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Notes on the stability of dynamic economic systems

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Abstract

The Lyapunov function method is the focus of this paper in proving marginal stability, asymptotical or globally asymptotical stability of discrete dynamic systems. We show that the slightly relaxed versions of the well known sufficient conditions are also necessary. © 2000 Elsevier Science Inc. All rights reserved.

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1. Introduction

Stationary discrete dynamical systems can be mathematically formulated as

$$\underline{x}_{t+1} = \underline{g}(\underline{x}_t), \tag{1}$$

where $\underline{x}_t$ is the state of the system at time period t, and $\underline{g}$ is the state-transition function. If $S \subseteq \mathbb{R}^n$ is the state space, it is usually assumed that $\mathscr{D}(\underline{g}) = S, \mathscr{R}(\underline{g}) \subseteq S$, and $\underline{g}$ is continuous. If $\underline{x}_0 \in S$ is an arbitrary initial state, then equality (1) uniquely determines the state trajectory, $\underline{x}_t$, $t \geqslant 0$. An equilibrium of the system is defined as a state $\underline{\bar{x}} \in S$ such that

$$\underline{\bar{x}} = \underline{g}(\underline{\bar{x}}). \tag{2}$$

Therefore the equilibrium-problem of system (1) is equivalent to the fixed point-problem of function $\underline{g}$, and any existence theorem of fixed-points of

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vector variable, vector valued functions can be used to establish the existence of equilibria of discrete dynamic systems.

In most applications the asymptotical behavior of the state $\underline{x}_t$ is investigated. These stability concepts are usually applied: marginal stability, asymptotical stability, and global asymptotical stability.

An equilibrium $\underline{\bar{x}}$ is called marginally stable if for arbitrary $\epsilon > 0$, there is a $\delta > 0$ such that $\|\underline{x}_0 - \underline{\bar{x}}\| < \delta$ implies that for all $t \geqslant 0$, $\|\underline{x}_t - \underline{\bar{x}}\| < \epsilon$. An equilibrium $\underline{\bar{x}}$ is called asymptotically stable if it is marginally stable and there exists a $\Delta > 0$ such that $\|\underline{x}_0 - \underline{\bar{x}}\| < \Delta$ implies that $\underline{x}_t \to \underline{\bar{x}}$ as $t \to \infty$. An equilibrium $\underline{\bar{x}}$ is called globally asymptotically stable if it is marginally stable and $\underline{x}_t \to \underline{\bar{x}}$ as $t \to \infty$ with arbitrary $\underline{x}_0 \in S$.

Notice that asymptotical stability can be viewed as the local convergence of the iteration process generated by function $\underline{g}$, and similarly, global asymptotical stability can be interpreted as the global convergence of iteration sequences with the additional condition that the entire iteration sequence must remain close to $\underline{\bar{x}}$.

There are many sufficient conditions that guarantee the marginal stability, asymptotical stability, or the global asymptotical stability of an equilibrium. Most of such conditions belong to one of the following classes: monotone iterations, conditions based on the Jacobian of $\underline{g}$, and the use of Lyapunov functions. Unfortunately, most conditions are only sufficient, and very few necessary stability conditions are known from the literature. Recently, Zhang and Zhang [1] have introduced a practical necessary condition based on the Jacobian of $\underline{g}$. However their result can be used only in very special cases. (See Ref. [2].)

In this paper we will focus on the Lyapunov function method and will introduce necessary stability conditions which are only slight modifications of the corresponding sufficient conditions. Based on the results of this paper the marginal stability, asymptotical stability, or the global asymptotical stability of an equilibrium can be analyzed for practical systems. For continuous systems, necessary conditions involving Lyapunov functions have been earlier presented in Ref. [3]. The results of this paper can be considered as the discrete time-scale counterparts of the classical theorems.

2. Sufficient conditions

Introduce the notation

$$\Omega = \{\underline{x} \mid \|\underline{x} - \underline{\bar{x}}\| \leqslant \epsilon_0\} \tag{3}$$

with some $\epsilon_0 > 0$ and assume that $\Omega \subseteq S$. Let $V : \Omega \mapsto \mathbb{R}$ be a real valued function defined on Ω. Introduce the following function properties:

(a) V has a unique minimum at $\underline{\bar{x}}$;

(b1) V is continuous at $\underline{\bar{x}}$;

(b2) V is continuous on Ω;
(c1) V is nonincreasing along any state sequence of system (1) which is in Ω;
(c2) V is quasi-strictly decreasing along state sequence of system (1) which is in Ω, that is, it is nonincreasing and if $\underline{x}_t \neq \bar{\underline{x}}$, then there is a $t^* > t$ such that $V(\underline{x}_{t^*}) < V(\underline{x}_t)$;
(c3) V is strictly decreasing along any state sequence of system (1) which is in Ω, that is, if $\underline{x}_t \neq \bar{\underline{x}}$, then $V(\underline{x}_{t+1}) < V(\underline{x}_t)$.
The following sufficient conditions are well known from system theory:

Theorem 2.1. (1) *If there is a real valued function V defined on Ω which satisfies conditions* (a), (b2), *and* (c1), *then the equilibrium* $\bar{\underline{x}}$ *is marginally stable*; (2) *If there is a real valued function V defined on Ω with properties* (a), (b2), *and* (c3), *then the equilibrium* $\bar{\underline{x}}$ *is asymptotically stable.*

The proof of this theorem can be found in most books on difference equations or on systems theory. For example, see Ref. [4] for details.

Assume next that the state space S is unbounded, and $V : S \mapsto \mathbb{R}$ is a real valued function defined on S. Introduce the following additional function property:
(d) $V(\underline{x}) \to \infty$ as $\|\underline{x}\| \to \infty$.
The following theorem, which is also well known from system theory, guarantees the global asymptotical stability of the equilibrium.

Theorem 2.2. *If there is a real valued function V defined on S such that it satisfies conditions* (a), (b2), (c3), *and* (d) *with Ω being replaced by S, then the equilibrium is globally asymptotically stable.*

The following corollary is useful in many applications.

Corollary 2.3. *If S is bounded, and there is a real valued function V defined on S which satisfies conditions* (a), (b2), *and* (c3) *with Ω being replaced by S, then the equilibrium is globally asymptotically stable.*

These results play a fundamental role in analysing the asymptotical behavior of discrete dynamic economic systems. For example, their applications in oligopoly theory are illustrated in Ref. [5].

3. Necessary conditions

In this section we will show that the slightly relaxed versions of the conditions of Theorems 2.1 and 2.2 are necessary. In particular we will prove the following result:

Theorem 3.1. *Assume that the equilibrium $\underline{\bar{x}}$ is an interior point of S.*
1. *If $\underline{\bar{x}}$ is marginally stable, then there is a real valued function defined on S which satisfies conditions* (a), (b1), *and* (c1), *where Ω is replaced by S*;
2. *If the stability is asymptotical, then there is a real valued function defined on a neighbourhood Ω of $\underline{\bar{x}}$ which satisfies properties* (a), (b2), *and* (c2);
3. *If the stability is globally asymptotical, then there is a real valued function defined on S, and there is a neighbourhood Ω of $\underline{\bar{x}}$, that satisfies properties* (a), (b2), (c2), *and* (d).

Proof. Introduce the notation $\underline{g}^1(\underline{x}) = \underline{g}(\underline{x})$, and for $k \geqslant 1, \underline{g}^{k+1}(\underline{x}) = \underline{g}(\underline{g}^k(\underline{x}))$. Since $\mathscr{D}(\underline{g}) = S, \mathscr{R}(\underline{g}) \subseteq S, \underline{g}^k$ is defined on S for all k, and $\mathscr{R}(\underline{g}^k) \subseteq S$. Define now function V as follows:

$$V(\underline{x}) = \sup\{\|\underline{x} - \underline{\bar{x}}\|; \|\underline{g}^1(\underline{x}) - \underline{\bar{x}}\|; \|\underline{g}^2(\underline{x}) - \underline{\bar{x}}\|; \ldots\}. \tag{4}$$

Notice first that $V(\underline{\bar{x}}) = 0$ and if $\underline{x} \neq \underline{\bar{x}}$, then $V(\underline{x}) > 0$ implies that property (a) always holds. Let $\underline{x}_0, \underline{x}_1, \underline{x}_2 \ldots$ be a state sequence of system (1). Then the definition of function V implies that

$$V(\underline{x}_t) = \sup\{\|\underline{g}^t(\underline{x}_0) - \underline{\bar{x}}\|; \|\underline{g}^{t+1}(\underline{x}_0) - \underline{\bar{x}}\|; \|\underline{g}^{t+2}(\underline{x}_0) - \underline{\bar{x}}\|; \ldots\}, \tag{5}$$

from which we conclude that condition (c1) is always satisfied. The definition of V also implies that condition (d) always holds.

Assume first that the equilibrium $\underline{\bar{x}}$ is marginally stable. If $\underline{x}_0 = \underline{x}$, then for all $t \geqslant 1, \underline{x}_t = \underline{g}^t(\underline{x})$. From the definition of marginal stability of the equilibrium we conclude that for all $\epsilon > 0$, there exists a $\delta > 0$ such that $\|\underline{x} - \underline{\bar{x}}\| < \delta$ implies that for all $t > 0$, $\|\underline{x}_t - \underline{\bar{x}}\| < \epsilon$. That is, $V(\underline{x}) \leqslant \epsilon$ showing that V is continuous at $\underline{\bar{x}}$. Hence part 1 is proven.

Assume next that the stability is asymptotical. We have seen earlier that function V always satisfies condition (a), and V is nonincreasing along any state sequence of Eq. (1). Let $\underline{x}_t \neq \underline{\bar{x}}$. Then $\|\underline{x}_t - \underline{\bar{x}}\| > 0$ and since the stability is asymptotical, there is a $t^* > t$ such that $\|\underline{x}_{t^*} - \underline{\bar{x}}\| < \frac{1}{2}\|\underline{x}_t - \underline{\bar{x}}\|$ showing that V is quasi-strictly decreasing. In order to complete the proof of the theorem, we will show that there is a neighbourhood Ω of $\underline{\bar{x}}$ such that V is continuous on Ω.

The asymptotical stability of $\underline{\bar{x}}$ implies that there is a $\delta_1 > 0$ such that if $\|\underline{x}_0 - \underline{\bar{x}}\| < \delta_1$, then $\|\underline{x}_t - \underline{\bar{x}}\| \to 0$ as $t \to \infty$. Define $\Omega = \{\underline{x} | \|\underline{x} - \underline{\bar{x}}\| < \delta_1\} \cap S$, which is an open neighbourhood of $\underline{\bar{x}}$, and let $\underline{x}^* \neq \underline{\bar{x}}$ be any point in Ω. For any $\underline{x} \in \Omega$, we define an integer valued function $N(\underline{x}) = \min\{t | \|\underline{g}^t(\underline{x}) - \underline{\bar{x}}\| = V(\underline{x}), t \geqslant 0\}$. $N(\cdot)$ is well defined because $V(\cdot)$ is nonincreasing along any state sequence. From the marginal stability of $\underline{\bar{x}}$, there exists a $\delta_2, 0 < \delta_2 < \frac{1}{2}\|\underline{x}^* - \underline{\bar{x}}\|$, such that $\|\underline{x} - \underline{\bar{x}}\| < \delta_2$ implies $\|\underline{g}^t(\underline{x}) - \underline{\bar{x}}\| < \frac{1}{2}\|\underline{x}^* - \underline{\bar{x}}\|$ for $t \geqslant 0$. Define $B_1 = \{\underline{x} | \|\underline{x} - \underline{\bar{x}}\| < \frac{1}{2}\|\underline{x}^* - \underline{\bar{x}}\|\}$, $B_2 = \{\underline{x} | \|\underline{x} - \underline{\bar{x}}\| < \delta_2\}$, and $B_3 = \{\underline{x} | \|\underline{x} - \underline{x}^*\| < \frac{1}{2}\|\underline{x}^* - \underline{\bar{x}}\|\}$. Since $\underline{g}^t(\underline{x}^*) \to \underline{\bar{x}}$ as $t \to \infty$, there exists a $t_1 > 0$, such that $\underline{g}^t(\underline{x}^*) \in B_2$ whenever $t \geqslant t_1$. Let C be an open neighbourhood of $\underline{g}^{t_1}(\underline{x}^*)$ which is entirely included in B_2. Let $E = (\underline{g}^{t_1})^{-1}(C) \cap B_3$, which is open. Obviously, $\underline{g}^{t_1}(E) \subseteq C \subseteq B_2$. The definition of δ_2 and B_2 implies that for all $t \geqslant t_1$, $\underline{g}^t(E) \subseteq B_1$. Then $\underline{g}^t(E) \cap E \subseteq B_1 \cap B_3 = \phi$ for $t \geqslant t_1$. Hence for any $\underline{x} \in E$, we have $N(\underline{x}) < t_1$, and thus

$$V(\underline{x}) = \max\{\|\underline{x} - \bar{\underline{x}}\|; \|\underline{g}^1(\underline{x}) - \bar{\underline{x}}\|; \|\underline{g}^2(\underline{x}) - \bar{\underline{x}}\|; \ldots ; \|\underline{g}^{t_1-1}(\underline{x}) - \bar{\underline{x}}\|\}. \tag{6}$$

Since the maximum of a given finite number of continuous functions is also continuous, we conclude that V is continuous at $\underline{x}^*$. Therefore V is continuous on Ω because $\underline{x}^*$ is arbitrarily chosen in Ω. □

4. An alternative approach

It is well known, that if $\underline{g}$ is differentiable in a neighbourhood of an equilibrium $\bar{\underline{x}}$, and $\|\underline{J}(\bar{\underline{x}})\| < 1$ with some norm, where $\underline{J}$ is the Jacobian of $\underline{g}$, then $\bar{\underline{x}}$ is asymptotically stable. This result is a simple consequence of the continuity of $\underline{J}$, the mean value theorem of derivatives, and the contraction principle (see, for example, Ref. [6]). However, $\|\underline{J}(\bar{\underline{x}})\| < 1$ is not necessary as it is shown by the selection of the function

$$\underline{g}(x_1, x_2) = \begin{pmatrix} x_1 e^{-x_1^2} + x_2 e^{-x_2^2} \\ x_2 e^{-x_2^2} \end{pmatrix}. \tag{7}$$

The details of this example are given in Ref. [7]. Further analysis of stability conditions based on the Jacobian of the state transition function will be presented in a subsequent paper.

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