappa'^2\Delta t')$$

In order for these equations to be consistent,

$$\kappa_k\beta_{iik}=\kappa_k[\beta_{iik}(\boldsymbol{\kappa},\boldsymbol{\kappa}')]_0\,exp\left\{-2\nu\left[\kappa^2\left(t-t_0+\frac{1}{2}\Delta t\right)\right.\right.$$
$$\left.\left.+\kappa\kappa'(t-t_0)\cos\theta+\kappa'^2\left(t-t_0+\frac{1}{2}\Delta t'\right)\right]\right\} \tag{35}$$

where the indices i and j have been contracted and the equation has been inner-multiplied by κ_k in order to convert it to scalar form. The subscript 0 refers to the values of β_{iik} at $t=t_0,\Delta t=\Delta t'=0$; and θ is the angle between $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}'$.

In order to connect β_{iik} and φ_{iki}, let $\boldsymbol{r}'=\Delta t'=0$ in equation (27) and compare with (8) to obtain

$$\varphi_{iki}(\boldsymbol{\kappa},\Delta t,t)=\int_{-\infty}^{\infty}\beta_{iik}(\boldsymbol{\kappa},\Delta t,\boldsymbol{\kappa}',0,t)\mathrm{d}\boldsymbol{\kappa}' \tag{36}$$

Substitution of equations (35) and (36) into (12) results in

$$\frac{\partial E}{\partial t}+2\nu\kappa^2E=W \tag{37}$$

where

$$E=2\pi\kappa^2\varphi_{ii}$$

and

$$W=\int_0^\infty i\kappa_k(\beta_{iik})_0(2\pi)^2\kappa^2\kappa'^2\,exp\{-2\nu[\kappa^2(t-t_0$$
$$+\Delta t/2)+\kappa'^2(t-t_0)]\}\cdot\left[\int_{-1}^{1}exp\,[-2\nu\kappa\kappa'(t-t_0)\right.$$
$$\left.\cos\theta]\mathrm{d}(\cos\theta)\right]\mathrm{d}\kappa'+\int_0^\infty[i(-\kappa_k)\beta_{iik}(-\boldsymbol{\kappa},$$
$$-\boldsymbol{\kappa}')]_0(2\pi)^2\kappa^2\kappa'^2\,exp\left\{-2\nu\left[\kappa^2\left(t-t_0+\frac{\Delta t}{2}\right)\right.\right.$$
$$\left.\left.+\kappa'^2(t-t_0+\Delta t)\right]\right\}\cdot\left[\int_{-1}^{1}exp\,[-2\nu\kappa\kappa'(t-t_0\right.$$
$$\left.+\Delta t)]\mathrm{d}(\cos\theta)\right]\mathrm{d}\kappa' \tag{38}$$

where $\mathrm{d}\boldsymbol{\kappa}'$ is written as $-2\pi\kappa'^2\mathrm{d}(\cos\theta)\mathrm{d}\kappa'$. The quantities $(\beta_{iik})_0$ depend on the initial conditions of the turbulence. In order that these results will reduce to previous results for $\Delta t=0$ (ref. 10), let

$$(2\pi)^2\,i\kappa_k\beta_{iik}(\boldsymbol{\kappa},\boldsymbol{\kappa}')_0=-\frac{1}{2}\beta_0(\kappa^4\kappa'^6-\kappa^6\kappa'^4) \tag{39}$$

Then,

$$(2\pi)^2\,i(-\kappa_k)\beta_{iik}(-\boldsymbol{\kappa},-\boldsymbol{\kappa}')_0=-\frac{1}{2}\beta_0(\kappa^4\kappa'^6-\kappa^6\kappa'^4) \tag{40}$$

Substituting equations (39) and (40) into (38) and carrying out the integrations with respect to θ and κ' result in

$$
\begin{aligned}
W= & -\frac{(\pi/2)^{1/2}}{256}\frac{\beta_0}{2}\,exp\left[-\frac{3}{2}\nu\kappa^2\left(t-t_0+\frac{2}{3}\Delta t\right)\right] \\
& \left[\frac{105\kappa^6}{\nu^{9/2}(t-t_0)^{9/2}}+\frac{45\kappa^8}{\nu^{7/2}(t-t_0)^{7/2}}\right. \\
& \left.-\frac{19\kappa^{10}}{\nu^{5/2}(t-t_0)^{5/2}}-\frac{3\kappa^{12}}{\nu^{3/2}(t-t_0)^{3/2}}\right] \\
& -\frac{(\pi/2)^{1/2}}{256}\frac{\beta_0}{2}\,exp\left[-\frac{3}{2}\nu\kappa^2\left(t-t_0+\frac{1}{3}\Delta t\right)\right] \\
& \left[\frac{105\kappa^6}{\nu^{9/2}(t-t_0+\Delta t)^{9/2}}+\frac{45\kappa^8}{\nu^{7/2}(t-t_0+\Delta t)^{7/2}}\right. \\
& \left.-\frac{19\kappa^{10}}{\nu^{5/2}(t-t_0+\Delta t)^{5/2}}-\frac{3\kappa^{12}}{\nu^{3/2}(t-t_0+\Delta t)^{3/2}}\right]
\end{aligned}
\tag{41}
$$

Equation (41) reduces to the expression for the energy-transfer function involving only one time (ref. 10) if $\Delta t=0$. Note that $\int_0^\infty W\mathrm{d}\kappa=0$ only for $\Delta t=0$.

Substituting equation (41) in (37) and integrating with respect to t result in

$$
\begin{aligned}
E= & \frac{J_0\kappa^4}{3\pi}\,exp\left[-2\nu\kappa^2\left(t-t_0+\frac{1}{2}\Delta t\right)\right] \\
& -\frac{\pi^{1/2}}{256\nu}\frac{\beta_0}{2}\,exp\left[-\frac{3}{2}\nu\kappa^2\left(t-t_0+\frac{2}{3}\Delta t\right)\right] \\
& \left[-\frac{15\sqrt{2}\kappa^6}{\nu^{7/2}(t-t_0)^{7/2}}-\frac{12\sqrt{2}\kappa^8}{\nu^{5/2}(t-t_0)^{5/2}}\right. \\
& +\frac{7\sqrt{2}\kappa^{10}}{3\nu^{3/2}(t-t_0)^{3/2}}+\frac{16\sqrt{2}\kappa^{12}}{3\nu^{1/2}(t-t_0)^{1/2}} \\
& \left.-\frac{32\kappa^{13}}{3}F\left(\kappa\frac{\nu(t-t_0)^{1/2}}{2}\right)\right] \\
& -\frac{\pi^{1/2}}{256\nu}\frac{\beta_0}{2}\,exp\left[-\frac{3}{2}\kappa^2\nu\left(t-t_0+\frac{1}{3}\Delta t\right)\right] \\
& \left[-\frac{15\sqrt{2}\kappa^6}{\nu^{7/2}(t-t_0+\Delta t)^{7/2}}-\frac{12\sqrt{2}\kappa^8}{\nu^{5/2}(t-t_0+\Delta t)^{5/2}}\right. \\
& +\frac{7\sqrt{2}\kappa^{10}}{3\nu^{3/2}(t-t_0+\Delta t)^{3/2}}+\frac{16\sqrt{2}\kappa^{12}}{3\nu^{1/2}(t-t_0+\Delta t)^{1/2}} \\
& \left.-\frac{32\kappa^{13}}{3}F\left(\kappa\left[\frac{\nu(t-t_0+\Delta t)}{2}\right]^{1/2}\right)\right]
\end{aligned}
\tag{42}
$$

where

$$F(\omega)=e^{-\omega^2}\int_0^\omega e^{x^2}\mathrm{d}x$$

$$\omega=\kappa\left[\frac{\nu(t-t_0)}{2}\right]^{1/2}\quad\text{or}\quad\kappa\left[\frac{\nu(t-t_0+\Delta t)}{2}\right]^{1/2}$$

For evaluating the function of integration (with respect to time) in equation (42), the theory of Lin (ref. 13) or of Batchelor (ref. 1) is used. According to those theories the coefficient of the first term in the expansion of E in powers of κ is independent of time, whereas the other terms may not be. Thus, the function of integration, since it cannot be a function of time, is set equal to the first term of the expansion, which is $J_0\kappa^4/3\pi$. The theories of Lin and Batchelor are based on the assumption that correlations are exponentially small for large values of r. This is consistent with the results of the present theory (refs. 10 and 11) and does not seem to be inconsistent with some results of Batchelor and Proudman (ref. 14), if the effects of the singularities arising in their analysis are assumed to be negligible. According to Batchelor and Proudman, correlations, in general, would be expected to be negative power functions of r for large values of r, and all terms in the expansion of E would, in general, be functions of time. However, their results by no means rule out the possibility of exponentially small correlations or of a constant first term in the expansion of E. The present theory gives exponentially small correlations for large r (refs. 10 and 11) and is consistent with a constant first term in the expansion of E for evaluating the function of integration in equation (42).

By using the relation

$$\frac{\overline{u_i u_i'(\Delta t,t)}}{2}=\int_0^\infty E\,\mathrm{d}\kappa$$

and introducing the dimensionless time

$$T\equiv\frac{\nu^{11/9}J_0^{2/9}(t-t_0)}{\beta_0^{2/9}}\tag{43}$$

the dimensionless time correlation becomes

$$\frac{\beta_0^{5/9}}{J_0^{14/9}\nu^{5/9}}\frac{\overline{u_iu_i'(\Delta t,t)}}{2}=\frac{1}{32(2\pi)^{1/2}}\left(T+\frac{1}{2}\Delta T\right)^{-5/2}-\frac{\pi}{512}$$

$$\Bigg\{-\frac{25/(3\sqrt{3})}{T^{7/2}\left(T+\frac{2}{3}\Delta T\right)^{7/2}}-\frac{25/(3\sqrt{3})}{(T+\Delta T)^{7/2}\left(T+\frac{1}{3}\Delta T\right)^{7/2}}$$

$$-\frac{140/(9\sqrt{3})}{T^{5/2}\left(T+\frac{2}{3}\Delta T\right)^{9/2}}-\frac{140/(9\sqrt{3})}{(T+\Delta T)^{5/2}\left(T+\frac{1}{3}\Delta T\right)^{9/2}}$$

$$+\frac{245/(27\sqrt{3})}{T^{3/2}\left(T+\frac{2}{3}\Delta T\right)^{11/2}}+\frac{245/(27\sqrt{3})}{(T+\Delta T)^{3/2}\left(T+\frac{1}{3}\Delta T\right)^{11/2}}$$

$$+\frac{6160/(81\sqrt{3})}{T^{1/2}\left(T+\frac{2}{3}\Delta T\right)^{13/2}}+\frac{6160/(81\sqrt{3})}{(T+\Delta T)^{1/2}\left(T+\frac{1}{3}\Delta T\right)^{13/2}}$$

$$-\frac{32}{3}\sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\ldots 11+2n}{(2n-1)(n-1)!\,2^{3n+13}}$$

$$\left[\frac{T^{\frac{2n-1}{2}}}{\left(T+\frac{1}{2}\Delta T\right)^{\frac{2n+13}{2}}}+\frac{(T+\Delta T)^{\frac{2n-1}{2}}}{\left(T+\frac{1}{2}\Delta T\right)^{\frac{2n+13}{2}}}\right]\Bigg\} \qquad (44)$$

where $0!\equiv 1$. For $\Delta T=0$, this becomes

$$\frac{\beta_0^{5/9}}{J_0^{14/9}\nu^{5/9}}\frac{\overline{u_iu_i(t)}}{2}=\frac{T^{-5/2}}{32(2\pi)^{1/2}}+0.2296T^{-7} \qquad (45)$$

which is the same as the expression for the turbulent energy obtained previously (ref. 10). As in the case of the final period, an average time $t_m=t+\Delta t/2$ can be introduced. Equation (44) becomes, when written in terms of the dimensionless T_m rather than T,

$$\frac{\beta_0^{5/9}}{J_0^{14/9}\nu^{5/9}}\frac{\overline{u_iu_i'(\Delta t,t_m)}}{2}=\frac{1}{32(2\pi)^{1/2}}T_m^{-5/2}$$

$$-\frac{\pi}{512}\Bigg\{-\frac{25/(3\sqrt{3})}{\left(T_m-\frac{1}{2}\Delta T\right)^{7/2}\left(T_m+\frac{1}{6}\Delta T\right)^{7/2}}$$

$$-\frac{25/(3\sqrt{3})}{\left(T_m+\frac{1}{2}\Delta T\right)^{7/2}\left(T_m-\frac{1}{6}\Delta T\right)^{7/2}}-\cdots\Bigg\} \qquad (46)$$

This expression for $\overline{u_iu_i'}$ does not become independent of ΔT when written in terms of T_m, as was the case for the final period. However, the

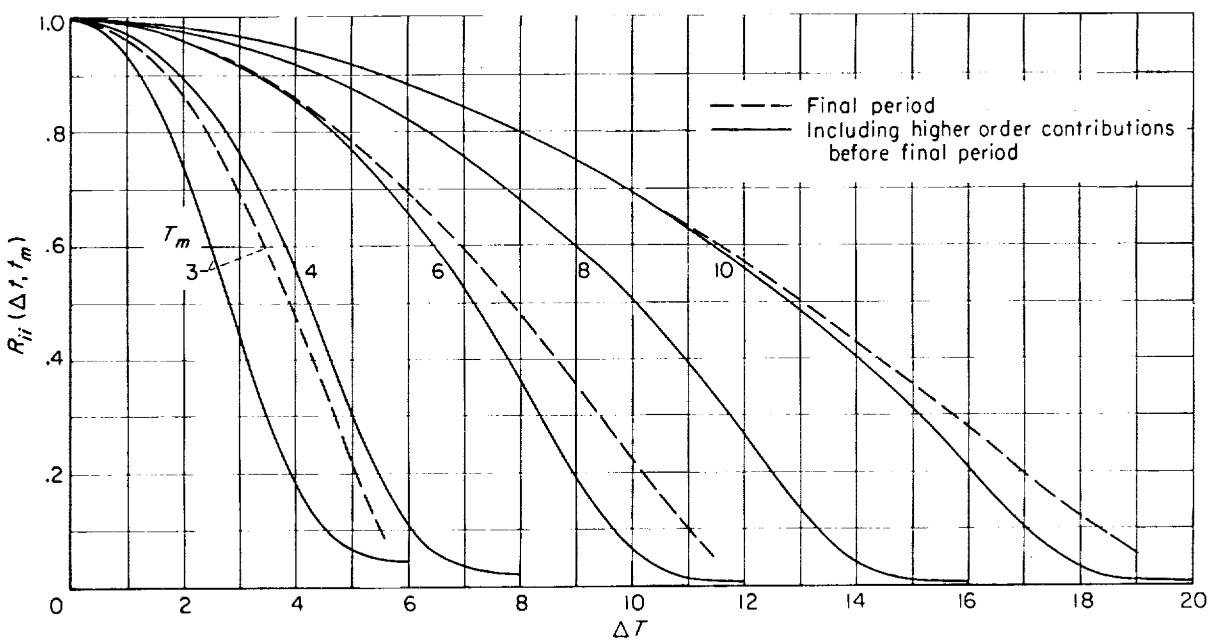

FIGURE 3.—Variation of time double-velocity correlation coefficient (eq. (47)) with dimensionless time interval and decay time, and comparison with values for final period. T defined by equation (43); correlation coefficient evaluated at $T_m=T+\Delta T/2$.

expression is still symmetric with respect to ΔT. This is in agreement with the results of Meecham (ref. 7), who came to that conclusion by kinematical considerations.

A more physically meaningful quantity than $\overline{u_iu_i'}$ is the time correlation coefficient, defined as follows:

$$R_{ii}\equiv\frac{\overline{u_iu_i'(\Delta t,t_m)}}{\left[\overline{u_iu_i\left(t_m+\frac{1}{2}\Delta t\right)}\;\overline{u_iu_i\left(t_m-\frac{1}{2}\Delta t\right)}\right]^{1/2}} \tag{47}$$

A plot of R_{ii}, obtained from equations (45), (46), and (47), is shown in figure 3. The final-period contributions are shown as dashed curves for comparison. As in the case of the final period, the values of R_{ii} decrease with dimensionless time separation. That is, of course, the type of behavior that would be expected on physical grounds; the time correlation coefficient can be considered as a measure of the sameness of the velocities at different times at a point in much the same way that the space correlation coefficient provides a similar measure for velocities at different points at one time. It is of interest that the values of $\overline{u_iu_i'}$ by themselves do not exhibit this behavior; in fact, they increase rather than decrease with time separation, as shown in figure 4. This unusual variation is apparently due to the nonlinear decay of the turbulence with time and would not be observed for stationary turbulence. For the

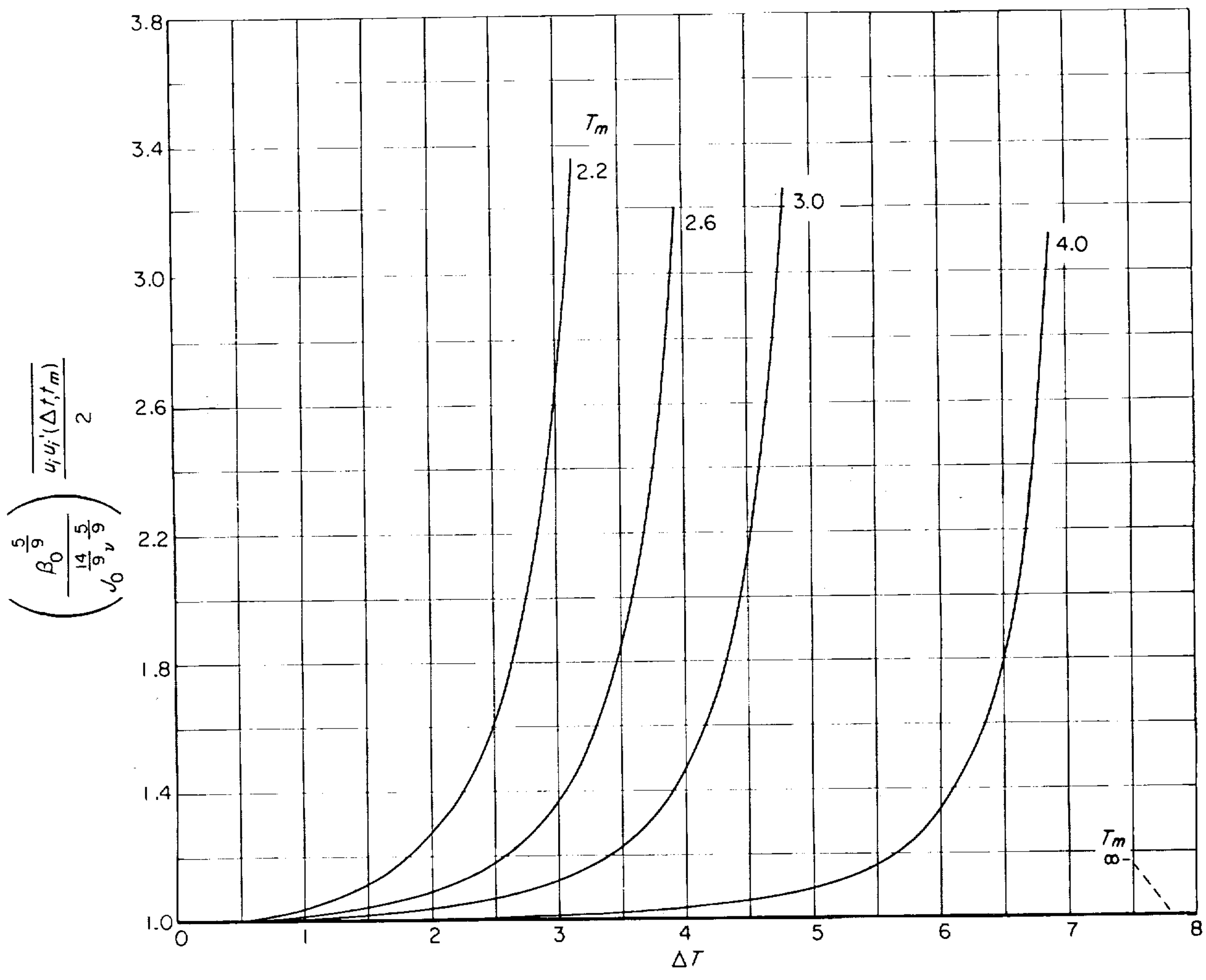

FIGURE 4.—Variations of dimensionless double-velocity correlation with time interval and decay time. Correlation evaluated at T_m.

decaying case, it appears that the correlation coefficient, as defined by equation (47), is much more meaningful than $\overline{u_i u_i'}$ by itself. The discussion of the time microscale given in connection with the final period would, of course, have been meaningless if it had been based on $\overline{u_i u_i'}$ rather than on the correlation coefficient.

Comparison of the dashed with the solid curves in figure 3 indicates that the general effect of the higher order inertia terms in the correlation equations is to decrease the correlation coefficient at a given value of time separation. This is opposite to the corresponding effect for space correlation (see, for instance, the experimental results of Stewart and Townsend (ref. 15) or the theoretical results of Deissler (ref. 11)). It is possible that the reduction of correlation coefficient by inertia terms is caused by the nonhomogeneity of the turbulence with time.

APPROXIMATE CALCULATION OF TURBULENT DIFFUSION FROM A SOURCE FOR SMALL VELOCITY FLUCTUATIONS

The time correlations considered in the preceding sections were concerned with velocities at different times at a fixed point in the fluid (Eulerian correlations). On the other hand, calculating the turbulent diffusion of particles from a source usually involves the Lagrangian correlations, which are based on the velocity of a moving fluid particle at different times, rather than on the velocity at a point. For small velocity fluctuations, however, it has been suggested by Burgers (ref. 4) that the two correlations should not differ greatly. This can be shown as follows:

First, consider the Eulerian time correlation $\overline{v(t)v(t')}$, where v is the component of the velocity in the y-direction; similar results could be obtained for the other velocity components. The Eulerian correlation can be expanded in a series as

$$\overline{v(t)v(t')}=[\overline{v(t)v(t')}]_{t'=t}+\left[\frac{\partial}{\partial t'}\overline{v(t)v(t')}\right]_{t'=t}(t'-t)+\frac{1}{2}\left[\frac{\partial^2}{\partial t'^2}\overline{v(t)v(t')}\right]_{t'=t}(t'-t)^2+\dots$$
$$=\overline{v^2(t)}+\overline{v(t)\left[\frac{\partial v(t')}{\partial t'}\right]_{t'=t}}(t'-t)+\frac{1}{2}\overline{v(t)\left[\frac{\partial^2 v(t')}{\partial t'^2}\right]_{t'=t}}(t'-t)^2+\dots \tag{48}$$

Similarly, the Lagrangian correlation is expanded as

$$[\overline{v(t)v(t')}]_L=\overline{v^2(t)}+\overline{v(t)\left[\frac{Dv(t')}{Dt'}\right]_{t'=t}}(t'-t)+\frac{1}{2}\overline{v(t)\left[\frac{D^2v(t')}{Dt'^2}\right]_{t'=t}}(t'-t)^2+\dots \tag{49}$$

The substantial or particle derivative can be written as

$$\frac{Dv(t')}{Dt'}=\frac{\partial v(t')}{\partial t'}+u\frac{\partial v(t')}{\partial x}+v\frac{\partial v(t')}{\partial y}+w\frac{\partial v(t')}{\partial z} \tag{50}$$

For small velocity fluctuations,

$$\frac{Dv(t')}{Dt'}\cong\frac{\partial v(t')}{\partial t'} \tag{51}$$

Also

$$\frac{D^2v(t')}{Dt'^2}\cong\frac{\partial}{\partial t'}\frac{Dv(t')}{Dt'}\cong\frac{\partial^2 v(t')}{\partial t'^2} \tag{52}$$

and so on for higher order derivatives. From equations (48), (49), (51), and (52)

$$\overline{v(t)v(t')}\cong[\overline{v(t)v(t')}]_L \tag{53}$$

is obtained, which was the relation to be proved. It should be noted that relation (53) is most accurate for small values of $t'-t$ as well as for small velocity fluctuations, inasmuch as the approximate relation (51) had to be applied a greater number of times to the higher order derivatives in equation (49) than to the lower order ones (see eq. (52)).

It should also be emphasized that equation (53) was obtained for the case of no mean motion. Thus, Eulerian time correlations measured with a stationary instrument in a moving stream will probably differ considerably from the Lagrangian correlations. However, if the instrument is moving with the stream, the two correlations will be approximately equal if the turbulence level is not too high (see ref. 12).

Next, the equation for the turbulent diffusion of particles originally concentrated at a source is considered. The theory of turbulent diffusion was originated by G. I. Taylor in 1922 (ref. 16) and has since been studied by a number of authors (e.g., refs. 17 and 18). The distance in the y-direction that a fluid particle originally at $y=0$ travels during the time interval $t'-t_1$ is

$$Y(t')=\int_{t_1}^{t'} v(t)\mathrm{d}t \tag{54}$$

Multiplication of this equation by $v(t')$ gives

$$v(t')Y(t')=\frac{1}{2}\frac{\mathrm{d}Y^2}{\mathrm{d}t'}=\int_{t_1}^{t'} v(t)v(t')\mathrm{d}t \tag{55}$$

Taking the particle average over all the "marked" particles that were originally concentrated at a source at $y=0$ and integrating with respect to t' result in

$$\overline{Y_2^2}=2\int_{t_1}^{t_2}\int_{t_1}^{t'} [\overline{v(t)v(t')}]_L\mathrm{d}t\mathrm{d}t' \tag{56}$$

Equation (56) gives the mean square of the distance that the marked fluid particles concentrated at $y=0$ at time t_1 have traveled by time t_2. It is evidently applicable to decaying as well as to stationary turbulence. Note that the double integral in equation (56) cannot be converted to a single integral as in the case of stationary turbulence.

If the approximate relation (53) is introduced and it is noted that, for isotropic turbulence, $\overline{v(t)v(t')}=[\overline{u_i(t)u_i(t')}]/3$, then equation (56) can be written in dimensionless form for isotropic turbulence as

$$\frac{\beta_0^{1/9}\nu^{17/9}}{J_0^{10/9}}\overline{Y_2^2}=\frac{2}{3}\int_{T_1}^{T_2}\int_{T_1}^{T'}\frac{\beta_0^{5/9}\overline{u_i(t)u_i(t')}}{J_0^{11/9}\nu^{5/9}}\mathrm{d}T\mathrm{d}T' \tag{57}$$

where the dimensionless time T is defined by equation (43) and the time correlation is obtained from equation (18) or (44) by remembering that $\Delta t=t'-t$ or $\Delta T=T'-T$.

FINAL PERIOD

For diffusion in the final period of decay, equation (57) can be integrated to give

$$\frac{\beta_0^{1/9}\nu^{17/9}}{J_0^{10/9}}\overline{Y_2^2}=\frac{1}{9\sqrt{\pi}}\left\{\frac{1}{\sqrt{2}}\left[\frac{1}{(T_1+\Delta T_2)^{1/2}}+\frac{1}{T_1^{1/2}}\right]-\frac{\sqrt{2}}{\left(T_1+\frac{1}{2}\Delta T_2\right)^{1/2}}\right\} \tag{58}$$

where T_1 is again the dimensionless time at which diffusion begins and ΔT_2 is the dimensionless time during which diffusion takes place (T_2-T_1). For large diffusion times,

$$\frac{\beta_0^{1/9}\nu^{17/9}}{J_0^{10/9}}\overline{Y_2^2}=\frac{1}{9\sqrt{2\pi}T_1^{1/2}} \tag{59}$$

That is, the turbulent diffusion distance reaches a constant value and becomes independent of ΔT_2 for large diffusion times. This differs from the case of stationary turbulence, where $\overline{Y_2^2}$ increases linearly with ΔT_2 for large diffusion times. The reason it reaches a constant value for decaying turbulence is that for large times the turbulence goes to zero, so that no more turbulent diffusion can take place.

Figure 5 shows dimensionless root-mean-square diffusion distance for the final period plotted against diffusion time for various values of T_1, the time at which diffusion begins. The curves have considerable curvature at early times but approach a linear form for large values of T_1. For early times, the diffusion distances are much larger than those for later times because of the higher turbulence level at early times.

TIMES BEFORE FINAL PERIOD

It might be argued that, strictly speaking, the approximate relation (53) should be used only in the final period, inasmuch as inertia terms were neglected in obtaining it. Inasmuch as no experimental diffusion data exist for the final period, however, some sort of approximation must be made for earlier times in order to compare the theory with experiment. The results might still be applicable for small times of diffusion; experimental data of Baldwin (ref. 12) for diffusion in a fully developed pipe flow indicated that equation (53 applies reasonably well for that case, although the turbulence probably did not correspond to that in a final period of decay.

For times earlier than those corresponding to the final period, equation (44) is used in (57) with ΔT replaced by $T'-T$. In this case the integration was carried out numerically on high-speed computing machinery. The resulting plot is shown in figure 6, where the final-period contributions are shown as dashed curves for comparison. The higher order inertia terms have a noticeable effect on the diffusion at early times; at later

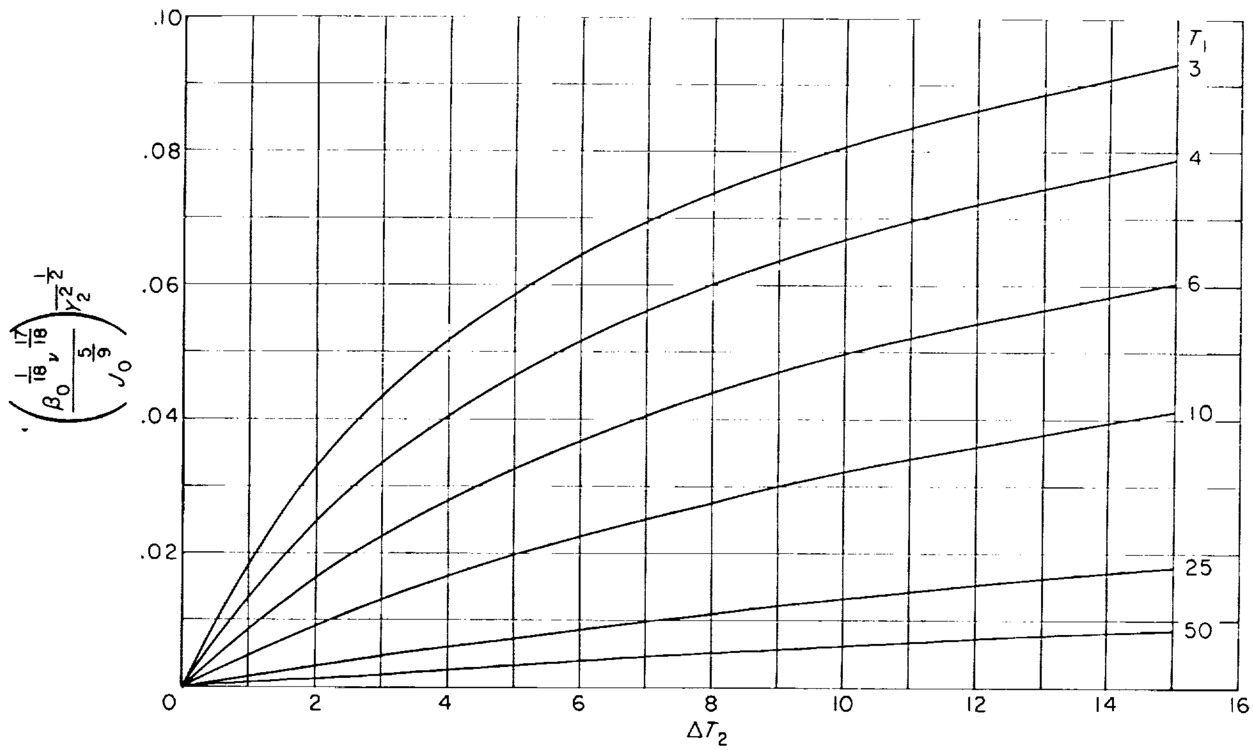

FIGURE 5.—Predicted root-mean-square turbulent diffusion distances for final period as a function of dimensionless diffusion time and decay time.

times the effect of those terms becomes negligible, and the solid curves approach those for the final period.

These curves should apply to the calculation of the width of the diffusion wake from a line source in a moving stream. In this case Δt would be replaced by the distance downstream from the source divided by the velocity of the mean stream. Comparison of the curves in figures 5 and 6 with those obtained experimentally (e.g., ref. 19) does, in fact, indicate a marked similarity. In order to obtain a more quantitative comparison, the constants J_0, β_0, and t_0, which depend on initial conditions, were evaluated from the decay data of Uberoi and Corrsin and equation (45). Equation (45) was found to represent the decay data closely when $J_0=1.05\times10^{-9}$ ft^7/sec^2, $\beta_0=1.81\times10^{-33}$ ft^{18}/sec^3, and $t_0=-0.407$ sec. With these values for the constants, diffusion data for an early and a late time are plotted in figure 7. Included for comparison are analytical results for the same values of T_1. The agreement between theory and experiment seems to be good for large values of T_1 and small values of ΔT_2, whereas some deviation is indicated for other conditions. This might have been expected from the nature of the approximations made in obtaining equation (53), which was used in the analysis. As discussed previously, that relation is most accurate for small velocity fluctuations (large T_1) and for small diffusion times.

CONCLUSIONS

The time correlation coefficient in a decaying homogeneous turbulent field, when evaluated at a time halfway between the times at which the two velocities are considered, decreased with time interval in much the same way that space correlation coefficients decrease with space interval. The time correlations by themselves, on the other hand, were independent of time separation in the final period and increased with time separation at

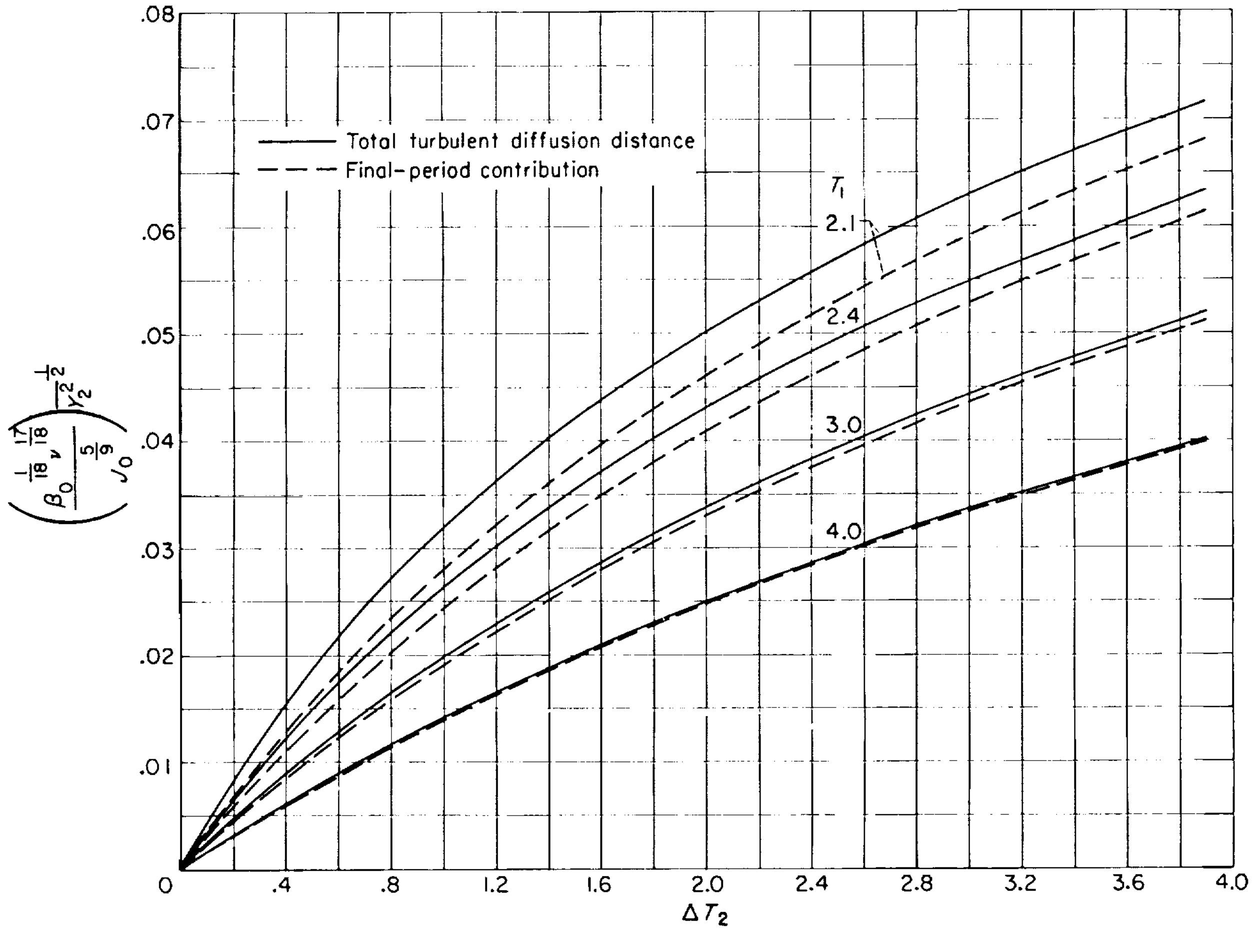

FIGURE 6.—Predicted root-mean-square diffusion distances for times before final period and comparison with final-period contributions.

earlier times, although they were symmetric with respect to Δt. The correlation *coefficient* (eq. (47)) appears to be a much more physically meaningful quantity than the correlation for a decaying turbulent field. The effect of the higher order inertia terms in the correlation equations for times before the final period was to reduce the value of the correlation coefficient at a given time interval below that for the final period. The ratio of time microscale to space microscale in the final period was a function of decay time and of initial conditions.

By assuming that the velocity fluctuations are sufficiently small for squares and products of velocities to be negligible, it can be shown that the Eulerian time correlation is approximately equal to the Lagrangian correlation. Turbulent root-mean-square diffusion distances were calculated by using this approximation and the equations for the time correlation obtained herein. The agreement between theory and experiment was good for large decay times (low turbulence levels) and for small diffusion times; for other conditions, some deviation was indicated. This was apparently due to the assumption of the equality of Eulerian and Lagrangian correlations, that assumption being most accurate for small velocity fluctuations and short diffusion times.

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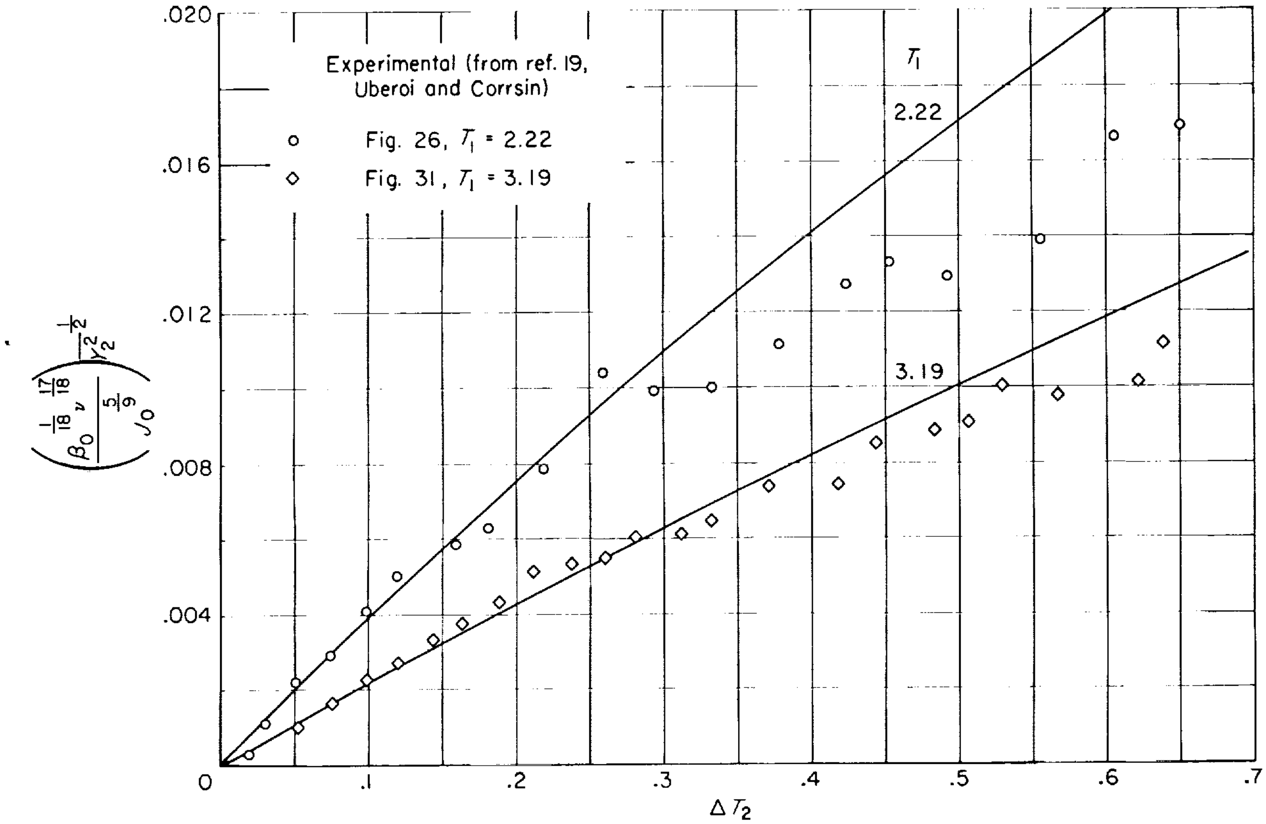

FIGURE 7.—Comparison of theory and experiment for width of turbulent diffusion wake from a line source in a decaying turbulent stream.

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