Boltzmann Equation

Detailed balance, which is a consequence of the time reversal symmetry observed for physical processes, requires

$$\begin{aligned}\frac{\partial n_\gamma(\mathbf{p}_\gamma)}{\partial t} = \frac{2}{(2\pi\hbar)^3}\int d^3\mathbf{p}_\mathrm{e}\int d^2\hat{\mathbf{n}}'\, V\,\frac{d^2\sigma}{d^2\hat{\mathbf{n}}'}\,\big[&(1-n_\mathrm{e}(\mathbf{p}_\mathrm{e}))\,n_\mathrm{e}(\mathbf{p}'_\mathrm{e})\,(1+n_\gamma(\mathbf{p}_\gamma))\,n_\gamma(\mathbf{p}'_\gamma)\\ &-(1-n_\mathrm{e}(\mathbf{p}'_\mathrm{e}))\,n_\mathrm{e}(\mathbf{p}_\mathrm{e})\,(1+n_\gamma(\mathbf{p}'_\gamma))\,n_\gamma(\mathbf{p}_\gamma)\big]\end{aligned} \qquad (X1)$$

where n_γ and n_e are the quantum mechanical occupation number for photons and electrons, respectively. The 2nd term in square brackets results from the scattering $(\mathbf{p}_\gamma,\mathbf{p}_\mathrm{e})\to(\mathbf{p}'_\gamma,\mathbf{p}'_\mathrm{e})$ while the 2nd term results from the inverse process. The $1+n_\gamma$ factor represents the increased scattering rate due to the stimulated emission of the the bosonic photons, while the $1-n_\gamma$ is the Pauli blocking factor giving the exclusion principle for fermionic electrons. The factor of 2 in the prefactor counts the two polarization states of the incoming electrons. We use the notation

$$\begin{aligned}&\mathbf{p}_\gamma=\frac{\epsilon}{c}\hat{\mathbf{n}} \qquad \mathbf{p}'_\gamma=\frac{\epsilon'}{c}\hat{\mathbf{n}}' \qquad |\hat{\mathbf{n}}|=|\hat{\mathbf{n}}'|=1 \qquad \Delta=\frac{\epsilon'}{\epsilon}-1 \qquad \alpha=\frac{\epsilon}{m_\mathrm{e}c^2}\\ &\mathbf{p}_\mathrm{e}=(m_\mathrm{e}c)\gamma\vec{\beta} \quad \beta=|\vec{\beta}| \quad \gamma=\frac{1}{\sqrt{1-\beta^2}}\\ &E=\sqrt{(m_\mathrm{e}c^2)^2+|c\mathbf{p}_\mathrm{e}|^2}=\gamma m_\mathrm{e}c^2 \qquad E'=\sqrt{(m_\mathrm{e}c^2)^2+|c\mathbf{p}'_\mathrm{e}|^2}\\ &\mu=\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}' \qquad \beta\mu_\mathrm{e}=\vec{\beta}\cdot\hat{\mathbf{n}}\end{aligned} \quad . \qquad (X2)$$

The value of $\mathbf{p}'_\mathrm{e}$ and ϵ' is determined by energy-momentum conservation, i.e. $E+\epsilon=E'+\epsilon'$ and $\mathbf{p}_\gamma+\mathbf{p}_\mathrm{e}=\mathbf{p}'_\gamma+\mathbf{p}'_\mathrm{e}$. The factor V in eq (X1) is a measure of the relative velocity between the ingoing electron and photon and is given by

$$V=c\sqrt{|\vec{\beta}_\gamma-\vec{\beta}_\mathrm{e}|^2-|\vec{\beta}_\gamma\times\vec{\beta}_\mathrm{e}|^2}=c(1-\hat{\mathbf{n}}\cdot\vec{\beta}_\mathrm{e}). \qquad (X3)$$

The first expression is the more general one while the last is specialized to the massless photon. Note if we had non-relativistic particles then $V=|\mathbf{v}_\gamma-\mathbf{v}_\mathrm{e}|$, i.e. the normal relative velocity.

Note that this form of the collision integral guarantees that a thermal distribution is a fixed point. Using a Fermi-Dirac distribution for the electrons and a Bose-Einstein distribution for the photon, i.e.

$$n_\mathrm{e}(E)=\frac{1}{\exp((E+\mu_\mathrm{e})/k_\mathrm{B}T)+1} \qquad n_\gamma(\epsilon)=\frac{1}{\exp((\epsilon+\mu_\gamma)/k_\mathrm{B}T)+1} \qquad (X4)$$

will cause the integrand of the collision integral to zero so long as the temperature is same for both. Of course we expect such a thermal distribution to be a stable fixed point since it is the highest entropy state and entropy increases. Since the number of photons and and electrons is conserved in this scattering the values of the chemical potentials, μ_e, and μ_γ does not effect this argument. In most astrophysical contexts the density of electrons is sufficiently low that Fermi-blocking is unimportant so we may set $1+n_\mathrm{e}\to 1$. When Fermi-blocking in ignored the equilibrium distribution for the electrons becomes a simple Boltzmann distribution, i.e.,

$$n_\mathrm{e}(E)=\exp(-(E+\mu_\mathrm{e})/k_\mathrm{B}T) \qquad (X5)$$

which is the limit of a Fermi-Dirac distribution when $\mu_\mathrm{e}\gg k_\mathrm{B}T$.

Henceforth we will ignore Fermi-blocking. We may rewrite the Boltzmann equation as

$$\begin{aligned}\frac{Dn_\gamma(\mathbf{p}_\gamma)}{c\,Dt} &= \frac{2c}{(2\pi\hbar)^3}\int d^3\mathbf{p}_\mathrm{e}\int d^2\hat{\mathbf{n}}'\,(1-\vec{\beta}_\mathrm{e}\cdot\hat{\mathbf{n}})\,\frac{d^2\sigma}{d^2\hat{\mathbf{n}}'}\,\left[n_\mathrm{e}(\mathbf{p}'_\mathrm{e})\,(1+n_\gamma(\mathbf{p}_\gamma))\,n_\gamma(\mathbf{p}'_\gamma)-n_\mathrm{e}(\mathbf{p}_\mathrm{e})\,(1+n_\gamma(\mathbf{p}'_\gamma))\,n_\gamma(\mathbf{p}_\gamma)\right]\\ &=\int d^2\hat{\mathbf{p}}'_\gamma\,\left[\frac{\epsilon^2}{\epsilon'^2}S(\mathbf{p}'_\gamma,\mathbf{p}_\gamma)\,(1+n_\gamma(\mathbf{p}_\gamma))\,n_\gamma(\mathbf{p}'_\gamma)-S(\mathbf{p}_\gamma,\mathbf{p}'_\gamma)\,(1+n_\gamma(\mathbf{p}'_\gamma))\,n_\gamma(\mathbf{p}_\gamma)\right]\end{aligned} \qquad (X6)$$

where

$$S(\mathbf{p}_\gamma,\mathbf{p}'_\gamma)=\frac{2}{(2\pi\hbar)^3}\int d^3\mathbf{p}_\mathrm{e}\,n_\mathrm{e}(\mathbf{p}_\mathrm{e})\,(1-\vec{\beta}_\mathrm{e}\cdot\hat{\mathbf{n}})\,\frac{d^2\sigma}{d^2\hat{\mathbf{n}}'}(\mathbf{p}_\mathrm{e},\mathbf{p}_\gamma\,\hat{\mathbf{n}}')\,\frac{\delta(\epsilon'-\epsilon(1+\bar{\Delta}))}{\epsilon'^2} \qquad (X7)$$

where $\bar{\Delta}(\mathbf{p}_\mathrm{e},\mathbf{p}_\gamma,\hat{\mathbf{n}}')$ gives the fractional change in the energy determined by energy-momentum conservation, i.e. is the solution to the equation

$$\sqrt{(m_\mathrm{e}c^2)^2+|c\mathbf{p}_\mathrm{e}|^2}+c|\mathbf{p}_\gamma|=\sqrt{(m_\mathrm{e}c^2)^2+|c\mathbf{p}_\mathrm{e}+c\mathbf{p}_\gamma-\epsilon\,(1+\bar{\Delta})\hat{\mathbf{n}}'|^2}+c\,|\mathbf{p}_\mathrm{e}|\,(1+\bar{\Delta}). \qquad (X8)$$

A unique real solution, greater than -1, exists for every $(\mathbf{p}_e, \mathbf{p}_\gamma, \hat{\mathbf{n}}')$.

If we assume the photon distribution function is isotropic (or more specifically expand to 0th order in the anisotropy) the Boltzmann equation becomes

$$\frac{Dn_\gamma(\mathbf{p}_\gamma)}{c\,Dt} = \int_{-1}^{\infty} d\Delta \left[(1+\Delta)\,\widehat{S}(\epsilon(1+\Delta), -\frac{\Delta}{1+\Delta}, \hat{\mathbf{n}})\,(1+n_\gamma(\epsilon))\,n_\gamma(\epsilon(1+\Delta)) - \overline{S}(\epsilon,\Delta,\hat{\mathbf{n}})\,(1+n_\gamma(\epsilon(1+\Delta)))\,n_\gamma(\epsilon)\right] \tag{X7}$$

where

$$\begin{aligned}\overline{S}(\epsilon,\Delta,\hat{\mathbf{n}}) &= \epsilon^3(1+\Delta)^2 \int d^2\hat{\mathbf{n}}'\, S(\frac{\epsilon}{c}\hat{\mathbf{n}}, \frac{\epsilon}{c}(1+\Delta)\hat{\mathbf{n}}') \\ \widehat{S}(\epsilon,\Delta,\hat{\mathbf{n}}') &= \epsilon^3(1+\Delta)^2 \int d^2\hat{\mathbf{n}}\, S(\frac{\epsilon}{c}\hat{\mathbf{n}}, \frac{\epsilon}{c}(1+\Delta)\hat{\mathbf{n}}')\end{aligned} . \tag{X8}$$

If the electron distribution is isotropic then there is

$$\overline{S}(\epsilon,\Delta,\hat{\mathbf{n}}) = \widehat{S}(\epsilon,\Delta,\hat{\mathbf{n}}) = \overline{S}(\epsilon,\Delta) \equiv \frac{\epsilon^3(1+\Delta)^3}{4\pi} \int d^2\hat{\mathbf{n}} \int d^2\hat{\mathbf{n}}'\, S(\frac{\epsilon}{c}\hat{\mathbf{n}}, \frac{\epsilon}{c}(1+\Delta)\hat{\mathbf{n}}') \tag{X9}$$

and there is no dependence on the direction $\hat{\mathbf{n}}$.

Changing the variable of integration on the 1st term in the Boltzmann equation from Δ to $1/(1+\Delta)-1$, renaming this new variable Δ, one finds

$$\frac{Dn_\gamma(\mathbf{p}_\gamma)}{c\,Dt} = \int_{-1}^{\infty} d\Delta \left[\frac{1}{(1+\Delta)^3}\widehat{S}(\frac{\epsilon}{1+\Delta}, \Delta, \hat{\mathbf{n}})\,(1+n_\gamma(\epsilon))\,n_\gamma(\frac{\epsilon}{1+\Delta}) - \overline{S}(\epsilon,\Delta,\hat{\mathbf{n}})\,(1+n_\gamma(\epsilon(1+\Delta)))\,n_\gamma(\epsilon)\right]. \tag{X10}$$

It is clear from the form of this equation that the total photon number is conserved, i.e.

$$\begin{aligned}\frac{dN_\gamma}{c\,dt} &= \frac{2}{(2\pi\hbar)^3}\int d^3\hat{\mathbf{p}}_\gamma\,\frac{dn_\gamma(\mathbf{p}_\gamma)}{dt} \\ &= \frac{4\pi\,c}{(2\pi\hbar)^3}\int_{-1}^{\infty} d\Delta \int_0^{\infty} d\epsilon\,\epsilon^2 \left[\frac{1}{(1+\Delta)^3}\overline{S}(\frac{\epsilon}{1+\Delta},\Delta)\,(1+n_\gamma(\epsilon))\,n_\gamma(\frac{\epsilon}{1+\Delta}) - \overline{S}(\epsilon,\Delta)\,(1+n_\gamma((1+\Delta)\epsilon))\,n_\gamma(\epsilon)\right] \\ &= \frac{4\pi\,c}{(2\pi\hbar)^3}\int_{-1}^{\infty} d\Delta \int_0^{\infty} d\epsilon\,\epsilon^2\,\overline{S}(\epsilon,\Delta)\,\left[(1+n_\gamma((1+\Delta)\epsilon))\,n_\gamma(\epsilon) - (1+n_\gamma((1+\Delta)\epsilon))\,n_\gamma(\epsilon)\right] = 0\end{aligned} \tag{X11}$$

where in the last line the variable of integration on the 1st term was changed from ϵ to $\epsilon/(1+\Delta)$.

Optical Depth

The number density of electrons is

$$N_e = \frac{2}{(2\pi\hbar)^3}\int d^3\mathbf{p}_e\, n_e(\mathbf{p}_e). \tag{X12}$$

If we define the optical depth

$$d\tau = N_e c\,\sigma_T dt \qquad \sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2 . \tag{X13}$$

Fokker-Planck Equation

If the electrons are moving slowly, $\beta \ll 1$, and the photons are not too energetic $\alpha \ll 1$ then conservation of energy and momentum guarantees that $\Delta \ll 1$, or more precisely that $\overline{S}(\epsilon,\Delta,\hat{\mathbf{n}})$ and $\widehat{S}(\epsilon,\Delta,\hat{\mathbf{n}})$ will be sharply peaked around $\Delta = 0$ with width of this peak being much less than unity. In other words the energy of a photon changes little in each scattering. The idea of a Fokker-Planck equation is to expand the integrand of the scattering integral in terms of the small energy change Δ. Note that we expand $\widehat{S}$ only wrt its 1st argument and not its 2nd. Expanding the above equation to 2nd order in Δ, but only expanding the dependence of $\overline{S}$ on the 1st argument.

First consider the case where the the electron di

Next consider the case of an isotropic electron momentum distribution and expand to 2nd order in Δ. The Boltzmann equation becomes

$$\frac{\partial n_\gamma(\epsilon)}{\partial\tau} = \frac{1}{\epsilon^2}\frac{\partial}{\partial\epsilon}\left[\epsilon^3\left(\frac{1}{2}\epsilon\,\overline{\overline{\Delta^2}}\,\frac{\partial n_\gamma(\epsilon)}{\partial\epsilon} + \left(-\overline{\overline{\Delta}} + 2\overline{\overline{\Delta^2}} + \frac{1}{2}\epsilon\overline{\overline{\Delta^{2\prime}}}\right)\,(1+n_\gamma(\epsilon))\,n_\gamma(\epsilon)\right)\right] \qquad (X13)$$

where

$$\overline{\overline{\Delta^n}} = \frac{1}{N_{\rm e}\sigma_{\rm T}}\int_{-1}^{\infty} d\Delta\,\Delta^n\,\overline{S}(\epsilon,\Delta) \qquad \overline{\overline{\Delta^{n\prime}}} = \frac{1}{N_{\rm e}\sigma_{\rm T}}\int_{-1}^{\infty} d\Delta\,\Delta^n\,\frac{\partial\overline{S}(\epsilon,\Delta)}{\partial\epsilon} \qquad (X13)$$

This Fokker-Plank equation agrees with eq 8 of Barbosa (1982).

As we will see to get all terms of the 2nd order in temperature we need to expand to 4th order in Δ. This expansion is

$$\begin{aligned}
\frac{\partial n_\gamma}{\partial\tau} = \frac{1}{\epsilon^2}\frac{\partial}{\partial\epsilon}\Bigg[\epsilon^3\Bigg(&\frac{1}{24}\left(\begin{aligned}-24\overline{\overline{\Delta}} + 48\overline{\overline{\Delta^2}} + 12\epsilon\overline{\overline{\Delta^{2\prime}}} - 80\overline{\overline{\Delta^3}} - 40\epsilon\overline{\overline{\Delta^{3\prime}}} - 4\epsilon^2\overline{\overline{\Delta^{3\prime\prime}}}\\ + 120\overline{\overline{\Delta^4}} + 90\epsilon\overline{\overline{\Delta^{4\prime}}} + 18\epsilon^2\overline{\overline{\Delta^{4\prime\prime}}} + \epsilon^3\overline{\overline{\Delta^{4\prime\prime\prime}}}\end{aligned}\right)\,(1+n_\gamma)\,n_\gamma\\
&+\frac{1}{24}\epsilon\left(12\overline{\overline{\Delta^2}} - 40\overline{\overline{\Delta^3}} - 8\epsilon\overline{\overline{\Delta^{3\prime}}} + 90\overline{\overline{\Delta^4}} + 36\epsilon\overline{\overline{\Delta^{4\prime}}} + 3\epsilon^2\overline{\overline{\Delta^{4\prime\prime}}}\right)\,\frac{\partial n_\gamma}{\partial\epsilon}\\
&+\frac{1}{12}\epsilon\left(-10\overline{\overline{\Delta^3}} - 2\epsilon\overline{\overline{\Delta^{3\prime}}} + 30\overline{\overline{\Delta^4}} + 12\epsilon\overline{\overline{\Delta^{4\prime}}} + \epsilon^2\overline{\overline{\Delta^{4\prime\prime}}}\right)\,n_\gamma\frac{\partial n_\gamma}{\partial\epsilon}\\
&+\frac{1}{12}\epsilon^2\left(2\overline{\overline{\Delta^3}} - 6\overline{\overline{\Delta^4}} - \epsilon\overline{\overline{\Delta^{4\prime}}}\right)\,\frac{\partial n_\gamma}{\partial\epsilon}\frac{\partial n_\gamma}{\partial\epsilon}\\
&+\frac{1}{24}\epsilon^2\left(-4\overline{\overline{\Delta^3}} + 18\overline{\overline{\Delta^4}} + 3\epsilon\overline{\overline{\Delta^{4\prime}}}\right)\,\frac{\partial^2 n_\gamma}{\partial\epsilon^2}\\
&+\frac{1}{6}\epsilon^2\left(-2\overline{\overline{\Delta^3}} + 6\overline{\overline{\Delta^4}} + \epsilon\overline{\overline{\Delta^{4\prime}}}\right)\,n_\gamma\frac{\partial^2 n_\gamma}{\partial\epsilon^2}\\
&+\frac{1}{24}\epsilon^3\left(\overline{\overline{\Delta^4}}\right)\,\frac{\partial^3 n_\gamma}{\partial\epsilon^3}\Bigg)\Bigg]
\end{aligned}$$

Compton Scattering

From Barbosa (1982) the cross-section in an arbitrary rest-frame is

$$\frac{d^2\sigma}{d^2\hat{\mathbf{n}}'} = \frac{3\sigma_{\rm T}}{16\pi}\frac{1-\beta^2}{[1-\hat{\mathbf{n}}'\cdot\vec{\beta}+\alpha\gamma^{-1}(1-\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}')]^2} \times\left[1+\left(1-\frac{(1-\beta^2)(1-\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}')}{(1-\hat{\mathbf{n}}\cdot\vec{\beta})\,(1-\hat{\mathbf{n}}'\cdot\vec{\beta})}\right)^2+\frac{\alpha^2(1-\beta^2)\,(1-\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}')^2}{(1-\hat{\mathbf{n}}'\cdot\vec{\beta})\,[1-\hat{\mathbf{n}}'\cdot\vec{\beta}+\alpha\gamma^{-1}(1-\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}')]}\right] \tag{N1}$$

where

$$\alpha=\frac{E_\gamma}{m_{\rm e}c^2}\qquad \sigma_{\rm T}=\frac{8\pi}{3}\left(\frac{e^2}{m_{\rm e}c^2}\right)^2\qquad \gamma=\frac{1}{\sqrt{1-\beta^2}}\qquad E'=E\,\frac{1-\hat{\mathbf{n}}\cdot\vec{\beta}}{1-\hat{\mathbf{n}}'\cdot\vec{\beta}+\alpha\gamma^{-1}(1-\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}')} \tag{N2}$$

Soft Photon Limit

We are interested in the case where the photons are soft, and in particular, $\alpha \ll 1$. Setting $\alpha = 0$ in the above expression yields

$$\frac{d^2\sigma^{(0)}}{d^2\hat{\mathbf{n}}'} = \frac{3\sigma_{\rm T}}{16\pi}\frac{1-\beta^2}{(1-\hat{\mathbf{n}}'\cdot\vec{\beta})^2}\left[1+\left(1-\frac{(1-\beta^2)(1-\hat{\mathbf{n}}\cdot\hat{\mathbf{n}}')}{(1-\hat{\mathbf{n}}\cdot\vec{\beta})\,(1-\hat{\mathbf{n}}'\cdot\vec{\beta})}\right)^2\right]\qquad E'=E\,\frac{1-\hat{\mathbf{n}}\cdot\vec{\beta}}{1-\hat{\mathbf{n}}'\cdot\vec{\beta}}. \tag{N3}$$

Note that in order for this to be a good approximation we must require that $\gamma(1-\hat{\mathbf{n}}'\cdot\vec{\beta}) \gg \alpha$ for all directions of $\hat{\mathbf{n}}'$ and $\vec{\beta}$. Thus we have also implicitly assumed that $\alpha\gamma \ll 1$, which implies that the energy of the photon in the center-of-mass frame obeys $E^{\rm c.m.} \lesssim m_{\rm e}c^2\alpha\gamma \ll m_{\rm e}c^2$. Our assumption prevents us from probing the high-energy limit of Compton scattering i.e. where $E^{\rm c.m.} \gtrsim m_{\rm e}c^2$. This explains why the total cross-section computed below never varies from the Thomson cross-section, $\sigma_{\rm T}$, whereas the Klein-Nishina cross-section falls below this value.

Let us define a spherical polar coordinate system, (θ', ϕ'), with pole in the direction of $\vec{\beta}$, so that

$$\hat{\mathbf{n}}'\cdot\vec{\beta}=\beta\,\cos\theta'\qquad \hat{\mathbf{n}}\cdot\vec{\beta}=\beta\,\mu_{\rm e}\qquad \hat{\mathbf{n}}\cdot\hat{\mathbf{n}}'=\mu_{\rm e}\cos\theta'+\sqrt{1-\mu_{\rm e}^2}\,\sin\theta'\,\cos\phi'. \tag{N4}$$

Since $d^2\hat{\mathbf{n}} = d\cos\theta'\,d\phi'$ the differential cross-section is

$$\frac{d^2\sigma^{(0)}}{d\cos\theta'\,d\phi'} = \frac{3\sigma_{\rm T}}{16\pi}\frac{1-\beta^2}{(1-\beta\,\cos\theta')^2}\left[1+\left(1-\frac{(1-\beta^2)(1-\mu_{\rm e}\cos\theta'-\sqrt{1-\mu_{\rm e}^2}\,\sin\theta'\,\cos\phi')}{(1-\beta\,\mu_{\rm e})\,(1-\beta\,\cos\theta')}\right)^2\right] \tag{N5}$$

To track the energy transfer between the photon and electron we define

$$\Delta=\frac{E'}{E}-1=\frac{\beta\,(\cos\theta'-\mu_{\rm e})}{1-\beta\,\cos\theta'}\qquad \cos\theta'=\frac{\Delta+\beta\mu_{\rm e}}{\beta\,(1+\Delta)}\qquad \Delta\in\left[-\frac{\beta\,(1+\mu_{\rm e})}{1+\beta},\frac{\beta\,(1-\mu_{\rm e})}{1-\beta}\right] \tag{N6}$$

We may replace the θ' dependence with a Δ dependence:

$$\frac{d^2\sigma^{(0)}}{d\cos\theta'\,d\phi'} = \frac{3\sigma_{\rm T}}{16\pi}\frac{(1-\beta^2)\,(1+\Delta)^2}{(1-\beta\,\mu_{\rm e})^2} \times\left[1+\left(1-\frac{(1-\beta^2)\,(1+\Delta-\mu_{\rm e}(\frac{\Delta}{\beta}+\mu_{\rm e})-\sqrt{1-\mu_{\rm e}^2}\,\sqrt{(1+\Delta)^2-(\frac{\Delta}{\beta}+\mu_{\rm e})^2}\,\cos\phi')}{(1-\beta\,\mu_{\rm e})^2}\right)^2\right] \tag{N7}$$

To change variable from (θ', ϕ') to (Δ, ϕ') we use the Jacobian

$$\left\|\frac{\partial(\cos\theta',\phi')}{\partial(\Delta,\phi')}\right\| = \left|\frac{\partial\cos\theta'}{\partial\Delta}\right| = \frac{1-\beta\,\mu_{\rm e}}{\beta\,(1+\Delta)^2} \tag{N8}$$

so

$$\frac{d^2\sigma^{(0)}}{d\Delta\,d\phi'} = \frac{3\sigma_{\rm T}}{16\pi}\,\frac{(1-\beta^2)}{\beta\,(1-\beta\,\mu_{\rm e})} \times \left[1+\left(1-\frac{(1-\beta^2)\,(1+\Delta-\mu_{\rm e}(\frac{\Delta}{\beta}+\mu_{\rm e})-\sqrt{1-\mu_{\rm e}^2}\,\sqrt{(1+\Delta)^2-(\frac{\Delta}{\beta}+\mu_{\rm e})^2}\,\cos\phi')}{(1-\beta\,\mu_{\rm e})^2}\right)^2\right]. \tag{N9}$$

If one is just interested in the energy transfer then one may integrate over ϕ':

$$\frac{d\sigma^{(0)}}{d\Delta} = \frac{3\sigma_{\rm T}}{8}\,\frac{(1-\beta^2)}{\beta\,(1-\beta\,\mu_{\rm e})} \times \left[1+\left(1-\frac{(1-\beta^2)\,(1+\Delta-\mu_{\rm e}(\frac{\Delta}{\beta}+\mu_{\rm e}))}{(1-\beta\,\mu_{\rm e})^2}\right)^2+\frac{1}{2}\frac{(1-\beta^2)^2(1-\mu_{\rm e}^2)\,\left((1+\Delta)^2-(\frac{\Delta}{\beta}+\mu_{\rm e})^2\right)}{(1-\beta\,\mu_{\rm e})^4}\right]. \tag{N10}$$

The total cross-section is

$$\sigma^{(0)}_{\rm tot} = \int_{-\frac{\beta(1+\mu_{\rm e})}{1+\beta}}^{\frac{\beta(1-\mu_{\rm e})}{1-\beta}} d\Delta\,\frac{d\sigma^{(0)}}{d\Delta} = \sigma_{\rm T} \tag{N11}$$

which is independent of $\mu_{\rm e}$.

Isotropic Cross-Section

If the distribution of photons or the distribution of electrons are isotropic then the distribution of $\mu_{\rm e}$ is uniform over the interval $[-1,1]$. To average over $\mu_{\rm e}$ at fixed Δ we must reverse the order of integration

$$\frac{1}{2}\int_{-1}^{1} d\mu_{\rm e}\int_{-\frac{\beta(1+\mu_{\rm e})}{1+\beta}}^{\frac{\beta(1-\mu_{\rm e})}{1-\beta}} d\Delta\,\frac{d\sigma^{(0)}}{d\Delta} = \frac{1}{2}\left[\int_{-\frac{2\beta}{1+\beta}}^{0} d\Delta\int_{-\frac{1+\beta}{\beta}\Delta-1}^{1} d\mu_{\rm e}\,\frac{d\sigma^{(0)}}{d\Delta}+\int_0^{\frac{2\beta}{1-\beta}} d\Delta\int_{-1}^{1-\frac{1-\beta}{\beta}\Delta} d\mu_{\rm e}\,\frac{d\sigma^{(0)}}{d\Delta}\right] \tag{N12}$$

hence we may write

$$\left\langle\frac{d\sigma^{(0)}}{d\Delta}\right\rangle_{\mu_{\rm e}} = \begin{cases}\left\langle\dfrac{d\sigma^{(0)}}{d\Delta}\right\rangle^+_{\mu_{\rm e}} & \Delta>0 \\ \left\langle\dfrac{d\sigma^{(0)}}{d\Delta}\right\rangle^-_{\mu_{\rm e}} & \Delta<0\end{cases} \qquad \begin{aligned}\left\langle\frac{d\sigma^{(0)}}{d\Delta}\right\rangle^+_{\mu_{\rm e}} &= \frac{1}{2}\int_{-1}^{1-\frac{1-\beta}{\beta}\Delta} d\mu_{\rm e}\,\frac{d\sigma^{(0)}}{d\Delta} \\ \left\langle\frac{d\sigma^{(0)}}{d\Delta}\right\rangle^-_{\mu_{\rm e}} &= \frac{1}{2}\int_{-\frac{1+\beta}{\beta}\Delta-1}^{1} d\mu_{\rm e}\,\frac{d\sigma^{(0)}}{d\Delta}\end{aligned}. \tag{N13}$$

Performing the integrals we find

$$\begin{aligned}\frac{1}{\sigma_{\rm T}}\left\langle\frac{d\sigma^{(0)}}{d\Delta}\right\rangle^+_{\mu_{\rm e}} &= \frac{3\left(3-5\beta^2+5\beta^4-3\beta^6\right)}{16\beta^6}\left(\ln\frac{1+\beta}{1-\beta}-\ln(1+\Delta)\right) \\ &\quad -\frac{3(1-\beta^2)(2\beta-(1-\beta)\Delta)}{64\beta^6(1+\Delta)^2}(4(3-\beta^2)+2(9+\beta-3\beta^2+\beta^3)\Delta+2(2+\beta+\beta^3)\Delta^2-(1+\beta)(1-\beta^2)\Delta^3) \\ \frac{1}{\sigma_{\rm T}}\left\langle\frac{d\sigma^{(0)}}{d\Delta}\right\rangle^-_{\mu_{\rm e}} &= \frac{3(3-5\beta^2+5\beta^4-3\beta^6)}{16\beta^6}\left(\ln\frac{1+\beta}{1-\beta}+\ln(1+\Delta)\right) \\ &\quad -\frac{3(1-\beta^2)\,(2\beta+(1+\beta)\Delta)}{64\beta^6(1+\Delta)^2}(4(3-\beta^2)+2(9-\beta-3\beta^2-\beta^3)\Delta+2(2-\beta-\beta^3)\Delta^2-(1-\beta)(1-\beta^2)\Delta^3)\end{aligned} \tag{N14}$$

which we may combine to give

$$\begin{aligned}\frac{1}{\sigma_{\rm T}}\left\langle\frac{d\sigma^{(0)}}{d\Delta}\right\rangle^{\pm}_{\mu_{\rm e}} &= \pm\frac{3(3-5\beta^2+5\beta^4-3\beta^6)}{16\beta^6}\left(\ln\frac{1\pm\beta}{1\mp\beta}-\ln(1+\Delta)\right) \\ &\quad \mp\frac{3(1-\beta^2)(\pm2\beta-(1\mp\beta)\Delta)}{64\beta^6(1+\Delta)^2}(4(3-\beta^2)+2(9\pm\beta-3\beta^2\pm\beta^3)\Delta+2(2\pm\beta\pm\beta^3)\Delta^2-(1\pm\beta)(1-\beta^2)\Delta^3)\end{aligned}. \tag{N15}$$

Note that one may transform between the $\pm$ functions by multiplying by -1 and replace β by $-\beta$. A simpler form is

$$\left\langle \frac{d\sigma^{(0)}}{d\Delta} \right\rangle_{\mu_e} = \sigma_{\rm T}\, F(\Delta, \beta\,{\rm sgn}(\Delta)) \tag{N16}$$

where, with $b = {\rm sgn}(\Delta)\,\beta$,

$$F(\Delta,b) = {\rm sgn}(\Delta) \times \mathcal{H}(1 - \frac{(1-b)\,\Delta}{2b}) \times \left[\frac{3(3-5b^2+5b^4-3b^6)}{32b^6} \ln\frac{1+b}{(1-b)(1+\Delta)} - \frac{3(1-b^2)(2b-(1-b)\Delta)}{128b^6(1+\Delta)^2} \times \right. \\ \left. \times\,(4(3-b^2) + 2(9+b-3b^2+b^3)\Delta + 2(2+b+b^3)\Delta^2 - (1+b)(1-b^2)\Delta^3) \right] \tag{N17}$$

where $\mathcal{H}()$ is the Lorentz-Heaviside "theta" function.

Weighted Average

The quantity which enters in the isotropic Boltzmann equation is the $1-\beta\mu_e$ weighted average, i.e.

$$\begin{aligned} &\left\langle (1-\beta\mu_e)\,\frac{d\sigma^{(0)}}{d\Delta} \right\rangle_{\mu_e} \\ &= \frac{3\sigma_{\rm T}}{8}\,\frac{(1-\beta^2)}{\beta} \left\langle 1 + \left(1 - \frac{(1-\beta^2)\,(1+\Delta-\mu_e(\frac{\Delta}{\beta}+\mu_e))}{(1-\beta\,\mu_e)^2}\right)^2 + \frac{1}{2}\frac{(1-\beta^2)^2(1-\mu_e^2)\,\left((1+\Delta)^2 - (\frac{\Delta}{\beta}+\mu_e)^2\right)}{(1-\beta\,\mu_e)^4} \right\rangle_{\mu_e} \\ &= \sigma_{\rm T}\,\overline{F}(\Delta, \beta\,{\rm sgn}(\Delta)) \end{aligned}$$

where

$$\overline{F}(\Delta,b) = {\rm sgn}(\Delta) \times \mathcal{H}(1 - \frac{(1-b)\,\Delta}{2b}) \times \left[\frac{3(1-b^2)^2(3-b^2)(2+\Delta)}{16b^6} \ln\frac{(1-b)(1+\Delta)}{1+b} \right. \\ \left. + \frac{3(1-b^2)(2b-(1-b)\Delta)}{32b^6(1+\Delta)}\,(4(3-3b^2+b^4) + 2(6+b-6b^2-b^3+2b^4)\Delta + (1-b^2)(1+b)\Delta^2) \right] \tag{N17}$$

which is quite similar to the unweighted average.

Moments of Energy Transfer

With the above expression we me compute averages of functions of Δ defined by

$$\overline{f(\Delta)}^{(0)} \equiv \frac{\int d\Delta\, f(\Delta)\, \frac{d\sigma^{(0)}}{d\Delta}}{\int d\Delta\, \frac{d\sigma^{(0)}}{d\Delta}} = \frac{1}{\sigma_{\rm T}} \int d\Delta\, f(\Delta)\, \frac{d\sigma^{(0)}}{d\Delta} \tag{N18}$$

for which we find

$$\begin{aligned} \overline{\Delta^1}^{(0)} &= \gamma^2\beta\,(\beta-\mu_e) \\ \overline{\Delta^2}^{(0)} &= \frac{1}{10}\gamma^4\beta^2(11(\beta-\mu_e)^2 + 3(1-\beta\mu_e)^2) \\ \overline{\Delta^3}^{(0)} &= \frac{1}{10}\gamma^6\beta^3(\beta-\mu_e)\,(9(1-\beta\mu_e)^2 + 13(\beta-\mu_e)^2) \\ \overline{\Delta^4}^{(0)} &= \frac{2}{35}\gamma^8\beta^4(3(1-\beta\mu_e)^4 + 33(\beta-\mu_e)^2(1-\beta\mu_e)^2 + 28(\beta-\mu_e)^4) \end{aligned} \tag{N19}$$

or averaging over μ_e with the prefactor from eq. ?

$$\langle\overline{f(\Delta)}^{(0)}\rangle_{\mu_e} = \frac{1}{2}\int_{-1}^{1} d\mu_e\,(1-\beta\mu_e)\,\overline{f(\Delta)}^{(0)} \tag{N20}$$

for which we find

$$\begin{aligned}
\langle\overline{\Delta^1}^{(0)}\rangle_{\mu_e} &= \frac{4}{3}\gamma^2\beta^2 \\
\langle\overline{\Delta^2}^{(0)}\rangle_{\mu_e} &= \frac{2}{15}\gamma^4\beta^2(5+16\beta^2) \\
\langle\overline{\Delta^3}^{(0)}\rangle_{\mu_e} &= \frac{4}{25}\gamma^6\beta^4(21+23\beta^2) \\
\langle\overline{\Delta^4}^{(0)}\rangle_{\mu_e} &= \frac{4}{525}\gamma^8\beta^4(147+1554\beta^2+859\beta^4) \\
\langle\overline{\Delta^5}^{(0)}\rangle_{\mu_e} &= \frac{32}{735}\gamma^{10}\beta^6(231+818\beta^2+271\beta^4) \\
\langle\overline{\Delta^6}^{(0)}\rangle_{\mu_e} &= \frac{8}{735}\gamma^{12}\beta^6(231+5067\beta^2+8973\beta^4+1969\beta^6)
\end{aligned} \tag{N21}$$

Thermal Moments

If the electrons have a non-degenerate thermal distribution then the occupation number depends only on the energy of the electron and not the momentum and has the Boltzmann form

$$n_e(\gamma) \propto e^{-\gamma U} \qquad U = \frac{m_e c^2}{kT_e} \tag{N20}$$

while the momentum-space measure in terms of γ is

$$d^3\mathbf{p} = 4\pi(m_e c)^3\gamma^2\beta^2 d(\gamma\beta) = 4\pi(m_e c)^3\gamma\,\sqrt{\gamma^2-1}\,d\gamma. \tag{N21}$$

Integrals over this thermal distribution may be written as modified Bessel function of the 2nd kind, $K_n(U)$, which obey the following identities

$$\frac{2n}{U}K_n(U) = K_{n+1}(U) - K_{n-1}(U) \qquad K_n'(U) = -\frac{1}{2}(K_{n+1}(U)+K_{n-1}(U)) \qquad K_{-n}(U) = K_n(U). \tag{N22}$$

Given that

$$\int_1^\infty d\gamma\,\sqrt{\gamma^2-1}\,e^{-\gamma U} = \frac{K_1(U)}{U} \tag{N23}$$

it follows that

$$\int_1^\infty d\gamma\,\gamma^{n+1}\,\sqrt{\gamma^2-1}\,e^{-\gamma U} = (-1)^n\left[\frac{d^n}{dU^n}\frac{K_1(U)}{U}\right]. \tag{N24}$$

We define the thermal average of any function of Δ by

$$\overline{\overline{f(\Delta)}}^{(0)} \equiv \frac{U}{K_2(U)}\int_1^\infty d\gamma\,\gamma\,\sqrt{\gamma^2-1}\,e^{-\gamma U}\,\langle\overline{f(\Delta)}\rangle_{\mu_e} \tag{N25}$$

so that using $\langle\overline{\Delta^n}\rangle_{\mu_e}$ computed above along with the identities we find

$$\begin{aligned}
\overline{\overline{\Delta}}^{(0)} &= \frac{4K_3(U)}{U\,K_2(U)} \\
\overline{\overline{\Delta^2}}^{(0)} &= \frac{2\,(U\,K_3(U)+21K_4(U))}{U^2K_2(U)} \\
\overline{\overline{\Delta^3}}^{(0)} &= \frac{84(3U\,K_4(U)+44K_5(U))}{5U^3K_2(U)} \\
\overline{\overline{\Delta^4}}^{(0)} &= \frac{12\,(546U\,K_5(U)+(7680+7U^2)\,K_6(U))}{5U^4K_2(U)} \\
\overline{\overline{\Delta^5}}^{(0)} &= \frac{96\,((2736+77U^2)\,K_7(U)+2916U\,K_8(U))}{7U^5K_2(U)} \\
\overline{\overline{\Delta^6}}^{(0)} &= \frac{24(4\,(44076+3781U^2)\,K_8(U)+U\,(424416+77U^2)\,K_9(U))}{7U^6K_2(U)}
\end{aligned} \quad . \tag{N26}$$

These 1st two expressions agree with eqs 32 & 33 of Barbosa (1982).

Using the Nth order asymptotic (large U) for the modified Bessel functions,

$$K_n(U) = \sqrt{\frac{\pi}{2U}}\, e^{-U} \left(1 + \sum_{j=1}^{N} \frac{\prod_{k=1}^{j}(4n^2 - (2k-1)^2)}{j!(8U)^j}\right) , \tag{N27}$$

we may expand the thermal moments for small temperature:

$$\begin{aligned}
\overline{\overline{\Delta}}^{(0)} &= 4\left(\frac{KT}{m_e c^2}\right) + 10\left(\frac{KT}{m_e c^2}\right)^2 + \frac{15}{2}\left(\frac{KT}{m_e c^2}\right)^3 + \mathcal{O}\left[\left(\frac{KT}{m_e c^2}\right)^4\right] \\
\overline{\overline{\Delta^2}}^{(0)} &= 2\left(\frac{KT}{m_e c^2}\right) + 47\left(\frac{KT}{m_e c^2}\right)^2 + \frac{1023}{4}\left(\frac{KT}{m_e c^2}\right)^3 + \mathcal{O}\left[\left(\frac{KT}{m_e c^2}\right)^4\right] \\
\overline{\overline{\Delta^3}}^{(0)} &= \frac{252}{5}\left(\frac{KT}{m_e c^2}\right)^2 + \frac{5208}{5}\left(\frac{KT}{m_e c^2}\right)^3 + \mathcal{O}\left[\left(\frac{KT}{m_e c^2}\right)^4\right] \\
\overline{\overline{\Delta^4}}^{(0)} &= \frac{84}{5}\left(\frac{KT}{m_e c^2}\right)^2 + \frac{7896}{5}\left(\frac{KT}{m_e c^2}\right)^3 + \mathcal{O}\left[\left(\frac{KT}{m_e c^2}\right)^4\right]
\end{aligned} . \tag{N28}$$

These 1st two expansions agree with eqs 35 & 36 of Barbosa (1982). At a given order this expansion becomes progressively worse as one considers larger powers of Δ. To obtain less than 10% errors with this 3rd order expansion one must limit the temperature to 310keV, 63keV, 15keV, and 4.9keV for Δ, Δ^2, Δ^3, and Δ^4, respectively. Where these higher order terms are important it is probably not a good idea to use these asymptotic series.

Extended Kompaneets Equation

$$\begin{aligned}
\frac{\partial n_\gamma}{\partial \tau} &= \frac{1}{\epsilon^2}\frac{\partial}{\partial \epsilon}\Bigg[\epsilon^3\Bigg(\frac{1}{3}\left(-3\overline{\overline{\Delta}}^{(0)} + 6\overline{\overline{\Delta^2}}^{(0)} - 10\overline{\overline{\Delta^3}}^{(0)} + 15\overline{\overline{\Delta^4}}^{(0)}\right)(1+n_\gamma)\,n_\gamma \\
&\quad + \frac{1}{12}\epsilon\left(6\overline{\overline{\Delta^2}}^{(0)} - 20\overline{\overline{\Delta^3}}^{(0)} + 45\overline{\overline{\Delta^4}}^{(0)}\right)\frac{\partial n_\gamma}{\partial \epsilon} + \frac{1}{12}\epsilon^2\left(-2\overline{\overline{\Delta^3}}^{(0)} + 9\overline{\overline{\Delta^4}}^{(0)}\right)\frac{\partial^2 n_\gamma}{\partial \epsilon^2} + \frac{1}{24}\epsilon^3\left(\overline{\overline{\Delta^4}}^{(0)}\right)\frac{\partial^3 n_\gamma}{\partial \epsilon^3} \\
&\quad + \frac{1}{6}\epsilon\left(-\overline{\overline{\Delta^3}}^{(0)} + 3\overline{\overline{\Delta^4}}^{(0)}\right)\left(5\,n_\gamma\frac{\partial n_\gamma}{\partial \epsilon} - \epsilon\left(\frac{\partial n_\gamma}{\partial \epsilon}\right)^2 + 2\epsilon\, n_\gamma\frac{\partial^2 n_\gamma}{\partial \epsilon^2}\right)\Bigg)\Bigg] + \mathcal{O}(\alpha, \Delta^5) \\
&= \frac{1}{\epsilon^2}\frac{\partial}{\partial \epsilon}\Bigg[\epsilon^3\Bigg(\frac{64\gamma^8\beta^6(77+83\beta^2)}{105}(1+n_\gamma)\,n_\gamma + \frac{\overline{\gamma^8\beta^2(35-105\beta^2+4417\beta^4+3333\beta^6)}}{105}\,\epsilon\,\frac{\partial n_\gamma}{\partial \epsilon} \\
&\quad + \frac{\overline{\gamma^8\beta^4(147+4634\beta^2+2899\beta^4)}}{525}\,\epsilon^2\,\frac{\partial^2 n_\gamma}{\partial \epsilon^2} + \frac{\overline{\gamma^8\beta^4(147+1554\beta^2+859\beta^4)}}{3150}\,\epsilon^3\,\frac{\partial^3 n_\gamma}{\partial \epsilon^3} \\
&\quad + \frac{8\,\overline{\gamma^8\beta^6(77+51\beta^2)}}{105}\,\epsilon\left(5\,n_\gamma\frac{\partial n_\gamma}{\partial \epsilon} - \epsilon\left(\frac{\partial n_\gamma}{\partial \epsilon}\right)^2 + 2\epsilon\, n_\gamma\frac{\partial^2 n_\gamma}{\partial \epsilon^2}\right)\Bigg)\Bigg] + \mathcal{O}(\alpha, \beta^6) \\
&= \frac{1}{\epsilon^2}\frac{\partial}{\partial \epsilon}\Bigg[\epsilon^3\Bigg(\frac{64(516\,K_6(U) + 77U\,K_7(U))}{U^4\,K_2(U)}(1+n_\gamma)\,n_\gamma + \frac{4\,(4104-5U^2)\,K_6(U) + U\,(4392+U^2)\,K_7(U)}{U^4 K_2(U)}\,\epsilon\,\frac{\partial n_\gamma}{\partial \epsilon} \\
&\quad + \frac{(12504+21U^2)\,K_6(U) + 4718U\,K_7(U)}{5U^4 K_2(U)}\,\epsilon^2\,\frac{\partial^2 n_\gamma}{\partial \epsilon^2} + \frac{(1128+7U^2)\,K_6(U) + 546U\,K_7(U)}{10U^4 K_2(U)}\,\epsilon^3\,\frac{\partial^3 n_\gamma}{\partial \epsilon^3} \\
&\quad + \frac{8(228\,K_6(U) + 77U\,K_7(U))}{U^4 K_2(U)}\left(5\,n_\gamma\frac{\partial n_\gamma}{\partial \epsilon} - \epsilon\left(\frac{\partial n_\gamma}{\partial \epsilon}\right)^2 + 2\epsilon\, n_\gamma\frac{\partial^2 n_\gamma}{\partial \epsilon^2}\right)\Bigg)\Bigg] + \mathcal{O}(\alpha, U^{-3}) \\
&= \frac{kT_e}{m_e c^2}\frac{1}{\epsilon^2}\frac{\partial}{\partial \epsilon}\left[\epsilon^3\left(\left(1 + \frac{5}{2}\frac{kT_e}{m_e c^2}\right)\epsilon\,\frac{\partial n_\gamma}{\partial \epsilon} + \frac{7}{10}\frac{kT_e}{m_e c^2}\left(6\epsilon^2\frac{\partial^2 n_\gamma}{\partial \epsilon^2} + \epsilon^3\frac{\partial^3 n_\gamma}{\partial \epsilon^3}\right)\right)\right] + \mathcal{O}\left(\alpha, \left(\frac{kT_e}{m_e c^2}\right)^3\right)
\end{aligned}$$

$$\frac{\Delta n_\gamma}{\tau} = \frac{xe^x}{(e^x-1)^2}\frac{kT_\mathrm{e}}{m_\mathrm{e}c^2}\Bigg(-\left(4+10\frac{kT_\mathrm{e}}{m_\mathrm{e}c^2}\right) + \left(1+\frac{47}{2}\frac{kT_\mathrm{e}}{m_\mathrm{e}c^2}\right)\, x\,\frac{e^x+1}{e^x-1} - \frac{42}{5}\frac{kT_\mathrm{e}}{m_\mathrm{e}c^2}\,x^2\frac{e^{2x}+4e^x+1}{(e^x-1)^2} + \frac{7}{10}\frac{kT_\mathrm{e}}{m_\mathrm{e}c^2}\,x^3\frac{(e^x+1)(e^{2x}+10e^x+1)}{(e^x-1)^3}\Bigg)$$

or we may think of this as an expansion

$$\Delta n_\gamma = \sum_{n=1}^{+}\sum_{m=0}^{+} Y_\mathrm{C}^{(n,m)} F_\mathrm{C}^{(n,m)}(x) \qquad Y_\mathrm{C}^{(n,m)} = \int d\tau\,\left(\frac{kT_\mathrm{e}}{m_\mathrm{e}c^2}\right)^n \left(\frac{kT_\gamma}{m_\mathrm{e}c^2}\right)^m \qquad x=\frac{\epsilon}{kT_\gamma}$$

and

$$F_\mathrm{C}^{(1,0)}(x) = \frac{xe^x}{(e^x-1)^2}\left(x\,\frac{e^x+1}{e^x-1}-4\right)$$

$$F_\mathrm{C}^{(2,0)}(x) = \frac{xe^x}{(e^x-1)^2}\left(-10+\frac{47}{2}x\,\frac{e^x+1}{e^x-1} - \frac{42}{5}x^2\frac{e^{2x}+4e^x+1}{(e^x-1)^2} + \frac{7}{10}x^3\frac{(e^x+1)(e^{2x}+10e^x+1)}{(e^x-1)^3}\right)$$

These distortions have asymptotic behaviour

$$F_\mathrm{C}^{(1,0)}(x) = x\,(x-4)\,e^{-x}+\mathcal{O}(e^{-2x}) \qquad F_\mathrm{C}^{(2,0)}(x) = x\,\left(\tfrac{7}{10}x^3-\tfrac{42}{5}x^2+\tfrac{47}{2}x-10\right)\,e^{-x}+\mathcal{O}(e^{-2x}) \qquad x\gg 1$$

$$F_\mathrm{C}^{(1,0)}(x) = -\tfrac{2}{x}+\mathcal{O}(x) \qquad F_\mathrm{C}^{(2,0)}(x) = \tfrac{17}{5x}+\mathcal{O}(x) \qquad x\ll 1$$

Zeros

Consider the case where the temperature along the beam is constant and much larger than the photon temperature so that

$$\Delta n_\gamma \propto= \sum_{n\geq 0}\left(\frac{kT_\mathrm{e}}{m_\mathrm{e}c^2}\right)^n F_\mathrm{C}^{(n,0)}(x).$$

The classical S-Z effect has the zero single of $F_\mathrm{C}^{(1,0)}(x)$ at $x=3.83002\ldots$. More generally, at least for small enough T_e, Δn_γ will continue a single zero, but not at the same value of x. Let us define the function $x_0(\frac{kT_\mathrm{e}}{m_\mathrm{e}c^2})$ to be the position of the zero as a function of temperature, i.e.

$$0=\sum_{n=0}^{\infty}\Theta_\mathrm{e}^n F_\mathrm{C}^{(n,0)}(x_0(\Theta_\mathrm{e}))$$

or since $F_\mathrm{C}^{(0,0)}(x)=0$

$$0=\sum_{n=1}^{\infty}\Theta_\mathrm{e}^{n-1} F_\mathrm{C}^{(n,0)}(x_0(\Theta_\mathrm{e})).$$

From this equation we can find all the derivatives of $x_0(\Theta_\mathrm{e})$ at $\Theta_\mathrm{e}=0$, e.g. by taking derivatives of this function wrt Θ_e. The 1st derivative yields

$$0=F_\mathrm{C}^{(2,0)}(x_0(0)) + F_\mathrm{C}^{(1,0)\prime}(x_0(0))\,x_0'(0).$$

or

$$x_0'(0) = -\frac{F_\mathrm{C}^{(2,0)}(x_0(0))}{F_\mathrm{C}^{(1,0)\prime}(x_0(0))}.$$

The 2nd derivative yields

$$0=2\,F_\mathrm{C}^{(3,0)}(x_0(0)) + 2F_\mathrm{C}^{(2,0)\prime}(x_0(0))\,x_0'(0) + F_\mathrm{C}^{(1,0)\prime\prime}(x_0(0))\,(x_0'(0))^2 + F_\mathrm{C}^{(1,0)\prime}(x_0(0))\,x_0''(0)$$

or

$$x_0''(0) = -\frac{2\,F_\mathrm{C}^{(3,0)}(x_0(0)) + 2F_\mathrm{C}^{(2,0)\prime}(x_0(0))\,x_0'(0) + F_\mathrm{C}^{(1,0)\prime\prime}(x_0(0))\,(x_0'(0))^2}{F_\mathrm{C}^{(1,0)\prime}(x_0(0))}$$

Non-Perturbative Rayleigh-Jeans Limit

One may expand the blackbody spectrum about zero frequency:

$$n_\gamma^{\rm BB}(x) = \frac{1}{x} + \frac{1}{2} - \frac{x}{12} + \mathcal{O}(x^3).$$

Substituting this expansion of the blackbody spectrum into the collision integral, (X10), we obtain a a similar expansion of distortion to the spectrum

$$\Delta n_\gamma^{\rm SZ}(x) = \sum_{k\geq -2} x^k \int d\tau \int_0^\infty d\beta\, f(\beta)\, s_k(\alpha,\beta).$$

So far we have not made any assumptions about the smallness of the electron temperature. Here we will use the cross-section expanded to 0th order in α, i.e we will use $\overline{S}(0,\Delta) = \sigma_{\rm T}\overline{F}(\Delta,\beta{\rm sgn}(\Delta))$ and will henceforth set to the 0 in the 1st argument of $s_k(0,\beta)$. Collecting terms we find that

$$\begin{aligned} s_{-2}(0,\beta) &= -\int_{-1}^\infty d\Delta\, \frac{\Delta}{(1+\Delta)^2}\, \overline{F}(\Delta,\beta{\rm sgn}(\Delta)) \\ s_{-1}(0,\beta) &= -\int_{-1}^\infty d\Delta\, \frac{\Delta^2(2+\Delta)}{(1+\Delta)^3}\, \overline{F}(\Delta,\beta{\rm sgn}(\Delta)) \end{aligned}.$$

If, for $\Delta < 0$ one changes the variable of integration from Δ to $\bar{\Delta} = -\frac{\Delta}{1+\Delta}$ and uses the symmetry property

$$\overline{F}(\Delta,\beta) = (1+\Delta)\overline{F}(\frac{-\Delta}{1+\Delta},-\beta)$$

one simplify the integral to one over positive only Δ:

$$\begin{aligned} s_{-2}(0,\beta) &= -\int_0^\infty d\Delta\, \frac{\Delta}{(1+\Delta)^2}\, \overline{F}(\Delta,\beta) + \int_{-1}^0 d\Delta\, \frac{\Delta}{(1+\Delta)^2}\, \overline{F}(\Delta,-\beta) \\ &= -\int_0^\infty d\Delta\, \frac{\Delta}{(1+\Delta)^2}\, \overline{F}(\Delta,\beta) + \int_0^\infty d\bar{\Delta} \frac{\bar{\Delta}}{1+\bar{\Delta}}\, \overline{F}(\frac{-\bar{\Delta}}{1+\bar{\Delta}},-\beta) \\ &= -\int_0^\infty d\Delta\, \frac{\Delta}{(1+\Delta)^2}\, \left[\overline{F}(\Delta,\beta) - (1+\Delta)\overline{F}(\frac{-\Delta}{1+\Delta},-\beta)\right] \\ &= -\int_0^\infty d\Delta\, \frac{\Delta}{(1+\Delta)^2}\, \left[\overline{F}(\Delta,\beta) - \overline{F}(\Delta,\beta)\right] = 0 \end{aligned}$$

and

$$\begin{aligned} s_{-1}(\beta) &= -\int_0^\infty d\Delta\, \frac{\Delta^2\,(2+\Delta)}{2(1+\Delta)^3}\, \overline{F}(\Delta,\beta) - \int_{-1}^0 d\Delta\, \frac{\Delta^2\,(2+\Delta)}{2(1+\Delta)^3}\, \overline{F}(\Delta,-\beta) \\ &= -\int_0^\infty d\Delta\, \frac{\Delta^2\,(2+\Delta)}{2(1+\Delta)^3}\, \overline{F}(\Delta,\beta) - \int_0^\infty d\bar{\Delta}\, \frac{\bar{\Delta}^2\,(2+\bar{\Delta})}{2(1+\bar{\Delta})^2}\, \overline{F}(\frac{-\bar{\Delta}}{1+\bar{\Delta}},-\beta) \\ &= -\int_0^\infty d\Delta\, \frac{\Delta^2\,(2+\Delta)}{2(1+\Delta)^3}\, \left[\overline{F}(\Delta,\beta) + (1+\Delta)\,\overline{F}(\frac{-\Delta}{1+\Delta},-\beta)\right] \\ &= -\int_0^\infty d\Delta\, \frac{\Delta^2\,(2+\Delta)}{2(1+\Delta)^3}\, (\overline{F}(\Delta,\beta) + \overline{F}(\Delta,\beta)) \\ &= -\int_0^\infty d\Delta\, \frac{\Delta^2\,(2+\Delta)}{(1+\Delta)^3}\, \overline{F}(\Delta,\beta) \\ &= -\left[\frac{9-15\beta^2+10\beta^4}{4\beta^4} - \frac{3\,(3-6\beta^2+4\beta^4-\beta^6)}{4\beta^5}\ln\frac{1+\beta}{1-\beta} - \frac{3(3-\beta^2)(1-\beta^2)^2}{16\beta^6}\left(\ln\frac{1+\beta}{1-\beta}\right)^2\right] \end{aligned}.$$

More generally one finds that the $s_k(\beta)$ with even k are zero due to the symmetry properties of $\overline{F}$, e.g. $s_0(\beta) = s_2(\beta) = 0$. Thus expanding in small frequency the lowest order contribution to the S-Z distortion is $\propto \frac{1}{x}$ and the next order correction is $\propto x$. We concentrate on the $\frac{1}{x}$, "Rayleigh-Jeans" term. Note that since $\overline{F}(\Delta,\beta) > 0$ it is clear that $s_{-1}(\beta) \leq 0$ for all β. Since we are mostly interested in non-relativistic electrons one may expand this function in β about 0

$$\begin{aligned} s_{-1}(0,\beta) = &-\frac{2}{3}\beta^2 - \frac{8}{75}\beta^4 - \frac{26}{525}\beta^6 - \frac{68}{2205}\beta^8 - \frac{34}{1575}\beta^{10} - \frac{5072}{315315}\beta^{12} \\ &- \frac{5366}{429975}\beta^{14} - \frac{96892}{9719325}\beta^{16} - \frac{3910586}{480134655}\beta^{18} - \frac{4146536}{611936325}\beta^{20} + \mathcal{O}(\beta^{22}) \end{aligned}$$

or expand in γ ($= \frac{1}{\sqrt{1-\beta^2}}$) about 1

$$\begin{aligned} s_{-1}(0,\frac{\sqrt{\gamma^2-1}}{\gamma}) = &-\frac{4}{3}(\gamma-1) + \frac{118}{75}(\gamma-1)^2 - \frac{312}{175}(\gamma-1)^3 + \frac{21566}{11025}(\gamma-1)^4 - \frac{23024}{11025}(\gamma-1)^5 + \frac{3441448}{1576575}(\gamma-1)^6 \\ &- \frac{10628096}{4729725}(\gamma-1)^7 + \frac{2024571392}{884458575}(\gamma-1)^8 - \frac{997789312}{430890075}(\gamma-1)^9 + \frac{509471814976}{218461268025}(\gamma-1)^{10} + \mathcal{O}((\gamma-1)^{11}) \end{aligned}$$

or expand in inverse powers of γ

$$\begin{aligned} s_{-1}(0,\frac{\sqrt{\gamma^2-1}}{\gamma}) = &-1 - \left(\frac{3}{4} - \frac{3}{2}\ln 2\gamma\right)\frac{1}{\gamma^2} - \left(\frac{27}{8} - \frac{21}{4}\ln 2\gamma + \frac{3}{2}\ln^2 2\gamma\right)\frac{1}{\gamma^4} - \left(\frac{429}{64} - \frac{201}{16}\ln 2\gamma + \frac{21}{4}\ln^2 2\gamma\right)\frac{1}{\gamma^6} \\ &- \left(\frac{1423}{128} - \frac{369}{16}\ln 2\gamma + \frac{45}{4}\ln^2 2\gamma\right)\frac{1}{\gamma^8} - \left(\frac{34037}{2048} - \frac{9397}{256}\ln 2\gamma + \frac{39}{2}\ln^2 2\gamma\right)\frac{1}{\gamma^{10}} + \mathcal{O}(\frac{1}{\gamma^{11}}) \end{aligned}$$

The β series is well-behaved, leading to at most 10% errors, while the γ expansion is highly divergent for only moderate values of γ. However when convolving with a thermal electron distribution function, the integral of the γ distribution is more tractable, being expressed in terms of modified Bessel functions rather than generalized hypergeometric functions.

Ultra-Relativistic Electrons

Low energy photons in the Rayleigh-Jeans region, when in collision with ultra-relativistic electrons, will gain so much energy that they will be shifted out of the Rayleigh-Jeans region. This being the case we expect that at each frequence the brightness will just be deminished by the factor $e^\tau - 1$ or in the small perturbation limit we are are working, by τ.

Rayleigh-Jeans Limit for Thermal Electrons

Of particular interest is a thermal electron distribution function where the $\frac{1}{x}$ term is

$$\Delta n_\gamma^{\rm SZ-RJ}(x) = \frac{1}{x}\int d\tau\, r_{-1}(\frac{kT_{\rm e}}{m_{\rm e}c^2})$$

where, changing the variable of integration from β to γ,

$$r_{-1}(\Theta_{\rm e}) = \int_1^\infty d\gamma\,\gamma\,\sqrt{\gamma^2-1}\; e^{-\gamma/\Theta_{\rm e}} s_{-1}(0,\frac{\sqrt{\gamma^2-1}}{\gamma}) \qquad .$$

We have not been able to express $r_{-1}(\Theta_{\rm e})$ in terms of simple functions, but it is easy to evaluated numerically. One may, however compute teh Taylor series of $r_{-1}(\Theta_{\rm e})$ by using the Taylor series of s_{-1} in γ about 1. Expanding in $\gamma-1$ to nth order, integrating wrt γ, and then Taylor expanding the result in $\Theta_{\rm e}$ will yield Taylor series which is accurate to order $\Theta_{\rm e}^n$. Using the 10th order $\gamma-1$ expansion above one finds

$$r_{-1}(\Theta_{\rm e}) = -2\frac{\begin{pmatrix} 36735421382656 \\ +\ 338455161521599\,\Theta_{\rm e} \\ +1945907797721088\,\Theta_{\rm e}^2 \\ +5282952472206930\,\Theta_{\rm e}^3 \\ +7234489912957440\,\Theta_{\rm e}^4 \\ -\ 378282822619680\,\Theta_{\rm e}^5 \end{pmatrix} K_5(\frac{1}{\Theta_{\rm e}}) - \begin{pmatrix} 36735421382656 \\ +\ 136379135164416\,\Theta_{\rm e} \\ +\ 741602461487847\,\Theta_{\rm e}^2 \\ -\ 31489295609856\,\Theta_{\rm e}^3 \\ +2060543644646400\,\Theta_{\rm e}^4 \\ -4158443518225920\,\Theta_{\rm e}^5 \\ +4539393871436160\,\Theta_{\rm e}^6 \end{pmatrix} K_6(\frac{1}{\Theta_{\rm e}})}{31208752575\,K_2(\frac{1}{\Theta_{\rm e}})} + \mathcal{O}(\Theta_{\rm e}^{11})$$

and Taylor expanding this in Θ_e we find

$$
\begin{aligned}
r_{-1}(\Theta_e) = & -2\Theta_e + \frac{17}{5}\Theta_e^2 - \frac{123}{20}\Theta_e^3 + \frac{1989}{140}\Theta_e^4 - \frac{14403}{320}\Theta_e^5 + \frac{20157}{112}\Theta_e^6 \\
& - \frac{423951}{512}\Theta_e^7 + \frac{5187387}{1232}\Theta_e^8 - \frac{29482141455}{1261568}\Theta_e^9 + \frac{36083969379}{256256}\Theta_e^{10} + \mathcal{O}(\Theta_e^{11})
\end{aligned}
$$

This series does not appear to converge.