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ABSTRACT

Small amplitude low frequency waves are treated in a plasma of infinite conductivity in the presence of magnetic and gravitational fields. In general, there are three kinds of wave motion: namely, acoustic, gravity, and hydromagnetic. In this analysis, three wave modes were found which we called the +mode, -mode, and the Alfven mode. Each mode is strongly coupled to each of the three kinds of motions, with the exception of the Alfven mode. The Alfven mode is independent of compressibility and gravity, since it can be separated from the dispersion relation.

The local dispersion relation is derived and expressed in a nondimensional form independent of the constants describing a particular atmosphere. This dispersion relation is shown to exhibit the behavior of stable and unstable modes of wave propagation, which are displayed on a number of graphs and in tabular form.

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The variation of the density with height is taken into account by a generalized W.K.B. method. Equations are found which give the height at which wave reflection occurs. This determines the maximum heights up to which the three modes can propagate.

I. INTRODUCTION

The great width of the coronal lines, the high level of ionization of Fe, Ca, and Ni, the washing out of the Fraunhofer spectrum by electron scattering, and the vast extent of the corona indicate an extremely high temperature, which astrophysicists believe is due to the transfer of energy from the convection zone by waves. Some authors[1-6] used acoustic waves to account for the corona heating, others[7-11]

[1] M. Schwarzschild, Ap. J. 107, 1 (1948).

[2] L. Biermann, Zs. f. Astrophysik 25, 161 (1948).

[3] E. Schatzman, Ann. de Astrophysique 12, 203 (1949).

[4] W. Unno and K. Kawabata, Pub. Astr. Soc. Japan 7, 21 (1955).

[5] R. Weymann, Ap. J. 132, 452 (1960).

[6] C. Jager, Bull. Astr. Inst. Netherlands 16, 71 (1961).

[7] H. Alfven, Monthly Notices 107, 211 (1947).

[8] T.G. Cowling, THE SUN, Univ. of Chicago Press; Chicago (1953).

[9] V.C.A. Ferraro, Ap. J. 119, 393 (1954).

[10] J.H. Piddington, Monthly Notices 116, 314 (1956).

[11] D.E. Osterbrock, Ap. J. 134, 347 (1961).

considered hydromagnetic waves as responsible, and another[12] used internal gravity waves for an explanation. One very special situation is treated by Yu[13].

However, in a magnetized atmosphere consisting of a plasma, it is in principle impossible to consider either one of these modes independently from the others. All modes interact with each other and have to be considered simultaneously. Therefore, we have investigated plasma wave propagation within a magnetized atmosphere under the influence of gravity in the magneto-hydrodynamic (M.H.D.) approximation, which is valid for low frequency waves. We will only consider the propagation of these waves in a magnetized atmosphere, and therefore neglect dissipative effects arising from viscosity, electrical resistivity, and heat conductivity.

With the addition of a magnetic field, the problem becomes much more complicated, not only by the introduction of another mode of propagation (Alfven mode), but also due to the fact that the magnetic field is a vector, thereby creating a third direction, which can be aligned arbitrarily with respect to the gravity vector $\underline{g}$ and the wave vector $\underline{k}$.

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12 W.A. Whitaker, Ap. J. 137, 914 (1963).

13 C.P. Yu, Phys. Fluids 8, 650 (1965).

2. FUNDAMENTAL EQUATIONS

The fundamental equations necessary to describe the wave motions are (in Gaussian units)

1) Euler equation:[14]

$$\rho \left[\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \nabla)\, \underline{v} \right] = -\nabla p + \rho \underline{g}$$

$$- \frac{1}{4\pi}\, \underline{H} \times (\nabla \times \underline{H}) \quad . \qquad (2.1)$$

In this equation $\underline{v}$ is the velocity vector of a particle of the oscillating medium, p is the pressure, ρ the density, $\underline{g}$ the gravity vector, and $\underline{H}$ the magnetic field strength. For convenience we choose $\underline{g}$ to be along the negative z direction (downward), and we choose to orient the coordinate system so that the arbitrarily directed unperturbed magnetic field in the absence of wave motion has no y component. Thus, we have

$$\underline{g} = -g\, \underline{e}_z \qquad (2.2)$$

and

$$\underline{H}_o = H_x \underline{e}_x + H_z \underline{e}_z \quad , \qquad (2.3)$$

where $\underline{e}_x$ and $\underline{e}_z$ are unit vectors in the x and z directions, respectively.

2) Continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0 \quad . \qquad (2.4)$$

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[14] T.G. Cowling, MAGNETOHYDRODYNAMICS, Interscience, New York, (1957).

3) <u>Second law of thermodynamics:</u> (adiabatic approximation)

$$\frac{ds}{dt} = \frac{\partial s}{\partial t} + \underline{v} \cdot \nabla s = 0 \quad , \tag{2.5}$$

where s is the specific entropy.

4) <u>Equation of state:</u>

$$\rho = \rho(p,s) \tag{2.6}$$

Thus, we have for the total differential

$$d\rho = \left.\frac{\partial \rho}{\partial s}\right)_p ds + \left.\frac{\partial \rho}{\partial p}\right)_s dp \quad . \tag{2.7}$$

5) <u>Ohm-Maxwell equation:</u>[15]

$$\frac{\partial H}{\partial t} = \nabla \times (\underline{v} \times \underline{H}) \tag{2.8}$$

We will assume that for the equilibrium conditions the variation of density, pressure, and entropy with height is exponential, as is the case for an ideal gas. Thus, we have

$$\begin{aligned} \rho_{eq} &= \rho_o e^{-z/h} \\ p_{eq} &= p_o e^{-z/h} \end{aligned} \tag{2.9}$$

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[15] T.G. Cowling, ibid.

where ρ_o and p_o are constants, and h is the so-called "scale height".

3. THE LINEARIZED EQUATIONS OF MOTION

We assume that all deviations of the perturbed quantities from their equilibrium values are small. Hence, we may put to first order:

$$\rho(\underline{r},t) = \left[\rho_o + \rho'(\underline{r},t)\right] e^{-z/h},$$

$$p(\underline{r},t) = \left[p_o + p'(\underline{r},t)\right] e^{-z/h},$$

$$s(\underline{r},t) = s_{eq} + s'(\underline{r},t),$$

and

$$\underline{H} = \underline{H}_o + \underline{H}'(\underline{r},t) \quad . \tag{3.1}$$

where $\underline{H}_o$ is constant. The quantities ρ', p', s', and $\underline{H}'$, together with the velocity $\underline{v} = \underline{v}(\underline{r},t)$, are considered to be perturbing quantitites of first order.

Consider the Euler equation (2.1) to zeroth order for equilibrium conditions:

$$0 = -\nabla p_{eq} + \rho_{eq}\underline{g} \quad . \tag{3.2}$$

By combining eq. (2.9) with eq. (3.2), we have

$$\underline{g} = -\frac{p_o}{h\rho_o}\,\underline{e}_z \qquad . \tag{3.3}$$

We substitute the quantities (3.1) into the fundamental equations (2.1), (2.4), (2.5), and (2.7), and neglect the products of the perturbing quantities. With the help of eq. (3.2), there results the linearized Euler equation:

$$\frac{\partial \underline{v}}{\partial t} = \frac{\nabla \underline{p}'}{\rho_o} + \frac{\rho'}{\rho_o}\,\underline{g} - \frac{1}{4\pi\rho_{eq}}\,\underline{H}_o \times (\nabla \times \underline{H}') \qquad . \tag{3.4}$$

The linearized continutity equation becomes

$$\frac{\partial \rho'}{\partial t} + \rho_o\,\nabla\cdot\underline{v} - \frac{\rho_o}{h}\,(\underline{v}\cdot\underline{e}_z) = 0 \qquad . \tag{3.5}$$

The second law of thermodynamics can be written in linearized form as:

$$\frac{\partial s'}{\partial t} + (\underline{v}\cdot\underline{e}_z)\,\frac{ds_{eq}}{dz} = 0 \qquad . \tag{3.6}$$

The linearized Ohm-Maxwell equation is:

$$\frac{\partial \underline{H}'}{\partial t} = \nabla \times (\underline{v} \times \underline{H}_o) \quad . \tag{3.7}$$

We linearize the equation of state by observing that the density is a continuous function of both the entropy and pressure, so that we can perform a Taylor expansion about the equilibrium state. Keeping only first order terms, eq. (2.7) becomes

$$\rho = \left[\rho_o + \left.\frac{\partial \rho_o}{\partial s_o}\right)_p s' + \left.\frac{\partial \rho_o}{\partial p_o}\right)_s p' \right] e^{-z/h} \tag{3.8}$$

By substitution, we have

$$\rho' = \left.\frac{\partial \rho_o}{\partial s_o}\right)_p s' + \frac{1}{a^2} p' \quad , \tag{3.9}$$

It is to be noted that the zero subscripts of the partial derivatives indicate that these quantities are evaluated at their equilibrium values. In eq. (3.9), $a = \left[\partial \rho_o / \partial p_o \right)_s \Big]^{-1/2}$ is known as the sound velocity in the medium.

Taking the time derivative of the Euler equation (3.4) and eq. (3.9), we obtain, respectively:

$$\frac{\partial^2 \underline{v}}{\partial t^2} + \frac{1}{\rho_o} \nabla \frac{\partial p'}{\partial t} - \frac{1}{h\rho_o} \frac{\partial p'}{\partial t} \underline{e}_z$$

$$- \frac{1}{\rho_o} \frac{\partial \rho'}{\partial t} \underline{g} + \frac{1}{4\pi\rho_{eq}} \underline{H}_o \times (\nabla \times \frac{\partial \underline{H}'}{\partial t}) = 0 \quad , \tag{3.10}$$

and

$$\frac{\partial \rho'}{\partial t} - \left.\frac{\partial \rho_\circ}{\partial s_\circ}\right)_p \frac{\partial s'}{\partial t} - \frac{1}{a^2}\frac{\partial p'}{\partial t} = 0 \quad . \tag{3.11}$$

Eqs. (3.5), (3.6), (3.7), (3.10), and (3.11) form a set of linear homogeneous equations for the unknowns $\underline{v}$, ρ', p', s', and $\underline{H}'$. From these equations and the plane wave assumption, it is possible to derive the dispersion relation.

In our case, however, it is expedient to first reduce the set of eqs. (3.5), (3.6), (3.7), (3.10), and (3.11) by eliminating the unknowns ρ', p', s', and $\underline{H}'$ resulting in only one equation for $\underline{v}$. For example, solving eq. (3.5) for $\partial\rho'/\partial t$, eq. (3.6) for $\partial s'/\partial t$, eq. (3.7) for $\partial \underline{H}'/\partial t$, and eq. (3.11) for $\partial\rho'/\partial t$; substituting $\partial\rho'/\partial t$ of eq. (3.5) and $\partial s'/\partial t$ of eq. (3.7) into eq. (3.11), and then substituting $\partial p'/\partial t$ of eq. (3.11), $\partial\rho'/\partial t$ of eq. (3.5), and $\partial \underline{H}'/\partial t$ of eq. (3.7) into eq. (3.10), we have for the equation of $\underline{v}$.

$$\frac{\partial^2 \underline{v}}{\partial t^2} + \frac{a^2}{h} \nabla (\underline{v}\cdot\underline{e}_z) - a^2 \nabla (\nabla\cdot\underline{v})$$

$$+ \nabla\left[\frac{a^2}{\rho_\circ} (\underline{v}\cdot\underline{e}_z) \left.\frac{\partial \rho_\circ}{\partial s_\circ}\right)_p \frac{ds_{eq}}{dz}\right] - \frac{a^2}{h^2} (\underline{v}\cdot\underline{e}_z)\, \underline{e}_z +$$

(continued)

$$+ \frac{a^2}{h} (\nabla\cdot\underline{v})\ \underline{e}_z - \frac{a^2}{h\rho_\circ} (\underline{v}\cdot\underline{e}_z) \left.\frac{\partial\rho_\circ}{\partial s_\circ}\right)_p \frac{ds_{eq}}{dz}\ \underline{e}_z$$

$$- \frac{(\underline{v}\cdot\underline{e}_z)}{h}\ \underline{g} + (\nabla\cdot\underline{v})\ \underline{g} + \frac{1}{4\pi\rho_{eq}} \underline{H}_\circ \times \left\{ \nabla \times \left[\nabla \times (\underline{v} \times \underline{H}_\circ) \right] \right\} = 0 \quad . \tag{3.12}$$

The vector differential equation (3.12) represents a set of three linear homogeneous differential equations for the unknowns v_x, v_y, and v_z.

4. PERFECT GAS ASSUMPTION

It is quite obvious that the case of a perfect gas deserves special consideration for the reason of being a very good approximation in most situations and for its mathematical simplicity.

For a perfect gas under adiabatic compression, we have

$$p\rho^{-\gamma} = p_\circ\rho_\circ^{-\gamma} \quad , \tag{4.1}$$

where $\gamma = c_p/c_v$ is the ratio of the specific heats, at constant pressure and density. For a momatomic gas, one has $\gamma = 5/3 = 1.6666...$. From eq. (4.1) and the definition of sound, we have

$$a^2 = \frac{dp_\circ}{d\rho_\circ} = \gamma \frac{p_\circ}{\rho_\circ} \quad . \tag{4.2}$$

With the help of eqs. (3.3) and (4.2), we can express the scale height as follows:

$$h = \frac{a^2}{g\,\gamma} \quad . \tag{4.3}$$

To express the term in the square brackets of eq. (3.12) for the case of an ideal gas, we consider the well-know thermodynamic relations

$$\left.\frac{\partial s}{\partial p}\right)_T = \frac{1}{\rho^2} \left.\frac{\partial \rho}{\partial T}\right)_p \quad \text{and} \quad \left.\frac{\partial \rho}{\partial s}\right)_p = \frac{T}{c_p} \left.\frac{\partial \rho}{\partial T}\right)_p \quad , \tag{4.4}$$

where T is the absolute temperature. By making use of eqs. (3.2) and (4.4), we can write

$$\frac{a^2}{\rho_\circ} \left.\frac{\partial \rho_\circ}{\partial s_\circ}\right)_p \frac{ds_{eq}}{dz} = \frac{a^2}{\rho_\circ} \left.\frac{\partial \rho_\circ}{\partial s_\circ}\right)_p \left.\frac{\partial s_\circ}{\partial p_\circ}\right)_T \frac{dp_{eq}}{dz}$$

$$= \frac{a^2}{\rho_\circ}\left[\frac{T}{c_p} \left.\frac{\partial \rho_\circ}{\partial T}\right)_p\right]\left[-\frac{g}{\rho_\circ} \left.\frac{\partial \rho_\circ}{\partial T}\right)_p\right]$$

$$= -\frac{T}{c_p}\, a^2 \frac{g}{\rho_\circ^2}\left[\left.\frac{\partial \rho_\circ}{\partial T}\right)_p\right]^2 = -\frac{g}{c_p T}\, a^2 \quad . \tag{(4.5}$$

It is expedient to put the velocity of sound in still another form. For the perfect gas assumption, we have

$$c_p - c_v = \frac{R}{A} = c_v(\gamma - 1) \quad , \tag{4.6}$$

where R is the gas constant, and A is the molecular weight. Thus, from Eq. (4.2) we obtain

$$a^2 = \gamma \frac{R}{A} T = c_p T(\gamma-1). \tag{4.7}$$

With eqs. (4.5) and (4.7), the term in the square brackets of eq. (3.12) becomes

$$\frac{a^2}{\rho_\circ} (\underline{v} \cdot \underline{e}_z) \left.\frac{\partial\rho_\circ}{\partial s_\circ}\right)_p \frac{ds_\circ}{dz} = -g (\gamma - 1) (\underline{v} \cdot \underline{e}_z) \tag{4.8}$$

By substitution of eq. (4.8) into eq. (3.12), we have the equation for $\underline{v}$ for acoustic magneto-gravity waves within a perfect gas

$$\frac{\partial^2 \underline{v}}{\partial t^2} - a^2 \text{ grad div } \underline{v} - \text{grad } (\underline{v} \cdot \underline{g})$$

$$-\underline{g} (\gamma - 1) \text{ div } \underline{v} + \frac{1}{4\pi\rho_{eq}} \underline{H}_\circ \times \text{curl curl } (\underline{v} \times \underline{H}_\circ) = 0. \tag{4.9}$$

5. PLANE WAVE ANALYSIS

For plane waves, the velocity describing the wave motion varies sinusoidally as follows:

$$\underline{v}(\underline{r},t) = \underline{v} \; e^{i(\underline{k}\cdot\underline{r} - \omega t)} \quad , \tag{5.1}$$

where $\underline{v}$ is a constant vector.

We assume the wavelength to be small compared to the scale height. This assumption allows us to consider ρ_{eq} in eq. (4.9) as a constant in the lowest approximation. In a better approximation, such as the W.K.B. approximation in section 11, we will assume ρ_{eq} to be slowly varying.

After inserting eq. (5.1) into eq. (4.9), there results

$$\omega^2 \underline{v} - a^2(\underline{k}\cdot\underline{v})\underline{k} + i(\underline{v}\cdot\underline{g})\underline{k} + i(\gamma-1)\ (\underline{k}\cdot\underline{v})\underline{g}$$

$$+ \frac{1}{4\pi\rho_{eq}} \underline{H}_o \times \left\{ \underline{k} \times \left[\underline{k} \times (\underline{v} \times \underline{H}_o) \right] \right\} = 0 \quad . \ (5.2)$$

Vector equation (5.2) represents a set of three linear homogeneous equations for the unknowns v_x, v_y, and v_z. The condition for a non-trival solution, the vanishing of the determinant of the coefficients, is the dispersion relation.

6. THE DISPERISON RELATION

The dispersion relation can be obtained from the three components of the equation of motion (5.2) by putting the determinant of the coefficients equal to zero. However, the dispersion relation can be derived more easily in a different way by forming the dot products of eq. (5.2) with $\underline{k}$, $\underline{g}$, and $\underline{H}_o$.

For this purpose, it is convenient to bring eq. (5.2) into a different form by expanding the multiple vector product of the last term in eq. (5.2):

$$\underline{H}_{\circ} \times \left\{ \underline{k} \times \left[\underline{k} \times (\underline{v} \times \underline{H}_{\circ}) \right] \right\} = (\underline{k} \cdot \underline{H}_{\circ})\ (\underline{v} \cdot \underline{H}_{\circ})\ \underline{k}$$
$$-H_{\circ}^{2}\ (\underline{k} \cdot \underline{v})\underline{k} - (\underline{k} \cdot \underline{H}_{\circ})^{2}\underline{v} + (\underline{k} \cdot \underline{v})\ (\underline{k} \cdot \underline{H}_{\circ})\ \underline{H}_{\circ} . \tag{6.1}$$

By substituting eq. (6.1) into eq. (5.2), we have

$$\omega^{2}\underline{v} - a^{2}(\underline{k} \cdot \underline{v})\underline{k} + i(\underline{v} \cdot \underline{g})\underline{k} + i(\gamma-1)(\underline{k} \cdot \underline{v})\underline{g}$$
$$+ \frac{(\underline{k} \cdot \underline{H}_{\circ})}{4\pi\rho_{eq}}\ (\underline{v} \cdot \underline{H}_{\circ})\underline{k} - \frac{H_{\circ}^{2}}{4\pi\rho_{eq}}\ (\underline{k} \cdot \underline{v})\underline{k} - \frac{(\underline{k} \cdot \underline{H}_{\circ})^{2}}{4\pi\rho_{eq}}\ \underline{v}$$
$$+ \frac{(\underline{k} \cdot \underline{v})}{4\pi\rho_{eq}}\ (\underline{k} \cdot \underline{H}_{\circ})\underline{H}_{\circ} = 0 \tag{6.2}$$

First, we then take the dot product of eq. (6.2) with $\underline{k}$ and obtain

$$\omega^{2}(\underline{v} \cdot \underline{k}) - a^{2}(\underline{k} \cdot \underline{v})k^{2} + i(\underline{v} \cdot \underline{g})k^{2}$$
$$+ i(\gamma - 1)\ (\underline{k} \cdot \underline{v})\ (\underline{g} \cdot \underline{k})$$
$$+ \frac{k^{2}}{4\pi\rho_{eq}} \left[(\underline{k} \cdot \underline{H}_{\circ})\ (\underline{v} \cdot \underline{H}_{\circ}) - H_{\circ}^{2}\ (\underline{k} \cdot \underline{v}) \right] = 0 \tag{6.3}$$

Second, we take the dot product of eq. (6.2) with $\underline{g}$, and obtain

$$\omega^2(\underline{v}\cdot\underline{g}) - a^2(\underline{k}\cdot\underline{v})(\underline{k}\cdot\underline{g}) + i(\underline{v}\cdot\underline{g})(\underline{k}\cdot\underline{g}) + i(\gamma-1)(\underline{k}\cdot\underline{v})g^2$$

$$+\frac{(\underline{k}\cdot\underline{g})}{4\pi\rho_{eq}}(\underline{k}\cdot\underline{H}_o)(\underline{v}\cdot\underline{H}_o) - \frac{(\underline{k}\cdot\underline{g})}{4\pi\rho_{eq}}H_o^2(\underline{k}\cdot\underline{v}) - \frac{(\underline{v}\cdot\underline{g})}{4\pi\rho_{eq}}(\underline{k}\cdot\underline{H}_o)^2$$

$$+\frac{(\underline{g}\cdot\underline{H}_o)}{4\pi\rho_{eq}}(\underline{k}\cdot\underline{H}_o)(\underline{k}\cdot\underline{v}) = 0 \qquad (6.4)$$

And third, we take the dot product of eq. (6.2) with $\underline{H}_o$, and obtain

$$\omega^2(\underline{v}\cdot\underline{H}_o) - a^2(\underline{k}\cdot\underline{v})(\underline{k}\cdot\underline{H}_o) + i(\underline{v}\cdot\underline{g})(\underline{k}\cdot\underline{H}_o)$$

$$+ i(\gamma-1)(\underline{k}\cdot\underline{v})(\underline{g}\cdot\underline{H}_o) = 0 \qquad (6.5)$$

Rearranging eqs. (6.3), (6.4), and (6.5), and noting that $(\underline{g}\cdot\underline{v}) = -g(\underline{v}\cdot\underline{e}_z)$, we have respectively

$$(\underline{v}\cdot\underline{k})\left[\omega^2 - a^2k^2 + i(\gamma-1)(\underline{g}\cdot\underline{k}) - \frac{H_o^2k^2}{4\pi\rho_{eq}}\right]$$

$$+ (\underline{v}\cdot\underline{g})\left[k^2 i\right] + (\underline{v}\cdot\underline{H}_o)\left[\frac{(\underline{k}\cdot\underline{H}_o)}{4\pi\rho_{eq}}k^2\right] = 0 \qquad (6.6)$$

$$(\underline{v}\cdot\underline{k})\left[-a^2(\underline{k}\cdot\underline{g}) + i(\gamma-1)g^2 - \frac{(\underline{k}\cdot\underline{g})}{4\pi\rho_{eq}} H_o^2 + \frac{(\underline{g}\cdot\underline{H}_o)}{4\pi\rho_{eq}} (\underline{k}\cdot\underline{H}_o)\right]$$

$$+ (\underline{v}\cdot\underline{g})\left[\omega^2 + i(\underline{k}\cdot\underline{g}) - \frac{(\underline{k}.\underline{H}_o)}{4\pi\rho_{eq}}\right]$$

$$+ (\underline{v}\cdot\underline{H}_o)\left[\frac{(\underline{k}\cdot\underline{g})}{4\pi\rho_{eq}} (\underline{k}\cdot\underline{H}_o)\right] = 0 \qquad (6.7)$$

and

$$(\underline{v}\cdot\underline{k})\left[-a^2(\underline{k}.\underline{H}_o)\right] + (\underline{v}\cdot\underline{g})\left[i(\underline{k}\cdot\underline{H}_o)\right]$$

$$+ (\underline{v}\cdot\underline{H}_o)\left[\omega^2\right] = 0 \qquad . \qquad (6.8)$$

The unknowns of the above three equations are $(\underline{v}\cdot\underline{k})$, $(\underline{v}\cdot\underline{g})$, and $(\underline{v}\cdot\underline{H}_o)$. Setting the determinant of the coefficients of eqs. (6.6), (6.7), and (6.8) equal to zero, we have the dispersion relation:

$$\omega^6 + \omega^4\left[-a^2k^2 - ig\gamma k_z - \frac{H_o^2}{4\pi\rho_{eq}} k^2 - \frac{(\underline{H}_o\cdot\underline{k})^2}{4\pi\rho_{eq}}\right]$$

$$+ \omega^2 \left\{ g^2(\gamma-1)(k^2-k_z^2) + ig\gamma\left[\frac{(\underline{H}_o\cdot\underline{k})^2}{4\pi\rho_{eq}} k_z\right.\right.$$

(cont.)

$$+ \frac{H_z(\underline{H}_o \cdot \underline{k})k^2}{4\pi\rho_{eq}} \Bigg] + 2\,a^2k^2 \;\; \frac{(\underline{H}_o \cdot \underline{k})^2}{4\pi\rho_{eq}} \;\; + \frac{H_o^2(\underline{H}_o \cdot \underline{k})^2 k^2}{(4\pi\rho_{eq})^2} \Bigg\}$$

$$- ig\gamma H_z \; \frac{(\underline{H}_o \cdot \underline{k})^3}{(4\pi\rho_{eq})^2} \, k^2 \; - \; g^2(\gamma - 1) \; \frac{(\underline{H}_o \cdot \underline{k})^2}{4\pi\rho_{eq}} \;\; (k^2 - k_z^2)$$

$$- \frac{a^2k^2(\underline{H}_o \cdot \underline{k})^4}{(4\pi\rho_{eq})^2} \;\; = 0 \qquad (6.9)$$

7. LIMITING CASES

Sound waves: We note that in the absence of gravity and the magnetic field, eq. (6.9) reduces to

$$\omega^2 = a^2k^2 \qquad , \qquad (7.1)$$

the dispersion relation valid for sound waves.

Acoustic-Gravity waves: In the absence of the magnetic field, eq. (6.9) reduces to

$$\omega^4 + \omega^2(-ig\gamma k_z - a^2k^2) + g(\gamma-1)(k^2-k_z^2) = 0 \quad (7.2)$$

the dispersion relation valid for acoustic-gravity waves. If we put $k_y = 0$ in eq. (7.2), we would have the dispersion relation valid for acoustic-gravity waves in a two-dimensional space, which was used by Hines[16].

Gravity waves: For the case of gravity waves in an incompressible medium, we divide eq. (7.2) by a^2, which gives

$$\frac{\omega^4}{a^2} + \omega^2\left(- \frac{ig\gamma k_z}{a^2} - k^2\right) + \frac{g^2}{a^2}(\gamma-1)(k^2-k_z^2) = 0 \quad (7.3)$$

From eq. (4.8) we obtain

$$\frac{g(\gamma-1)}{a^2} = -\frac{1}{\rho_o} \left.\frac{\partial\rho_o}{\partial s_o}\right)_p \frac{ds_{eq}}{dz} \quad (7.4)$$

We define θ to be the angle between the wave vector $\underline{k}$ and the vertical $\underline{e}_z$. From this definition it follows that

$$k^2 - k_z^2 = k^2 \sin^2\theta \quad (7.5)$$

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[16] C.O. Hines, Canadian J. Phys. 38, 1441 (1960).

By substituting eqs. (7.4) and(7.5) into eq. (7.3), we have

$$\frac{\omega^4}{a^2} + \omega^2 \left(- \frac{i g \gamma k_z}{a^2} - k^2 \right)$$

$$- \frac{g}{\rho_o} \left. \frac{\partial \rho_o}{\partial s_o} \right)_p \frac{ds_{eq}}{dz} k^2 \sin^2\theta = 0 \quad . \tag{7.6}$$

For an incompressible medium, we let the velocity of sound, a, become infinite. Eq. (7.6) becomes

$$\omega^2 = - \frac{g}{\rho_o} \left. \frac{\partial \rho_o}{\partial s_o} \right)_p \frac{ds_{eq}}{dz} \sin^2\theta \quad , \tag{7.7}$$

which is the dispersion relation for gravity waves in an incompressible medium[17].

<u>Magnetosonic waves</u>: Letting g = 0 in eq. (6.9), we obtain:

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[17] See for example, L.D. Landau and E.M. Lifshitz, FLUID MECHANICS, Pergamon Press, London, (1959).

$$\omega^6 + \omega^4 \left[-a^2k^2 - \frac{H_o^2}{4\pi\rho_{eq}} k^2 - \frac{(\underline{H}_o \cdot \underline{k})^2}{4\pi\rho_{eq}} \right]$$

$$+ \omega^2 \left[2a^2k^2 \frac{(\underline{H}_o \cdot \underline{k})^2}{4\pi\rho_{eq}} + \frac{H_o^2 (\underline{H}_o \cdot \underline{k})^2}{(4\pi\rho_{eq})^2} k^2 \right]$$

$$-a^2 \frac{(\underline{H}_o \cdot k)^4}{(4\pi\rho_{eq})^2} k^2 = 0 \quad . \tag{7.8}$$

Eq. (7.8) is separable into two parts, indicating two independent modes of wave propagation. We have

$$\omega^2 = \frac{(\underline{H}_o \cdot \underline{k})^2}{4\pi\rho_{eq}} \tag{7.9}$$

and

$$\omega^4 + \omega^2 \left[-a^2k^2 - \frac{H_o^2}{4\pi\rho_{eq}} k^2 \right] + a^2 \frac{(\underline{H}_o \cdot \underline{k})^2}{4\pi\rho_{eq}} k^2 = 0 \quad . \tag{7.10}$$

Eq. (7.9) expresses the dispersion relation for the Alfven mode. This mode is purely transverse and is independent of the compressibility of the medium.

Eq. (7.10) is the dispersion relation for longitudinal magnetosonic waves[18].

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18 See for example, S. Gartenhaus, ELEMENTS OF PLASMA PHYSICS, Holt, New York, (1964).

Alfven waves: Last of all, we consider the special case of a magnetic field acting on an incompressible medium. We see that by dividing eq. (7.10) by a^2k^2 and letting the velocity of sound become infinite, gives again eq. (7.9). Thus, for this case, only the Alfven mode exists.

8. WAVE MODES AND THE DIMENSIONLESS DISPERSION RELATION

In order to describe the three vectors $\underline{k}$, $\underline{g}$, and $\underline{H}_o$ with respect to each other, we must introduce three angles. (See Figure 1.)

Thus, first we define the angle η, between the z-direction and the magnetic field vector, hence

$$(\underline{e}_z \cdot \underline{H}_o) = H_o \cos\eta \quad . \tag{8.1}$$

Second, we define the angle ϕ, between the wave propagation vector and the magnetic field vector, hence

$$(\underline{k} \cdot \underline{H}_o) = H_o k \cos\phi \quad . \tag{8.2}$$

And third, as in Section 7, we define the angel θ, between the z- direction and the wave propagation vector, hence

$$(\underline{e}_z \cdot \underline{k}) = k \cos\theta \quad . \tag{8.3}$$

The explicit angular dependence of the dispersion relation is presented by substituting these angles into eq. (6.9) with the result:

$$\omega^6 + \omega^4\left[- a^2k^2 - ig\gamma k \cos\theta - \frac{H_\circ^2 k^2}{4\pi\rho_{eq}} (1 - \cos^2\phi)\right]$$

$$+ \omega^2 \left[g^2(\gamma-1)k^2 \sin^2\theta + ig\gamma \frac{H_\circ^2 k^3}{4\pi\rho_{eq}} (\cos\theta \cos\phi + \cos\eta) \cos\phi\right.$$

$$\left. + 2a^2 \frac{H_\circ^2 k^4}{4\pi\rho_{eq}} \cos^2\phi + \frac{H_\circ^4 k^4}{4\pi\rho_{eq}} \cos^2\phi\right]$$

$$-ig\gamma \frac{H_\circ^4 k^5}{(4\pi\rho_{eq})^2} \cos^3\phi \cos\eta - g^2(\gamma-1) \frac{H_\circ^2 k^4}{4\pi\rho_{eq}} \cos^2\phi \sin^2\theta$$

$$-a^2 \frac{H_\circ^4 k^6}{(4\pi\rho_{eq})^2} \cos^4\phi = 0 \quad . \tag{8.4}$$

Eq. (6.9) may be separated into two parts, indicating two independent modes of wave propagation. We have

$$\omega^2 = \frac{(\underline{H}_o \cdot \underline{k})^2}{4\pi\rho_{eq}} \quad . \tag{8.5}$$

and

$$\omega^4 + \omega^2 \left(-ig\gamma k_z - a^2 k^2 - \frac{H_o^2}{4\pi\rho_{eq}} k^2 \right)$$

$$+ g^2(\gamma-1)\ (k^2-k_z^2) + a^2 \frac{(\underline{H}_o \cdot \underline{k})^2}{4\pi\rho_{eq}} k^2$$

$$+ ig\gamma \frac{H_z(\underline{H}_o \cdot \underline{k})}{4\pi\rho_{eq}} k^2 = 0 \quad . \tag{8.6}$$

As in the magnetosonic case, we see the appearance of the Alfven mode, eq. (8.5). The second mode, eq. (8.6), is however, due to the coupling of gravity, magnetic field, and compressibility.

In order to obtain the angular dependence of eq. (8.6), we substitute into it the angles (8.1), (8.2), and (8.3), which gives

$$\omega^4 + \omega^2 \left(-ig\gamma k \cos\theta - a^2k^2 - \frac{H_\circ^2 k^2}{4\pi\rho_{eq}} \right)$$

$$+ g^2(\gamma-1)\ k^2\sin^2\theta + a^2 \frac{H_\circ^2}{4\pi\rho_{eq}} k^4 \cos^2\phi$$

$$+ ig\gamma \frac{H_\circ^2}{4\pi\rho_{eq}} k^3 \cos\eta \cos\phi = 0 \quad . \tag{8.7}$$

Solving eq. (8.7) for ω^2, we have

$$\omega_\pm^2 = 1/2\ a^2k^2 + \frac{H_\circ^2}{8\pi\rho_{eq}} k^2 + 1/2\ ig\gamma k \cos\theta$$

$$\pm 1/2 \Bigg[a^4k^4 - g^2\gamma^2k^2 \cos^2\theta + \frac{H_\circ^4 k^4}{(4\pi\rho_{eq})^2} + 2a^2 \frac{H_\circ^2 k^4}{4\pi\rho_{eq}}$$

$$+ 2\ ia^2g\gamma k^3 \cos\theta + 2\ ig\gamma \frac{H_\circ^2 k^3}{4\pi\rho_{eq}} \cos\theta$$

$$- 4g^2(\gamma-1)\ k^2 \sin^2\theta - 4\ a^2 \frac{H_\circ^2 k^4}{4\pi\rho_{eq}} \cos^2\phi$$

$$-4\ ig\gamma \frac{H_\circ^2 k^3}{4\pi\rho_{eq}} \cos\phi \cos\eta \Bigg]^{1/2} \quad . \tag{8.8}$$

In order to put this dispersion relation into dimensionless form, we must first define a characteristic wave number k_c and a characteristic frequency ω_c. It is convenient to define

$$k_c = \frac{H_o g}{a^3 (4\pi\rho_{eq})^{1/2}} \quad \text{and} \quad \omega_c = \frac{H_o g}{a^2 (4\pi\rho_{eq})^{1/2}} . \tag{8.9}$$

Let us consider the quantity $\frac{H_o^2}{4\pi\rho_{eq} a^2}$. From eq. (4.2) we have

$$\frac{H_o^2}{4\pi\rho_{eq} a^2} = \frac{2}{\gamma\beta} , \tag{8.10}$$

where

$$\beta = \frac{p_{eq}}{H_o^2/8\pi} . \tag{8.11}$$

By substituting eq. (8.9) and (8.10) into eq. (8.8), we have

$$
\begin{aligned}
W_{\pm}^{2} = {} & \left(1/2\, K^{2} + \frac{K^{2}}{\gamma\beta}\right) + i \left[\left(\frac{\gamma}{2}\right)^{3/2} \beta^{1/2} K \cos\theta \right] \\
& \pm \left\{ \left[1/4\, K^{4} - 1/8\, \beta \gamma^{3} K^{2} \cos^{2}\theta + \frac{K^{4}}{\gamma^{2}\beta^{2}} + \frac{K^{4}}{\gamma\beta} \right.\right. \\
& \left. - 1/2\, \gamma\beta\, (\gamma - 1) K^{2} \sin^{2}\theta - \frac{2K^{4}}{\gamma\beta} \cos^{2}\phi \right] \\
& + i \left[\left(\frac{\gamma}{2}\right)^{3/2} \beta^{1/2} \cos\theta - \left(\frac{2\gamma}{\beta}\right)^{1/2} \cos\phi \cos\eta \right. \\
& \left.\left. + \left(\frac{\gamma}{2\beta}\right)^{1/2} \cos\theta \right] K^{3} \right\}^{1/2} . \qquad (8.12)
\end{aligned}
$$

where $W = \frac{\omega}{\omega_c}$ and $K = \frac{k}{k_c}$. (8.13)

there is, however, a restriction on the three angles in eq. (8.12) from spherical trigonmetry given by

$$0° \leq (\theta + \eta + \phi) \leq 360° .$$

As can be seen from eq. (8.12), the dispersion relation is complex in general, implying stable and unstable wave motion. For unstable wave motion, the growth rates are given by the imaginary part of the frequency.

9. LIMITS OF STABILITY

As can be seen from the dispersion relation (8.12), there exist two independent modes, which we call the +mode and the -mode. Each mode splits up into two submodes due to the fact that the frequency W in eq. (8.12) is squared. If the dispersion relation is complex, one submode increases and the other decreases exponentially with time. If the wave motion is to be stable, there must be no wave growth. This implies that the imaginary part of the frequency must be zero. The wave number, K_{s1}, below which the atmosphere ceases to be stable shall be called the wave number stability limit.

To derive the relation for this condition, we rewrite eq. (8.12) into the following form

$$W_{\pm}^2 = A + iB \pm (C + iD)^{1/2} \quad , \tag{9.1}$$

where

$$A = 1/2K^2 + \frac{K^2}{\gamma\beta} \quad ,$$

$$B = \left(\frac{\gamma}{2}\right)^{3/2} \beta^{1/2} K \cos\theta \quad ,$$

$$C = 1/4\, K^4 - 1/8\, \beta\gamma^3 K^2 \cos^2\theta + \frac{K^4}{\gamma^2\beta^2} + \frac{K^4}{\gamma\beta}$$

$$- 1/2\, \gamma\beta(\gamma - 1) K^2 \sin^2\theta - \frac{2K^4}{\gamma\beta} \cos^2\phi \quad ,$$

and

$$D = \left[\left(\frac{\gamma}{2}\right)^{3/2} \beta^{1/2} \cos\theta - \left(\frac{2\gamma}{\beta}\right)^{1/2} \cos\phi \cos\eta \right.$$

$$\left. + \left(\frac{\gamma}{2\beta}\right)^{1/2} \cos\theta \right] K^3 \quad . \tag{9.2}$$

Separating $W_\pm^2$ into a real and an imaginary part, we have, using Euler's identity

$$W_\pm^2 = E + iF = (E^2 + F^2)^{1/2} \quad e^{i\alpha} \quad , \tag{9.3}$$

where

$$\alpha = \arctan F/E \quad . \tag{9.4}$$

Taking the square root of both sides of eq. (9.3) and expanding, we have

$$W_\pm = (E^2 + F^2)^{1/4} \quad e^{i\ \alpha/2} \quad . \tag{9.5}$$

The condition for stability, which implies that the imaginary part of the frequency is zero, is given by

$$e^{i\frac{\alpha}{2}} = e^{i2\pi n} \quad , \quad n = 0, 1, 2, \ldots \tag{9.6}$$

or

$$\alpha = 4\pi n \quad . \tag{9.7}$$

Therefore, due to eq. (9.4)

$$F/E = \tan(4\pi n) = 0 \ . \tag{9.8}$$

Since E is finite, this imples that F = 0 for stable wave motion.

To derive the condition for F = 0, we expand eq. (9.1) and obtain

$$W_{\pm}^2 = \left[A \pm (C^2 + D^2)^{1/4} \quad \cos \frac{\sigma}{2} \right]$$

$$+ \, i \left[B \pm (C^2 + D^2)^{1/2} \quad \sin \frac{\sigma}{2} \right]$$

$$= E + iF \qquad , \qquad (9.9)$$

where

$$\sigma = \arctan (D/C), \qquad (9.10)$$

for which the principal root ($-\pi/2 \leq \sigma \leq \pi/2$) has to be taken. Thus, the condition for F = 0 becomes

$$B = \mp (C^2 + D^2)^{1/4} \quad \sin \frac{\sigma}{2}$$

$$= \mp \left[1/2 \ (C^2 + D^2)^{1/2} - \frac{C}{2}\right]^{1/2} \qquad . \qquad (9.11)$$

By substituting the expressions B,C, and D, (Eqs. (9.2)) into eq. (9.11), we have the condition for the imaginary part of the frequency to be zero; that is, the condition for stability

$$\left(\frac{\gamma}{2}\right)^{3/2} \beta^{1/2} K \cos\theta = \mp \frac{1}{2^{1/2}} \Bigg\{ \Bigg[\Bigg(\frac{1}{4} K^4 - 1/8\,\beta\gamma^3 K^2 \cos^2\theta$$

$$+ \frac{K^4}{\gamma^2\beta^2} + \frac{K^4}{\gamma\beta} - \frac{1}{2}\gamma\beta(\gamma-1) K^2 \sin^2\theta - \frac{2K^4}{\gamma\beta} \cos^2\phi \Bigg)^2$$

$$+ \Bigg(\Bigg[\left(\frac{\gamma}{2}\right)^{3/2} \beta^{1/2} \cos\theta - \left(\frac{2\gamma}{\beta}\right)^{1/2} \cos\phi \cos\eta$$

$$+ \left(\frac{\gamma}{2\beta}\right)^{1/2} \cos\theta \Bigg] K^3 \Bigg)^2 \Bigg]^{1/2} - 1/4\, K^4 - 1/8\,\beta\gamma^3 K^2 \cos^2\theta$$

$$+ \frac{K^4}{\gamma^2\beta^2} + \frac{K^4}{\gamma\beta} - 1/2\,\gamma\beta(\gamma-1) K^2 \sin^2\theta$$

$$- \frac{2K^4}{\gamma\beta} \cos^2\phi \Bigg\}. \qquad (9.12)$$

In the following cases, we will analyze the conditions for stable wave modes using eqs. (9.11) and (9.12).

Case I: In this case the wave vector and the magnetic field are in the vertical direction; $\theta = 0°$, $\eta = 0°$, and $\phi = 0°$. First, we consider the -mode, which means that we choose the plus sign in eq. (9.11). Since $B > 0$, we can write

$$B = \left[1/2 \ (C^2 + D^2)^{1/2} - \frac{C}{2} \right]^{1/2} \quad (9.13)$$

Upon squaring each side and rearranging, we have

$$4 \ B^2 \ (B^2 + C) = D^2. \quad (9.14)$$

Substituting the expression B,C, and D (Eq. (9.2)) into eq. (9.14), and putting $\theta = 0°$, $\eta = 0°$, and $\phi = 0°$, we obtain an identity This shows that $\mathrm{Im}(W_-) = 0$ for all values of K. Hence, stable wave propagation occurs for the -mode for all values of K.

However, in the +mode case, the minus sign is chosen in eq. (9.11), and since B is positive, it implies that $\mathrm{Im}(W_+) \neq 0$ for all values of K (except the trivial case of $K = 0$). Hence, wave propagation is unstable for the +mode for all values of K.

Case II: The wave vector and the magnetic field are horizontal. $\theta = 90°$, $\eta = 90°$, and $\phi = 0°$. In this case we see that $B = 0$ and $D = 0$, so our conclusion regarding stability will be valid for both the +mode and the -mode. Eq. (9.11) upon squaring and rearranging, becomes

$$(C^2)^{1/2} = C \quad .$$

This means that for values of K for which $C > 0$, then $Im(W_\pm) = 0$, and the modes are stable, but for values of K for which $C < 0$, then $Im(W_\pm) \neq 0$, and the modes are unstable. The wave number stability limit is given by

$$K_{s1} = \left[\frac{1/2 \ \gamma\beta \ (\gamma-1)}{1/4 + \frac{1}{\gamma^2\beta^2} - \frac{1}{\gamma\beta}} \right]^{1/2} \quad . \tag{9.15}$$

Thus, the waves of both the +mode and the -mode are unstable for $K < K_{s1}$ and stable for $K > K_{s1}$.

For this case, the dispersion relation (8.12) is plotted for the +mode and the -mode for $K > K_{s1}$ in Fig. 2 and Fig. 3, respectively. $Im(W_+)$ and $Im(W_-)$ are plotted in Fig. 4.

<u>Case III</u>: This is the case of a vertical wave vector and a horizontal magnetic field; $\theta = 0°, \eta = 90°$, and $\phi = 90°$. As in Case I, we first consider the -mode, which means that we choose the plus sign in eq. (9.11). Since $B > 0$, we can again consider eq. (9.14). Substituting the expressions B, C, and D (eq. (9.2) into eq. (9.14) and putting $\theta = 0°$, $\eta = 90°$, and $\phi = 90°$, we obtain an identity. This shows that $Im(W_-) = 0$ for all values of K. Hence, the mode should be stable for all values of K. However,

in this case, the -mode does not actually exist, since $Re(W_-) = 0$ also.

For the +mode situation, the minus sign is chosen in eq. (9.11), and since B is positive, this implies that $Im(W_+) \neq 0$ for all values of K (except the trivial case of $K = 0$). Hence, wave propagation is unstable for the +mode for all values of K.

Case IV: In this case the wave vector is horizontal and the magnetic field is vertical; $\theta = 90°, \eta = 0°$, and $\phi = 90°$. We see that $B = 0$ and $D = 0$ as in Case II, so our results, as before, will be valid for both the +mode and the -mode. Eq. (9.11) becomes, upon squaring and re-arranging

$$(C^2)^{1/2} = C \quad .$$

This means that for values of K for which $C > 0$, then $Im(W_\pm) = 0$, and the modes are stable, but for values of K for which $C < 0$, then $Im(W_\pm) \neq 0$, and the modes are unstable. This wave number stability limit is given by

$$K_{s1} = \left[\frac{1/2 \ \gamma\beta \ (\gamma - 1)}{1/4 + \frac{1}{\gamma^2\beta^2} + \frac{1}{\gamma\beta}}\right]^{1/2} \tag{9.16}$$

Thus, the wave of both the +mode and the -mode is unstable for $K < K_{s1}$ and stable for $K > K_{s1}$.

For this case, the dispersion relation (8.12) is plotted for the +mode and the -mode for $K > K_{s1}$ in Fig. 5 and Fig. 6, respectively. $Im(W_+)$ and $Im(W_-)$ are plotted in Fig. 7.

Case V: The wave vector, the magnetic field, and the vertical are aligned at 45° with respect to each other; $\theta = 45°, \eta = 45°$, and $\phi = 45°$. First, we consider the -mode, which means that we choose the plus sign in eq. (9.11). Since $B > 0$, we obtain again eq. (9.14). Substituting the expressions B,C, and D (eq. (9.2)) into eq. (9.14) and putting $\theta = 45°, \eta = 45°$, and $\phi = 45°$, we obtain only one stable wave number K_s, given by

$$K_s = \left[\frac{\gamma^3 \beta^3 (\gamma - 1)}{4(\sqrt{2}-1)(\gamma\beta - 2)} \right] , \qquad (9.17)$$

for which $Im(W_-) = 0$.

In the +mode situation, the minus sign is chosen in eq. (9.11), and since B is positive, it implies that $Im(W_+) \neq 0$ for all values of K (except the trivial case of $K = 0$). Hence, wave propagation is unstable for the +mode for all values of K. $Im(W_-)$ is plotted in Fig. 8, which shows the interesting case of a stable motion at one single wave number.

Table 1 lists the ranges of stability for the different wave modes for several combinations of the angles θ, η, ϕ.

10. GROUP AND PHASE VELOCITIES

Group velocities can be calculated directly from the dispersion relation (8.12)

$$V_{g\pm} = \frac{\partial W_\pm}{\partial K} = \frac{1}{2W_\pm}\Bigg\{ K + \frac{2}{\gamma\beta} K + i\left(\frac{\gamma}{2}\right)^{3/2} \beta^{1/2} \cos\theta$$

$$\pm \frac{1}{2\Delta}\Bigg[K^3 - 1/4\beta\gamma^3 K \cos^2\theta + \frac{4K^3}{\gamma^2\beta^2} + \frac{4K^3}{\gamma\beta}$$

$$- \gamma\beta(\gamma - 1) K \sin^2\theta - \frac{8K^3}{\gamma\beta}\cos^2\phi - 3i\left(\frac{2\gamma}{\beta}\right)^{1/2} K^2 \cos\phi \cos\eta$$

$$+ 3i\left(\frac{\gamma}{2}\right)^{3/2} \beta^{1/2} K^2 \cos\theta + 3i\left(\frac{\gamma}{2\beta}\right)^{1/2} K^2 \cos\theta\Bigg]\Bigg\} . \tag{10.1}$$

where

$$\Delta = \Bigg\{\Bigg[1/4\, K^4 - 1/8\, \beta\, \gamma^3 K^2 \cos^2\theta + \frac{K^4}{\gamma^2\beta^2} + \frac{K^4}{\gamma\beta}$$

$$- 1/2\, \gamma\beta(\gamma - 1) K^2 \sin^2\theta - \frac{2K^4}{\gamma\beta} \cos^2\phi\Bigg]$$

$$+ i\Bigg[\left(\frac{\gamma}{2}\right)^{3/2} \beta^{1/2} \cos\theta - \left(\frac{2\gamma}{\beta}\right)^{1/2} \cos\phi \cos\eta$$

$$+ \left(\frac{\gamma}{2\beta}\right)^{1/2} \cos\theta\Bigg] K^3\Bigg\}^{1/2} .$$

Also, phase velocities can be calculated directly from eq. (8.12) by

$$V_{p\pm} = \frac{W_\pm}{K} \quad . \qquad (10.2)$$

V_g and V_p are dimensionless quantities in units of the sound velocity a.

11. W.K.B. APPROXIMATION

We assumed above that ρ_{eq} could be considered a constant in the lowest approximation, if the wavelength was small compared with the scale height. A better approximation is obtained by taking into account the slowly varying change of the equilibrium density in the vertical direction by a generalized W.K.B. method, which shall be presented here.

It is known, for instance from the application of the W.K.B. approximation to the solution of Schrödinger's equation, that the W.K.B. method gives reasonable good results even if the condition of a small wavelength is no longer satisfied. Therefore, we can also expect that our results, which are derived by using the W.K.B. method, are applicable with reasonable accuracy, even if the wavelength cannot be considered small against the scale height.

In the plane wave analysis of Section 5, the z-dependence was automatically taken into account by eq. (5.1).

In the W.K.B. method, however, we assume for the velocity components a dependence of the form

$$f(z)\, e^{i(\underline{k}_\perp \cdot \underline{r} - \omega t)} \quad , \qquad (11.1)$$

where $\underline{k}_{\perp} = k_x \underline{e}_x + k_x \underline{e}_y$. $f(z)$ describes the unknown z-dependence to be treated by the W.K.B. method. In eq. (5.1) it is seen that we had chosen

$$f(z) = e^{ik_z z} \quad . \tag{11.2}$$

Now, however, we have, in keeping with the W.K.B. analysis

$$f(z) = A(z)\, e^{\Phi(z)} \quad , \tag{11.3}$$

where $-i\Phi(z)$ is the phase,and $A(z)$ is a slowly varying function of z. We can consider d^2A/dz^2 small against A and, therefore, neglect it.

We will begin the W.K.B. analysis from the equation of motion (4.9)

$$\frac{\partial^2 \underline{v}}{\partial t^2} - a^2 \text{ grad div } \underline{v} - \text{ grad } (\underline{v}\cdot\underline{g}) - \underline{g}(\gamma-1) \text{ div } \underline{v}$$

$$+ \frac{1}{4\pi\rho_{eq}} \underline{H}_o \times \text{curl curl } (\underline{v} \times \underline{H}_o) = 0 \quad .$$

For the velocity dependence, we introduce the expression from eq. (11.1)

$$\underline{v}(\underline{r},t) = \underline{v}_c\, f(z)\, e^{i(\underline{k}_{\perp}\cdot\underline{r} - \omega t)} \tag{11.4}$$

where $\underline{v}_c$ is a constant vector in the $\underline{v}$ direction.

Using the expression (11.4), we perform the various mathematical operations indicated in eq. (4.9) as follows:

$$\frac{\partial^2 \underline{v}}{\partial t^2} = -\omega^2 \underline{v}_c \, f \, e^{i(\underline{k}_\perp \cdot \underline{r} - \omega t)} , \tag{11.5}$$

$$\text{grad div } \underline{v} = \underline{k}_\perp \left[i(\underline{v}_c \cdot \underline{e}_z) \frac{df}{dz} - (\underline{k}_\perp \cdot \underline{v}_c) \, f \right]$$

$$+ \underline{e}_z \left[i(\underline{k}_\perp \cdot \underline{v}_c) \frac{df}{dz} + (\underline{v}_c \cdot \underline{e}_z) \frac{d^2 f}{dz^2} \right] e^{i(\underline{k}_\perp \cdot \underline{r} - \omega t)} , \tag{11.6}$$

$$\text{grad } (\underline{v} \cdot \underline{e}_z) = \left[\underline{k}_\perp i (\underline{v}_c \cdot \underline{e}_z) f \right.$$

$$\left. + \underline{e}_z (\underline{v}_c \cdot \underline{e}_z) \frac{df}{dz} \right] e^{i(\underline{k}_\perp \cdot \underline{r} - \omega t)} , \tag{11.7}$$

$$\text{div } \underline{v} = \left[i(\underline{k}_\perp \cdot \underline{v}_c) \, f \right.$$

$$\left. + (\underline{v}_c \cdot \underline{e}_z) \frac{df}{dz} \right] e^{i(\underline{k}_\perp \cdot \underline{r} - \omega t)} , \tag{11.8}$$

and

$$\underline{H}_\circ \times \text{curl curl } (\underline{v} \times \underline{H}_\circ) = \Big\{ \underline{k}_\perp \Big[f H_\circ^2 \, (\underline{k}_\perp \cdot \underline{v}_c)$$

$$- f(\underline{H}_\circ \cdot \underline{v}_c)\,(\underline{k}_\perp \cdot \underline{H}_\circ) + i \frac{df}{dz} (\underline{H}_\circ \cdot \underline{v}_c)\,(\underline{H}_\circ \cdot \underline{e}_z)$$

$$- i \frac{df}{dz} H_\circ^2 \,(\underline{v}_c \cdot \underline{e}_z) \Big] + \underline{v}_c \Big[f\,(\underline{H}_\circ \cdot \underline{k}_\perp)^2$$

$$- 2i \frac{df}{dz} (\underline{H}_\circ \cdot \underline{e}_z)\,(\underline{k}_\perp \cdot H_\circ) - \frac{d^2f}{dz^2} (\underline{H}_\circ \cdot \underline{e}_z)^2 \Big]$$

$$+ \underline{H}_\circ \Big[- f(\underline{H}_\circ \cdot \underline{k}_\perp)\,(\underline{k}_\perp \cdot \underline{v}_c) + i \frac{df}{dz} (\underline{H}_\circ \cdot \underline{e}_z)\,(\underline{k}_\perp \cdot \underline{v}_c)$$

$$+ i \frac{df}{dz} (\underline{H}_\circ \cdot \underline{k}_\perp)\,(\underline{v}_c \cdot \underline{e}_z) + \frac{d^2f}{dz^2} (\underline{H}_\circ \cdot \underline{e}_z)\,(\underline{v}_c \cdot \underline{e}_z) \Big]$$

$$+ \underline{e}_z \Big[i \frac{df}{dz} (\underline{H}_\circ \cdot \underline{v}_c)\,(\underline{k}_\perp \cdot \underline{H}_\circ) - i \frac{df}{dz} H_\circ^2 \,(\underline{k}_\perp \cdot \underline{v}_c)$$

$$+ \frac{d^2f}{dz^2} (\underline{H}_\circ \cdot \underline{v}_c)\,(\underline{H}_\circ \cdot \underline{e}_z) - \frac{d^2f}{dz^2} H_\circ^2 \,(\underline{v}_c \cdot \underline{e}_z) \Big] \Big\} e^{i(\underline{k}_\perp \cdot \underline{r} - \omega t)} . \tag{11.9}$$

By substituting eqs. (11.5) - (11.9) into eq. (4.9), we have, upon collecting terms and canceling the exponential term

$$
\begin{aligned}
&\underline{v}_c \Bigg[\omega^2 f - \frac{(\underline{H}_\circ \cdot \underline{k}_\perp)^2}{4\pi\rho_{eq}} f + 2i \frac{(\underline{H}_\circ \cdot \underline{e}_z)\ (\underline{H}_\circ \cdot \underline{k}_\perp)}{4\pi\rho_{eq}} \\
&+ \frac{(\underline{H}_\circ \cdot \underline{e}_z)^2}{4\pi\rho_{eq}} \frac{d^2 f}{dz^2} \Bigg] + \underline{k}_\perp \Bigg[ia^2 (\underline{v}_c \cdot \underline{e}_z) \frac{df}{dz} - a^2 (\underline{k}_\perp \cdot \underline{v}_c) f \\
&- ig\ (\underline{v}_c \cdot \underline{e}_z) f + \frac{(\underline{H}_\circ \cdot \underline{v}_c)(\underline{H}_\circ \cdot \underline{k}_\perp)}{4\pi\rho_{eq}} f - \frac{H_\circ^2}{4\pi\rho_{eq}} (\underline{k}_\perp \cdot \underline{v}_c) f \\
&- i \frac{(\underline{H}_\circ \cdot \underline{v}_c)(\underline{H}_\circ \cdot \underline{e}_z)}{4\pi\rho_{eq}} \frac{df}{dz} + i \frac{H_\circ^2}{4\pi\rho_{eq}} (\underline{v}_c \cdot \underline{e}_z) \frac{df}{dz} \Bigg] \\
&+ \underline{H}_\circ \Bigg[\frac{(\underline{H}_\circ \cdot \underline{k}_\perp)(\underline{k}_\perp \cdot \underline{v}_c)}{4\pi\rho_{eq}} f - i \frac{(\underline{H}_\circ \cdot \underline{e}_z)(\underline{k}_\perp \cdot \underline{v}_c)}{4\pi\rho_{eq}} \frac{df}{dz} \\
&- i \frac{(\underline{H}_\circ \cdot \underline{k}_\perp)(\underline{v}_c \cdot \underline{e}_z)}{4\pi\rho_{eq}} \frac{df}{dz} - \frac{(\underline{H}_\circ \cdot \underline{e}_z)(\underline{v}_c \cdot \underline{e}_z)}{4\pi\rho_{eq}} \frac{d^2 f}{dz^2} \Bigg] \\
&+ \underline{e}_z \Bigg[ia^2 (\underline{k}_\perp \cdot \underline{v}_c) \frac{df}{dz} + a^2 (\underline{v}_c \cdot \underline{e}_z) \frac{d^2 f}{dz^2} - i(\gamma - 1) g (\underline{k}_\perp \cdot \underline{v}_c) f \\
&- g\gamma\ (\underline{v}_c \cdot \underline{e}_z) \frac{df}{dz} - i \frac{(\underline{H}_\circ \cdot \underline{v}_c)(\underline{k}_\perp \cdot \underline{H}_\circ)}{4\pi\rho_{eq}} \frac{df}{dz} + i \frac{H_\circ^2}{4\pi\rho_{eq}} (\underline{k}_\perp \cdot \underline{v}_c) \frac{df}{dz} \\
&- \frac{(\underline{H}_\circ \cdot \underline{v}_c)(\underline{H}_\circ \cdot \underline{e}_z)}{4\pi\rho_{eq}} \frac{d^2 f}{dz^2} + \frac{H_\circ^2}{4\pi\rho_{eq}} (\underline{v}_c \cdot \underline{e}_z) \frac{d^2 f}{dz^2} \Bigg] = 0.
\end{aligned}
\tag{11.10}
$$

From eq. (11.3), we have

$$\frac{df}{dz} = \left(\frac{dA}{dz} + A\,\frac{d\Phi}{dz}\right) e^{\Phi} \quad , \qquad \text{and} \tag{11.11}$$

$$\frac{d^2 f}{dz^2} = \left[\frac{d^2 A}{dz^2} + 2\,\frac{dA}{dz}\,\frac{d\Phi}{dz} + A\left(\frac{d\Phi}{dz}\right)^2 + A\,\frac{d^2\Phi}{dz^2}\right] e^{\Phi} . \tag{11.12}$$

By inserting eqs. (11.11) and (11.12) into eq. (11.10), we obtain terms multiplied by A, d^2A/dz^2, $(dA/dz)(d\Phi/dz)$ $A(d\Phi/dz)^2$, $A\,d^2\Phi/dz^2$, dA/dz, and $A\,d\Phi/dz$. In the spirit of the W.K.B. method, we consider A(z) to be a function slowly varying over a wave length 1/k, and for this reason, we neglect terms multiplied by d^2A/dz^2 and dA/dz. However, since $d\Phi/dz \propto k$, terms multiplied by $(dA/dz)(d\Phi/dz)$ cannot be neglected in the W.K.B. approximation, which becomes exact in the mathematical limit $k \to \infty$.

We then separate eq. (11.10) into two equations; one equation containing terms multiplied by $(dA/dz)(d\Phi/dz)$ and $A\,d^2\Phi/dz^2$, and the other equation containing terms multiplied by A, $A(d\Phi/dz)$, and $A(d\Phi/dz)^2$.

Hence, we have, respectively

$$\underline{v}_c\left[2\,\frac{dA}{dz}\,\frac{d\Phi}{dz}\,\frac{(\underline{H}_\circ\cdot\underline{e}_z)^2}{4\pi\rho_{eq}} + A\,\frac{d^2\Phi}{dz^2}\,\frac{(\underline{H}_\circ\cdot\underline{e}_z)^2}{4\pi\rho_{eq}}\right]$$

$$+\ \underline{H}_\circ\left\{(\underline{v}_c\cdot\underline{e}_z)\left[-2\frac{dA}{dz}\,\frac{d\Phi}{dz}\,\frac{(\underline{H}_\circ\cdot\underline{e}_z)}{4\pi\rho_{eq}} - A\,\frac{d^2\Phi}{dz^2}\,\frac{(\underline{H}_\circ\cdot\underline{e}_z)}{4\pi\rho_{eq}}\right]\right\}$$

$$+\ \underline{e}_z\left\{\frac{(\underline{H}_\circ\cdot\underline{v}_c)}{4\pi\rho_{eq}}\left[-A\,\frac{d^2\Phi}{dz^2}\,(\underline{H}_\circ\cdot\underline{e}_z) - 2\,\frac{dA}{dz}\,\frac{d\Phi}{dz}\,(\underline{H}_\circ\cdot\underline{e}_z)\right]\right.$$

$$+\ (\underline{v}_c\cdot\underline{e}_z)\left[2\,\frac{dA}{dz}\,\frac{d\Phi}{dz}\,a^2 + \frac{dA}{dz}\,\frac{d^2\Phi}{dz^2}\,a^2 + \frac{dA}{dz}\,\frac{d^2\Phi}{dz^2}\,H_\circ^2\right.$$

$$\left.\left. +\ 2\,\frac{dA}{dz}\,\frac{d\Phi}{dz}\,H_\circ^2\right]\right\} = 0, \tag{11.13}$$

and

$$\underline{v}_c\left[\omega^2 - \frac{(\underline{H}_\circ\cdot\underline{k}_\perp)^2}{4\pi\rho_{eq}} + \left(\frac{d\Phi}{dz}\right)^2\frac{(\underline{H}_\circ\cdot\underline{e}_z)^2}{4\pi\rho_{eq}} + 2i\,\frac{d\Phi}{dz}\,\frac{(\underline{H}_\circ\cdot\underline{e}_z)(\underline{H}_\circ\cdot\underline{k}_\perp)}{4\pi\rho_{eq}}\right]$$

$$+\ \underline{k}_\perp\left\{(\underline{k}_\perp\cdot\underline{v}_c)\left[-a^2 - \frac{H_\circ^2}{4\pi\rho_{eq}}\right] + \frac{(\underline{H}_\circ\cdot\underline{v}_c)}{4\pi\rho_{eq}}\left[(\underline{H}_\circ\cdot\underline{k}_\perp)\right.\right.$$

(cont.)

$$
\left.\left. -i\frac{d\phi}{dz}(\underline{H}_\circ\cdot\underline{e}_z)\right] + (\underline{v}_c\cdot\underline{e}_z)\left[-ig + ia^2\frac{d\phi}{dz} + i\frac{d\phi}{dz}\frac{H_\circ^2}{4\pi\rho_{eq}}\right]\right\}
$$

$$
+ \underline{H}_\circ\left\{\frac{(\underline{v}_c\cdot\underline{k}_\perp)}{4\pi\rho_{eq}}\left[(\underline{H}_\circ\cdot\underline{k}_\perp) - i\frac{d\phi}{dz}(\underline{H}_\circ\cdot\underline{e}_z)\right]\right.
$$

$$
\left. + \frac{(\underline{v}_c\cdot\underline{e}_z)}{4\pi\rho_{eq}}\left[-\left(\frac{d\phi}{dz}\right)^2(\underline{H}_\circ\cdot\underline{e}_z) - i\frac{d\phi}{dz}(\underline{H}_\circ\cdot\underline{k}_\perp)\right]\right\}
$$

$$
+ \underline{e}_z\left\{(\underline{v}_c\cdot\underline{k}_\perp)\left[-ig(\gamma-1) + i\frac{d\phi}{dz}a^2 + i\frac{d\phi}{dz}\frac{H_\circ}{4\pi\rho_{eq}}\right]\right.
$$

$$
+ \frac{(\underline{H}_\circ\cdot\underline{v}_c)}{4\pi\rho_{eq}}\left[-\left(\frac{d\phi}{dz}\right)^2(\underline{H}_\circ\cdot\underline{e}_z) - i\frac{d\phi}{dz}(\underline{H}_\circ\cdot\underline{k}_\perp)\right]
$$

$$
\left. + (\underline{v}_c\cdot\underline{e}_z)\left[\left(\frac{d\phi}{dz}\right)^2 a^2 + \left(\frac{d\phi}{dz}\right)^2\frac{H_\circ}{4\pi\rho_{eq}} - \gamma g\frac{d\phi}{dz}\right]\right\} = 0.
$$

(11.14)

Eq. (11.13) is a three-dimensional vector equation, which is equivalent to three scalar equations for the three velocity components of $\underline{v}_c$, which are v_{cx}, v_{cy}, and v_{cz}. The condition for the solution of these three equations is the vanishing of the determinant of the coefficients. However, here it is advantageous to take the dot product of eq. (11.13) with $\underline{k}_{\perp}$, $\underline{H}_o$, and $\underline{e}_z$. The determinant of the coefficients of $(\underline{v}_c \cdot \underline{k}_{\perp})$, $(\underline{v}_c \cdot \underline{H}_o)$, and $(\underline{v}_c \cdot \underline{e}_z)$ of the three equations thus formed is , where i:j denotes the term of the i^{th} row and the j^{th} column:

$$1:1 = \frac{(\underline{H}_o \cdot \underline{e}_z)}{4\pi\rho_{eq}} \left[2 \frac{dA}{dz} \frac{d\Phi}{dz} + A \frac{d^2\Phi}{dz^2} \right]$$

$$1:2 = 0$$

$$1:3 = \frac{(\underline{H}_o \cdot \underline{e}_z)\ (\underline{H}_o \cdot \underline{k}_{\perp})}{4\pi\rho_{eq}} \left[-2 \frac{dA}{dz} \frac{d\Phi}{dz} - A \frac{d^2\Phi}{dz^2} \right]$$

$$2:1 = 0$$

$$2:2 = 0$$

$$2:3 = a^2 \frac{(\underline{H}_o \cdot \underline{e}_z)}{4\pi\rho_{eq}} \left[2 \frac{dA}{dz} \frac{d\Phi}{dz} + A \frac{d^2\Phi}{dz^2} \right]$$

$$3{:}1 = 0$$

$$3{:}2 = \frac{(\underline{H}_o \cdot \underline{e}_z)}{4\pi\rho_{eq}} \left[- 2 \frac{dA}{dz} \frac{d\Phi}{dz} - A \frac{d^2\Phi}{dz^2}\right]$$

and $$3{:}3 = \left(a^2 + \frac{H_o^2}{4\pi\rho_{eq}}\right) \left[2 \frac{dA}{dz} \frac{d\Phi}{dz} - A \frac{d^2\Phi}{dz^2}\right] . \tag{11.15}$$

It follows from setting this determinant to zero that

$$2 \frac{dA}{dz} \frac{d\Phi}{dz} = - A \frac{d^2\Phi}{dz^2}$$

which gives $$A = \left(\frac{d\Phi}{dz}\right)^{-1/2} , \tag{11.16}$$

This is the expected relation between the amplitude and the phase from employing the W.K.B. method.

We treat eq. (11.14) in a similar manner as eq. (11.13). The condition for the solution of the three scalar equations contained in the vector eq. (11.14) is for the determinant of the coefficients to be zero. Again, we obtain three equations from eq. (11.14) by taking the dot product of eq. (11.14) with $\underline{k}_\perp$, $\underline{H}_o$, and $\underline{e}_z$. The determinant of the coefficients of $(\underline{v}_c \cdot \underline{k}_\perp)$, $(\underline{v}_c \cdot \underline{H}_o)$ and $(\underline{v}_c \cdot \underline{e}_z)$ of the three equations is, after dividing by A:

$$1{:}1 = \omega^2$$

$$1:2 = -a^2 \frac{(\underline{H}_o \cdot \underline{k}_\perp)}{4\pi\rho_{eq}} + \frac{(\underline{H}_o \cdot \underline{e}_z)}{4\pi\rho_{eq}} \left[i a^2 \frac{d\Phi}{dz} - ig(\gamma - 1) \right] ,$$

$$1:3 = \frac{(\underline{H}_o \cdot \underline{k}_\perp)}{4\pi\rho_{eq}} \left[i a^2 \frac{d\Phi}{dz} - ig \right]$$

$$+ \frac{(\underline{H}_o \cdot \underline{e}_z)}{4\pi\rho_{eq}} \left[a^2 \left(\frac{d\Phi}{dz}\right)^2 - \gamma g \frac{d\Phi}{dz} \right] ,$$

$$2:1 = \frac{(\underline{H}_o \cdot \underline{k}_\perp)}{4\pi\rho_{eq}} k_\perp^2 - i \frac{d\Phi}{dz} (\underline{H}_o \cdot \underline{e}_z) k_\perp^2 ,$$

$$2:2 = \omega^2 - a^2 k_\perp^2 - \frac{H_o^2 k_\perp^2}{4\pi\rho_{eq}}$$

$$+ \left(\frac{d\Phi}{dz}\right)^2 \frac{(\underline{H}_o \cdot \underline{e}_z)^2}{4\pi\rho_{eq}} + i \frac{d\Phi}{dz} \frac{(\underline{H}_o \cdot \underline{e}_z)(\underline{H}_o \cdot \underline{k}_\perp)}{4\pi\rho_{eq}} ,$$

$$2:3 = -igk_\perp^2 + i a^2 \frac{d\Phi}{dz} k_\perp^2 - \left(\frac{d\Phi}{dz}\right)^2 \frac{(\underline{H}_o \cdot \underline{k}_\perp)(\underline{H}_o \cdot \underline{e}_z)}{4\pi\rho_{eq}}$$

$$+ i \frac{d\Phi}{dz} \frac{H_o^2 k_\perp^2}{4\pi\rho_{eq}} - i \frac{d\Phi}{dz} \frac{(\underline{H}_o \cdot \underline{k}_\perp)}{4\pi\rho_{eq}} ,$$

$$3:1 = -\left(\frac{d\Phi}{dz}\right)^2 \frac{(\underline{H}_o \cdot \underline{e}_z)}{4\pi\rho_{eq}} - i \frac{d\Phi}{dz} \frac{(\underline{H}_o \cdot \underline{k}_\perp)}{4\pi\rho_{eq}} ,$$

$$3{:}2 = -\, i\, g(\gamma - 1) + i\, a^2 \frac{d\Phi}{dz} + i \frac{d\Phi}{dz} \frac{H_o^2}{4\pi\rho_{eq}}$$

$$-i \frac{d\Phi}{dz} \frac{(\underline{H}_o \cdot \underline{e}_z)^2}{4\pi\rho_{eq}} + \frac{(\underline{H}_o \cdot \underline{k}_\perp)\,(\underline{H}_o \cdot \underline{e}_z)}{4\pi\rho_{eq}}$$

and
$$3{:}3 = \omega^2 + a^2 \left(\frac{d\Phi}{dz}\right)^2 + \left(\frac{d\Phi}{dz}\right)^2 \frac{H_o^2}{4\pi\rho_{eq}} - \gamma g \frac{d\Phi}{dz}$$

$$- \frac{(\underline{H}_o \cdot \underline{k}_\perp)^2}{4\pi\rho_{eq}} + i \frac{d\Phi}{dz} \frac{(\underline{H}_o \cdot \underline{e}_z)\,(\underline{H}_o \cdot \underline{k}_\perp)}{4\pi\rho_{eq}} .$$

Setting this determinant to zero gives

$$\left(\frac{d\Phi}{dz}\right)^6 \left[a^2 \frac{(\underline{H}_o \cdot \underline{e}_z)^4}{(4\pi\rho_{eq})^2}\right] + \left(\frac{d\Phi}{dz}\right)^5 \left[4ia^2 \frac{(\underline{H}_o \cdot \underline{k}_\perp)\,(\underline{H}_o \cdot \underline{e}_z)^3}{(4\pi\rho_{eq})^2}\right.$$

$$\left. - g\gamma \frac{(\underline{H}_o \cdot \underline{e}_z)^4}{(4\pi\rho_{eq})^2}\right] + \left(\frac{d\Phi}{dz}\right)^4 \left[2a^2\omega^2 \frac{(\underline{H}_o \cdot \underline{e}_z)^2}{4\pi\rho_{eq}}\right.$$

$$+ \omega^2 H_o^2 \frac{(\underline{H}_o \cdot \underline{e}_z)^2}{(4\pi\rho_{eq})^2} - 3ig\gamma \frac{(\underline{H}_o \cdot \underline{k}_\perp)\,(\underline{H}_o \cdot \underline{e}_z)^3}{(4\pi\rho_{eq})^2}$$

$$\left. - a^2 k_\perp^2 \frac{(\underline{H}_o \cdot \underline{e}_z)^4}{(4\pi\rho_{eq})^2} - 6\, a^2 \frac{(\underline{H}_o \cdot \underline{k}_\perp)^2 (\underline{H}_o \cdot \underline{e}_z)^2}{(4\pi\rho_{eq})^2}\right] + \left(\frac{d\Phi}{dz}\right)^3 \left[- 2g\gamma\omega^2 \frac{(\underline{H}_o \cdot \underline{e}_z)^2}{4\pi\rho_{eq}}\right.$$

$$+ 4ia^2\omega^2 \frac{(\underline{H}_o \cdot \underline{k}_\perp)\,(\underline{H}_o \cdot \underline{e}_z)}{4\pi\rho_{eq}} + 2i\omega^2 H_o^2 \frac{(\underline{H}_o \cdot \underline{k}_\perp)\,(\underline{H}_o \cdot \underline{e}_z)}{(4\pi\rho_{eq})^2}$$

$$+ g\gamma k_{\perp}^2 \frac{(\underline{H}_\circ \cdot \underline{e}_z)^4}{(4\pi\rho_{eq})^2} + 3g\gamma \frac{(\underline{H}_\circ \cdot \underline{k}_{\perp})^2 (\underline{H}_\circ \cdot \underline{e}_z)^2}{(4\pi\rho_{eq})^2}$$

$$- 4ia^2 k_{\perp}^2 \frac{(\underline{H}_\circ \cdot \underline{k}_{\perp})(\underline{H}_\circ \cdot \underline{e}_z)^3}{(4\pi\rho_{eq})^2} - 4ia^2 \frac{(\underline{H}_\circ \cdot \underline{k}_{\perp})^3 (\underline{H}_\circ \cdot \underline{e}_z)}{(4\pi\rho_{eq})^2} \Bigg]$$

$$+ \left(\frac{d\Phi}{dz}\right)^2 \Bigg[a^2\omega^4 + \omega^4 H_\circ^2 + \omega^4 \frac{(\underline{H}_\circ \cdot \underline{e}_z)^2}{4\pi\rho_{eq}} - 3ig\gamma\omega^2 \frac{(\underline{H}_\circ \cdot \underline{e}_z)(\underline{H}_\circ \cdot \underline{k}_{\perp})}{4\pi\rho_{eq}}$$

$$-2a^2 k_{\perp}^2 \omega^2 \frac{(\underline{H}_\circ \cdot \underline{e}_z)^2}{4\pi\rho_{eq}} - 2a^2\omega^2 \frac{(\underline{H}_\circ \cdot \underline{k}_{\perp})^2}{4\pi\rho_{eq}} - \omega^2 H_\circ^2 k_{\perp}^2 \frac{(\underline{H}_\circ \cdot \underline{e}_z)^2}{(4\pi\rho_{eq})^2}$$

$$- \omega^2 H_\circ^2 \frac{(\underline{H}_\circ \cdot \underline{k}_{\perp})^2}{(4\pi\rho_{eq})^2} + 3ig\gamma k_{\perp}^2 \frac{(\underline{H}_\circ \cdot \underline{k}_{\perp})(\underline{H}_\circ \cdot \underline{e}_z)^3}{(4\pi\rho_{eq})^2} + ig\gamma \frac{(\underline{H}_\circ \cdot \underline{k}_{\perp})^3 (\underline{H}_\circ \cdot \underline{e}_z)}{(4\pi\rho_{eq})^2}$$

$$+g^2(\gamma-1)k_{\perp}^2 \frac{(\underline{H}_\circ \cdot \underline{e}_z)^2}{4\pi\rho_{eq}} + 6a^2 k_{\perp}^2 \frac{(\underline{H}_\circ \cdot \underline{k}_{\perp})^2 (\underline{H}_\circ \cdot \underline{e}_z)^2}{(4\pi\rho_{eq})^2} + a^2 \frac{(\underline{H}_\circ \cdot \underline{k}_{\perp})^4}{(4\pi\rho_{eq})^2} \Bigg]$$

$$+ \frac{d\Phi}{dz} \Bigg[- g\gamma\omega^4 + 2i\omega^4 \frac{(\underline{H}_\circ \cdot \underline{k}_{\perp})(\underline{H}_\circ \cdot \underline{e}_z)}{4\pi\rho_{eq}} + g\gamma\omega^2 \frac{(\underline{H}_\circ \cdot \underline{k}_{\perp})^2}{4\pi\rho_{eq}}$$

$$+ g\gamma\omega^2 k_{\perp}^2 \frac{(\underline{H}_\circ \cdot \underline{e}_z)^2}{4\pi\rho_{eq}} - 4ia^2\omega^2 k_{\perp}^2 \frac{(\underline{H}_\circ \cdot \underline{k}_{\perp})(\underline{H}_\circ \cdot \underline{e}_z)}{4\pi\rho_{eq}} - 2iH_\circ^2 \omega^2 k_{\perp}^2 \frac{(\underline{H}_\circ \cdot \underline{k}_{\perp})(\underline{H}_\circ \cdot \underline{e}_z)}{(4\pi\rho_{eq})^2}$$

(cont.)

$$- 3g\gamma k_\perp^2 \frac{(\underline{H}_o \cdot \underline{k}_\perp)^2 (\underline{H}_o \cdot \underline{e}_z)^2}{(4\pi\rho_{eq})^2} + 2ig^2(\gamma-1)k_\perp^2 \frac{(\underline{H}_o \cdot \underline{k}_\perp)(\underline{H}_o \cdot \underline{e}_z)}{4\pi\rho_{eq}}$$

$$\left. + 4ia^2k_\perp^2 \frac{(\underline{H}_o \cdot \underline{k}_\perp)^3 (\underline{H}_o \cdot \underline{e}_z)}{(4\pi\rho_{eq})^2} \right] + \omega^6 - \omega^4 a^2 k_\perp^2 - \omega^4 \frac{H_o^2 k_\perp^2}{4\pi\rho_{eq}}$$

$$- \omega^4 \frac{(\underline{H}_o \cdot \underline{k}_\perp)^2}{4\pi\rho_{eq}} + \omega^2 g^2 k_\perp^2 (\gamma-1) + ig\gamma\omega^4 k_\perp^2 \frac{(\underline{H}_o \cdot \underline{k}_\perp)(\underline{H}_o \cdot \underline{e}_z)}{4\pi\rho_{eq}}$$

$$+ 2\omega^2 a^2 k_\perp^2 \frac{(\underline{H}_o \cdot \underline{k}_\perp)^2}{4\pi\rho_{eq}} + \omega^2 H_o^2 k_\perp^2 \frac{(\underline{H}_o \cdot \underline{k}_\perp)^2}{(4\pi\rho_{eq})^2} - ig\gamma k_\perp^2 \frac{(\underline{H}_o \cdot \underline{k}_\perp)^3 (\underline{H}_o \cdot \underline{e}_z)}{(4\pi\rho_{eq})^2}$$

$$-g^2(\gamma-1)k_\perp^2 \frac{(\underline{H}_o \cdot \underline{k}_\perp)^2}{4\pi\rho_{eq}} - a^2 k_\perp^2 \frac{(\underline{H}_o \cdot \underline{k}_\perp)^4}{(4\pi\rho_{eq})^2} = 0 \qquad (11.18)$$

Eq. (11.18) is a first order differential equation of the sixth degree, and it can be factored into two first order differential equations; one of the second degree and one of the fourth degree. The theory of differential equations allows us to equate each factor to zero. Upon rearranging terms, we have

$$\left(\frac{d\phi}{dz}\right)^2 + 2i\frac{(\underline{H}_o \cdot \underline{k}_\perp)}{(\underline{H}_o \cdot \underline{e}_z)} \frac{d\phi}{dz} - \frac{(\underline{H}_o \cdot \underline{k}_\perp)^2}{(\underline{H}_o \cdot \underline{e}_z)^2}$$

$$+ \frac{4\pi\omega^2}{(\underline{H}_o \cdot \underline{e}_z)^2} \rho_{eq}(z) = 0 , \qquad (11.19)$$

and $$\left(\frac{d\Phi}{dz}\right)^4 + \left(\frac{d\Phi}{dz}\right)^3 \left[2i \frac{(\underline{H}_o \cdot \underline{k}_\perp)}{(\underline{H}_o \cdot \underline{e}_z)} - \frac{g\gamma}{a^2} \right]$$

$$+ \left(\frac{d\Phi}{dz}\right)^2 \left[\frac{4\pi\omega^2}{(\underline{H}_o \cdot \underline{e}_z)^2} \rho_{eq}(z) - \frac{\omega^2 H_o^2}{a^2 (\underline{H}_o \cdot \underline{e}_z)^2} - k_\perp^2 - \frac{(\underline{H}_o \cdot \underline{k}_\perp)^2}{(\underline{H}_o \cdot \underline{e}_z)^2} \right.$$

$$\left. - i \frac{g\gamma}{a^2} \frac{(\underline{H}_o \cdot \underline{k}_\perp)}{(\underline{H}_o \cdot \underline{e}_z)} \right] + \frac{d\Phi}{dz} \left[- \frac{g\gamma\omega^2 4\pi}{a^2 (\underline{H}_o \cdot \underline{e}_z)^2} \rho_{eq}(z) \right.$$

$$\left. - 2ik_\perp^2 \frac{(\underline{H}_o \cdot \underline{k}_\perp)}{(\underline{H}_o \cdot \underline{e}_z)} + \frac{g\gamma k_\perp^2}{a^2} \right] + \frac{4\pi\omega^4}{a^2 (\underline{H}_o \cdot \underline{e}_z)^2} \rho_{eq}(z)$$

$$- \frac{4\pi\omega^2 k_\perp^2}{(\underline{H}_o \cdot \underline{e}_z)^2} \rho_{eq}(z) - \frac{\omega^2}{a^2} \frac{H_o^2 k_\perp^2}{(\underline{H}_o \cdot \underline{e}_z)^2} + \frac{4\pi g^2 (\gamma-1) k_\perp^2}{a^2 (\underline{H}_o \cdot \underline{e}_z)^2} \rho_{eq}(z)$$

$$+ k_\perp^2 \frac{(\underline{H}_o \cdot \underline{k}_\perp)^2}{(\underline{H}_o \cdot \underline{e}_z)^2} + ig\gamma \frac{k_\perp^2}{a^2} \frac{\underline{H}_o \cdot \underline{k}_\perp)}{(\underline{H}_o \cdot \underline{e}_z)} = 0 \quad . \tag{11.20}$$

The solution of the differential equation (11.19) gives the phase for the W.K.B. solution for the Alfven mode. The solution of the differential equation (11.20) gives the phase for the W.K.B. solution of the two remaining modes.

The differential equations (11.19) and (11.20) can be brought into a form that can be solved by the separation of variables, after first solving both equations as algebraic equations of the second and fourth degree in the unknown $d\Phi/dz$. This can always be done in closed form for an algebraic equation up to the fourth degree.

12. VERTICAL PENETRATION DEPTHS

The condition $d\Phi/dz = 0$ implies that the z-component of the wave number, k, is zero. This gives the condition for the termination of the wave propagation in the z-direction, that is, the vertical penetration depth. At this point, $\underline{k}_\perp = \underline{k}$.

Setting $d\phi/dz = 0$ in eq. (11.19) and using the definition of ρ_{eq}, (the first equation of (2.9)), gives upon solving for the penetration depth, z_{pd}

$$z_{pd} = h \ln \left[\frac{4\pi\rho_o \omega^2}{(\underline{H}_o \cdot \underline{k}_\perp)^2} \right] , \tag{12.1}$$

where h is the scale height defined by eq. (4.3). From eq. (8.5), the dispersion relation for the Alfven mode at $z = 0$ is

$$\omega^2 = \frac{(\underline{H}_o \cdot \underline{k})^2}{4\pi\rho_o} . \tag{12.2}$$

We define two angles. $\phi_\perp$ is the angle between $\underline{H}_o$ and $\underline{k}_\perp$, and ϕ_o is the angle between $\underline{H}_o$ and $\underline{k}$ at $z = 0$. By the substitution of eq. (12.2) and the angles $\phi_\perp$ and ϕ_o into eq. (12.1), we have

$$z_{pd} = h \ln \left[\frac{k^2 \cos^2\phi_o}{k_\perp^2 \cos^2\phi_\perp} \right] . \tag{12.3}$$

But from the definition of the angle θ, we have

$$\frac{k_\perp^2}{k^2} = \sin^2\theta \quad . \tag{12.4}$$

Thus, eq. (12.3) becomes

$$z_{pd} = h \ln \left[\frac{\cos^2\phi_o}{\sin^2\theta \cos^2\phi_\perp} \right] \quad , \tag{12.5}$$

which is the equation for the penetration depth for the Alfven mode.

As an example, we will consider the case of a vertical magnetic field. First. let us suppose that the wave vector is initially directed vertically. We have $\theta = 0°$, $\phi_o = 0°$, and $\phi_\perp = 90°$, which give $z_{pd} = +\infty$. Thus, we can say that no reflection exists for the Alfven mode for vertical directed waves in a vertical magnetic field. Second, let us suppose that the wave vector is initially directed along the horizontal. We have $\theta = 90°$ and $\phi_o = \phi_\perp$, which give $z_{pd} = 0$. This result is obvious, since initially we had no z-component of the wave vector.

Setting $d\Phi/dz = 0$ in eq. (11.20), gives upon solving for the penetration depth

$$z_{pd} = h \ln \left[\frac{4\pi\rho_o \omega^2 - 4\pi\rho_o \omega^2 a^2 k_\perp^2 + 4\pi\rho_o g^2 (\gamma-1) k_\perp^2}{\omega^2 H_o^2 k_\perp^2 - a^2 k_\perp^2 (\underline{H}_o \cdot \underline{k}_\perp)^2 - ig\gamma(\underline{H}_o \cdot \underline{k}_\perp)(\underline{H}_o \cdot \underline{e}_z) k_\perp^2} \right] \quad . \tag{12.6}$$

In order to put this equation into a nondimensional form, we define a ratio of pressures at z = 0, similar to eq. (8.10),

$$\beta_\circ = \frac{p_\circ}{H_\circ^2/8\pi} \quad . \tag{12.7}$$

Using eq. (12.7) and the characteristic wave number and the characteristic frequency defined by (8.9), we have, upon substitution into eq. (12.6)

$$z_{pd} = h \ln \left\{ \frac{\frac{\gamma\beta_\circ}{2}\left[W^4 - W^2K_\perp^2 + \frac{\gamma\beta_\circ}{2}(\gamma-1)\right]}{W^2K_\perp^2 - K_\perp^4 \cos^2\phi_\perp - i\gamma^{3/2}\left(\frac{\beta_\circ}{2}\right)^{1/2} K_\perp^3 \cos\phi_\perp \cos\eta} \right\} \tag{12.8}$$

where the expression given by the dispersion relation (8.12) has to be substituted for W.

13. PARTICLE ORBITS

In this section we consider the orbit of an individual particle of the medium undergoing wave motion. The particle orbits can be obtained from the vector equation (6.2). Upon separating this equation into its components $\underline{e}_x$, $\underline{e}_y$, and $\underline{e}_z$, equating each component to zero, and rearranging the terms, we have respectively the following three scalar equations:

$$\mu_{11}\, v_x + \mu_{12}\, v_y + (\mu_{13} + i\varepsilon_{13})\, v_z = 0 \tag{13.1}$$

$$\mu_{21}\, v_x + \mu_{22}\, v_y + (\mu_{23} + i\varepsilon_{23})\, v_z = 0 \tag{13.2}$$

$$(\mu_{31} + i\varepsilon_{31})\, v_x + (\mu_{32} + i\varepsilon_{32})\, v_y + (\mu_{33} + i\varepsilon_{33})\, v_z = 0\,, \tag{13.3}$$

where

$$\mu_{11} = \omega^2 - a^2 k_x^2 + 2\,\frac{(\underline{k}\cdot\underline{H}_\circ)}{4\pi\rho_{eq}}\, k_x H_x - \frac{H_\circ^2}{4\pi\rho_{eq}}\, k_x^2 - \frac{(\underline{k}.\underline{H}_\circ)^2}{4\pi\rho_{eq}}\,,$$

$$\mu_{12} = -a^2 k_x^2 k_y^2 - \frac{H_\circ^2}{4\pi\rho_{eq}}\, k_x k_y + \frac{(\underline{k}\cdot\underline{H}_\circ)}{4\pi\rho_{eq}}\, k_y H_x\,,$$

$$\mu_{13} = -a^2 k_x k_z + \frac{(\underline{k}\cdot\underline{H}_\circ)}{4\pi\rho_{eq}}\,(k_x H_z + k_z H_x) - \frac{H_\circ^2}{4\pi\rho_{eq}}\, k_x k_z\,,$$

$$\mu_{21} = -a^2 k_x k_y + \frac{(\underline{k}\cdot\underline{H}_\circ)}{4\pi\rho_{eq}} k_y H_x - \frac{H_\circ^2}{4\pi\rho_{eq}} k_x k_y \quad ,$$

$$\mu_{22} = \omega^2 - a^2 k_y^2 - \frac{H_\circ^2}{4\pi\rho_{eq}} k_y^2 - \frac{(\underline{k}\cdot\underline{H}_\circ)^2}{4\pi\rho_{eq}} \quad ,$$

$$\mu_{23} = -a^2 k_y k_z + \frac{(\underline{k}\cdot\underline{H}_\circ)}{4\pi\rho_{eq}} k_y H_z - \frac{H_\circ^2}{4\pi\rho_{eq}} k_y k_z \quad ,$$

$$\mu_{31} = -a^2 k_x k_z + \frac{(\underline{k}\cdot\underline{H}_\circ)}{4\pi\rho_{eq}} (k_z H_x + k_x H_z) - \frac{H_\circ^2}{4\pi\rho_{eq}} k_x k_z \quad ,$$

$$\mu_{32} = -a^2 k_y k_z + \frac{(\underline{k}\cdot\underline{H}_\circ)}{4\pi\rho_{eq}} k_y H_z - \frac{H_\circ^2}{4\pi\rho_{eq}} k_y k_z \quad ,$$

$$\mu_{33} = \omega^2 - a^2 k_z^2 + 2\,\frac{(\underline{k}\cdot\underline{H}_\circ)}{4\pi\rho_{eq}} k_z H_z - \frac{H_\circ^2}{4\pi\rho_{eq}} k_z^2 - \frac{(\underline{k}\cdot\underline{H}_\circ)^2}{4\pi\rho_{eq}} \quad .$$

$$\varepsilon_{13} = -g k_x \quad ,$$

$$\varepsilon_{23} = -g k_y \quad ,$$

$$\varepsilon_{31} = -(\gamma-1)\, g k_x ,$$

$$\varepsilon_{32} = -(\gamma-1)\, g k_y \quad ,$$

and

$$\varepsilon_{33} = -\gamma g k_z \quad .$$

To obtain the particle orbits, we assume that according to eq. (5.1), the z-component of the velocity has the form

$$v_z = A\, e^{i(\underline{k} \cdot \underline{r} - \omega t)} \quad , \quad (A = \text{Const.}) \qquad (13.4)$$

We solve eq. (13.2) for v_z and substitute the result into eq. (13.1). This gives

$$v_x = \frac{(\mu_{13} - \frac{\mu_{12}}{\mu_{22}} \mu_{23}) + i(\epsilon_{13} - \frac{\mu_{12}}{\mu_{22}} \epsilon_{23})}{(\mu_{11} + \frac{\mu_{12}}{\mu_{22}} \mu_{21})} v_z \qquad (13.5)$$

We then solve eq. (13.2) for v_x and substitute the result into eq. (13.1). This gives

$$v_y = \frac{(\mu_{13} - \frac{\mu_{11}}{\mu_{21}} \mu_{23}) + i(\epsilon_{13} - \frac{\mu_{11}}{\mu_{21}} \epsilon_{23})}{(\mu_{12} - \frac{\mu_{11}}{\mu_{21}} \mu_{22})} v_z \qquad (13.6)$$

For small deviations of the particles from their equilibrium positions $\underline{r}_o = x_o \underline{e}_x + y_o \underline{e}_y + z_o \underline{e}_z$, we can put

$$x = x_o + \xi$$
$$y = y_o + \eta$$
$$z = z_o + \zeta \quad ,$$

or respectively

$$\xi = x - x_o$$
$$\eta = y - y_o$$
$$\zeta = z - z_o$$

where ξ , η, and ζ are considered to be small quantities. By substituting these expressions into eqs. (13.4), (13.5), and (13.6), and by putting $\underline{r}$ $\underline{r}_o$ on the right-hand-side of eq. (13.4) according to lowest order pertubation theory, we have after integrating for the real parts

$$\xi = \frac{-\frac{A}{\omega}}{(\mu_{11}\mu_{22} - \mu_{12}\mu_{21})} \Big[(\mu_{13}\mu_{22} - \mu_{12}\mu_{23}) \sin (\underline{k}\cdot\underline{r}_o - \omega t)$$

$$+ (\mu_{22}\varepsilon_{13} - \mu_{12}\varepsilon_{23}) \cos (\underline{k}\cdot\underline{r}_o - \omega t) \Big] , \tag{13.8}$$

$$\eta = \frac{-\frac{A}{\omega}}{(\mu_{12}\mu_{21} - \mu_{11}\mu_{22})} \Big[(\mu_{13}\mu_{21} - \mu_{11}\mu_{23}) \sin (\underline{k}\cdot\underline{r}_o - \omega t)$$

$$+ (\mu_{21}\varepsilon_{13} - \mu_{11}\varepsilon_{23}) \cos (\underline{k}\cdot\underline{r}_o - \omega t) \Big] , \tag{13.9}$$

and

$$\zeta = - \frac{A}{\omega} \sin (\underline{k}\cdot\underline{r}_o - \omega t) \tag{13.10}$$

Eqs. (13.8), (13.9), and (13.10) are the equations for the particle orbits in parametric form, where time is the parameter.

We can consider the equilibirum point to be at the origin without modifying the shape of the particle's path. So, we set $x_o = y_o = z_o = 0$, which yields

$$\xi = x, \ \eta = y, \text{ and } \zeta = z.$$

Upon introducing the phase $\chi_o = (\underline{k} \cdot \underline{r}_o - \omega t)$, we have for eqs. (13.8), (13.9), and (13.10)

$$\begin{aligned} x &= \Lambda_1 \sin \chi_o + T_1 \cos \chi_o \\ y &= \Lambda_2 \sin \chi_o + T_2 \cos \chi_o \\ z &= \Lambda_3 \sin \chi_o \quad , \end{aligned} \tag{13.11}$$

where

$$\Lambda_1 = -\frac{A}{\omega}\left(\frac{\mu_{13}\mu_{22} - \mu_{12}\mu_{23}}{\mu_{11}\mu_{22} - \mu_{12}\mu_{21}}\right),$$

$$\Lambda_2 = -\frac{A}{\omega}\left(\frac{\mu_{13}\mu_{21} - \mu_{11}\mu_{23}}{\mu_{12}\mu_{21} - \mu_{11}\mu_{22}}\right),$$

$$\Lambda_3 = -\frac{A}{\omega} \quad ,$$

$$T_1 = -\frac{A}{\omega}\left(\frac{\mu_{22}\varepsilon_{13} - \mu_{12}\varepsilon_{23}}{\mu_{11}\mu_{22} - \mu_{12}\mu_{21}}\right) \quad ,$$

and

$$T_2 = -\frac{A}{\omega}\left(\frac{\mu_{21}\varepsilon_{13} - \mu_{11}\mu_{23}}{\mu_{12}\varepsilon_{21} - \mu_{11}\mu_{22}}\right) \quad .$$

Eqs. (13.11) describe the particle orbit as an ellipse in parametric form, but do not describe an ellipse located in the x-y plane nor one aligned with its principal axes along the x- and y-coordinates. However, it may be put into the familiar form of an ellipse by means of a three dimensional rotation. Toward this end, we consider the general orthogonal transformation

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad , \tag{13.12}$$

where the transformation matrix (α_{ik}) is determined from the condition that the ellipse shall be located in the x'-y' plane and alligned with its principal axes along the x'- and y'-coordinates. Since this matrix describes a general orthogonal transformation, its elements have to satisfy the six orthogonality and normalization conditions

$$\sum_{i=1}^{3} \alpha_{ij}\alpha_{ik} = \delta_{jk} \qquad j,k, = 1,\ 2,\ 3 \quad . \tag{13.13}$$

By substituting eqs. (13.11) into eqs. (13.12), we have

$$x' = \Gamma_1 \sin \chi_o + \Omega_1 \cos \chi_o$$

$$y' = \Gamma_2 \sin \chi_o + \Omega_2 \cos \chi_o$$

$$z' = \Gamma_3 \sin \chi_o + \Omega_3 \cos \chi_o \quad , \qquad (13.14)$$

where

$$\Gamma_j = \sum_{i=1}^{3} \alpha_{ji} \Lambda_i \qquad j = 1, 2, 3$$

and

$$\Omega_j = \sum_{i=1}^{2} \alpha_{ji} T_i \qquad j = 1, 2, 3 \quad .$$

In order for the particle orbit to be located in the x'-y' plane, it is required that z' = 0. Since this condition must be satisfied for all possible phases χ_o, this implies that

$$\Gamma_3 = 0 \tag{13.15}$$

and

$$\Omega_3 = 0 \quad . \tag{13.16}$$

Furthermore, in order to align the x' and y' coordinates along the principal axis of the ellipse, it is required that

$$\Omega_1 = 0 \tag{13.17}$$

and

$$\Gamma_2 = 0 \quad . \tag{13.18}$$

Eqs. (13.16) and (13.17) are not independent from each other, because they represent two linear homogeneous equations for the quantitites τ_1 and τ_2. In order that eqs. (13.16) and (13.17) are satisfied simultaneously, it is required that the determinant of their coefficients vanish. This condition is given by

$$\alpha_{31}\,\alpha_{12} - \alpha_{32}\,\alpha_{11} = 0 \ . \tag{13.19}$$

The α_{ik}'s represent nine unknowns, and the orthogonality and normalization conditions, eqs. (13.13), give us six conditions toward the determinantion of these unknowns. Eqs. (13.15), (13.18), and (13.19) give us the required three additional conditions necessary to determine all the nine unknowns α_{ik}. Hence, the transformation matrix is uniquely determined, and from eq. (13.14) follows the equation for the particle orbit, which is in general an ellipse given by

$$\left(\frac{x'}{r_1}\right)^2 + \left(\frac{y'}{\Omega_2}\right)^2 = 1 \tag{13.20}$$

It is to be noted that the orbits are frequency or wave number dependent according to eqs. (13.1) and (13.2). The dispersion relation, eq. (6.9), relates the frequency to the wave number.

For the special case of the Alfven wave, because of the simplicity of this mode, the particle orbit can be derived in a more direct way. From the conditions of transversality which are valid for pure Alfven waves, we have

$$\nabla \cdot \underline{v} = 0$$

which, with the plane wave solution (5.1), becomes

$$\underline{k} \cdot \underline{v} = 0 \tag{13.21}$$

and furthermore, for Alfven waves we have

$$\underline{v} \cdot \underline{H}_o = 0 \quad . \tag{13.22}$$

By **substituting** eqs. (13.21) and (13.22) into eq. (6.2), we obtain

$$\omega^2 \underline{v} + i\,(\underline{v}\cdot\underline{g})\underline{k} - \frac{(\underline{k}\cdot\underline{H}_o)^2}{4\pi\rho_{eq}}\,\underline{v} = 0 \quad . \tag{13.23}$$

From the dispersion relation for Alfven waves, eq. (8.5), it can be seen that eq. (13.23) implies

$$\underline{v}\cdot\underline{g} = 0 \tag{13.24}$$

Thus, the particle velocity vector is both perpendicular to the manetic field vector $\underline{H}_o$, eq. (13.22), and to the gravity vector $\underline{g}$, eq. (13.24). Hence, the particle orbit is a straight line directed along the velocity vector $\underline{v}$.

14. CONCLUSION

We have considered small amplitude low frequency waves in a plasma of infinite conductivity under the influence of a constant external magnetic and gravitational field. We have shown the existence of three wave modes, which are unstable in many cases. One mode, identical with the Alfven mode, and therefore stable, is independent of compressibility and gravity.

The instabilities do not appear by considering the hydromagnetic wave motion independent from the internal gravity wave motion. Our results show instabilities which may be related to the interchange instability in curved magnetic field configurations, which has been theoretically investigated by Rosenbluth and Longmire[19]. In their case an "effective" gravitational force was introduced to take into account the centrifugal force on particle orbits due to field line curvature.

As can be seen from Table 1, there is a large anisotropy with regard to stable wave propagation. This behavior does not occur in a treatment neglecting the wave mode coupling. Therefore, our results suggest that a search be made for these instabilities in the solar atmosphere. It may be possible that the violent nature of solar flares which occur in ionized layers premeated by a magnetic field in regions above sunspots is related to these instabilities.

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[19] M.N. Rosenbluth and C.L. Longmire, Ann. Phys. (N.Y.) 1, 120 (1957).

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θ	η	ϕ	+mode	-mode
0°	0°	0°	U	S
90°	90°	0°	SL	SL
90°	0°	90°	SL	SL
0°	90°	90°	U	NP
90°	90°	90°	SL	SL
90°	90°	45°	SL	SL
90°	45°	90°	SL	SL
45°	90°	90°	U	U
90°	45°	45°	U	U
45°	90°	45°	U	U
45°	45°	90°	U	U
45°	45°	45°	U	SWN
0°	45°	45°	U	U
45°	0°	45°	U	SWN

TABLE 1. Ranges of stability of certain combination of angles.

S .... Stable wave propagation occurs for all K.

NP.... No propagation.

SL.... A stability limit exists for which a stable mode occurs for $K > K_{s1}$ and instability occurs for $K < K_{s1}$.

SWN... A single discrete wave number exists for which stable wave motion is possible, otherwise unstable.

U .... Unstable wave propagation.

FIGURE CAPTIONS:

Figure 1. Definition of the angles θ, η, and ϕ.

Figure 2. The local dispersion relation $Re(W_+)$ for $K > K_{s1}$. $\theta = 90°$, $\eta = 90°$, and $\phi = 0°$.

Figure 4. The growth rates, $Im(W_+)$ and $Im(W_-)$, for $\theta = 90°$, $\eta = 90°$, and $\phi = 0°$.

Figure 5. The local dispersion relation $Re(W_+)$ for $K > K_{s1}$. $\theta = 90°$, $\eta = 0°$, and $\phi = 90°$.

Figure 6. The local dispersion relation $Re(W_-)$ for $K > K_{s1}$. $\theta = 90°$, $\eta = 0°$, and $\phi = 90°$.

Figure 7. The growth rates, $Im(W_+)$ and $Im(W_-)$, for $\theta = 90°$, $\eta = 0°$, and $\phi = 90°$.

Figure 8. The growth rate, $Im(W_-)$, for $\theta = 45°$, $\eta = 45°$, and $\phi = 45°$.

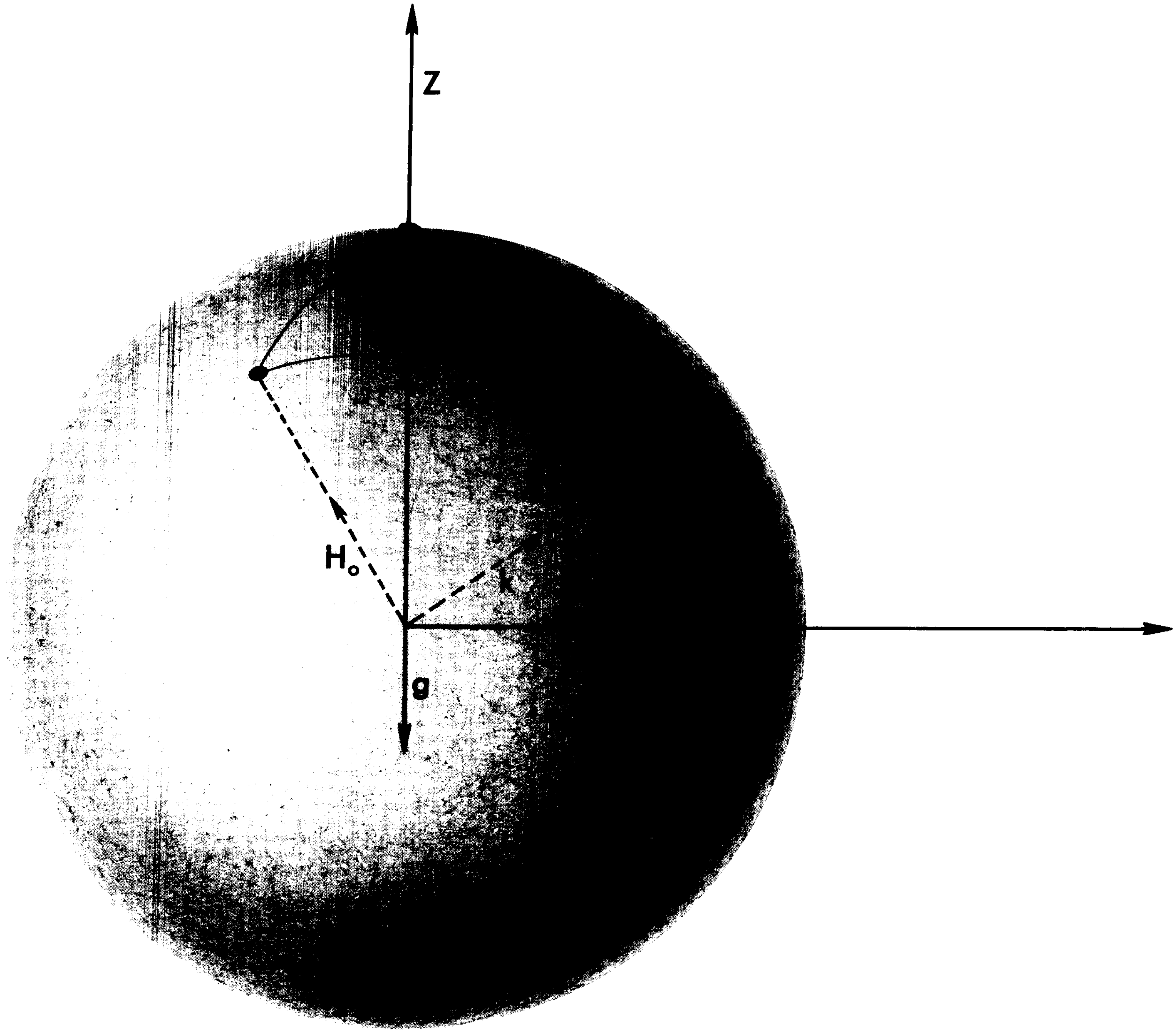
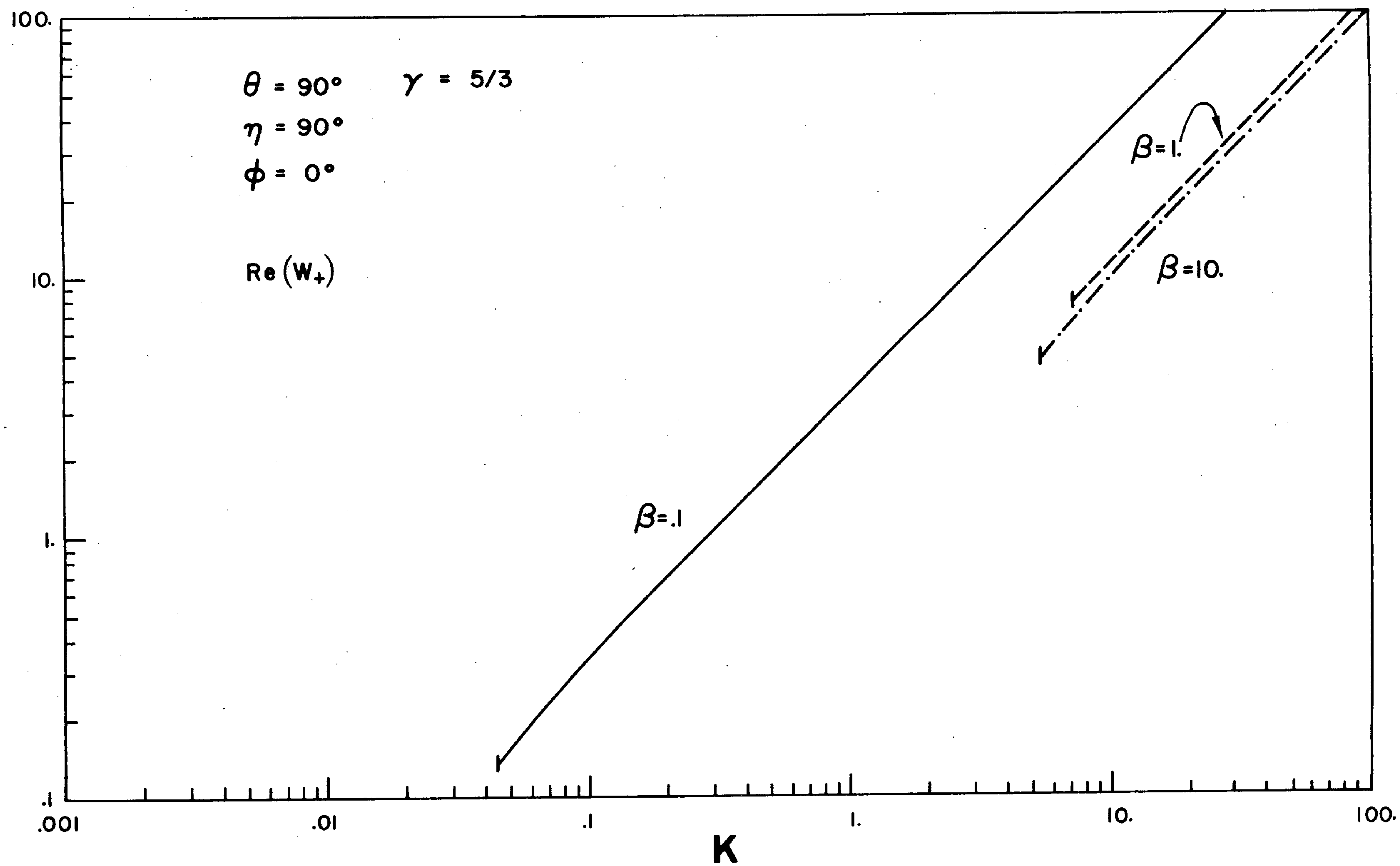
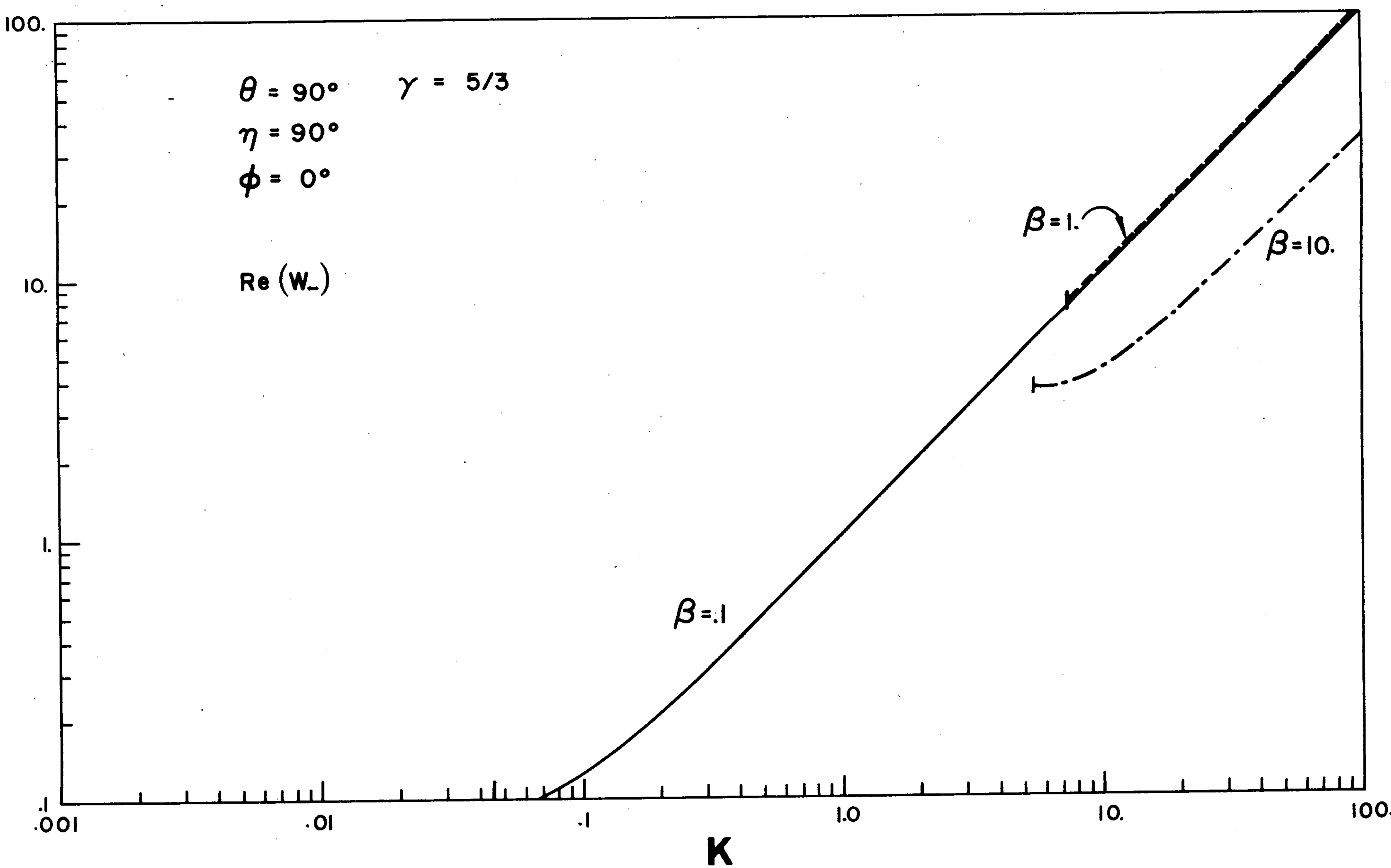


Figure 1.



$\theta = 90°$ $\gamma = 5/3$
$\eta = 90°$
$\phi = 0°$
Re(W_+)
β=1.
β=10.
β=.1
100.
10.
1.
.1
.001
.01
.1
1.
10.
100.
K

Figure 2.



100.
10.
1.
.1
.001
.01
.1
1.0
10.
100.
K
θ = 90°
γ = 5/3
η = 90°
ϕ = 0°
Re (W₋)
β=1.
β=10.
β=.1

Figure 3.

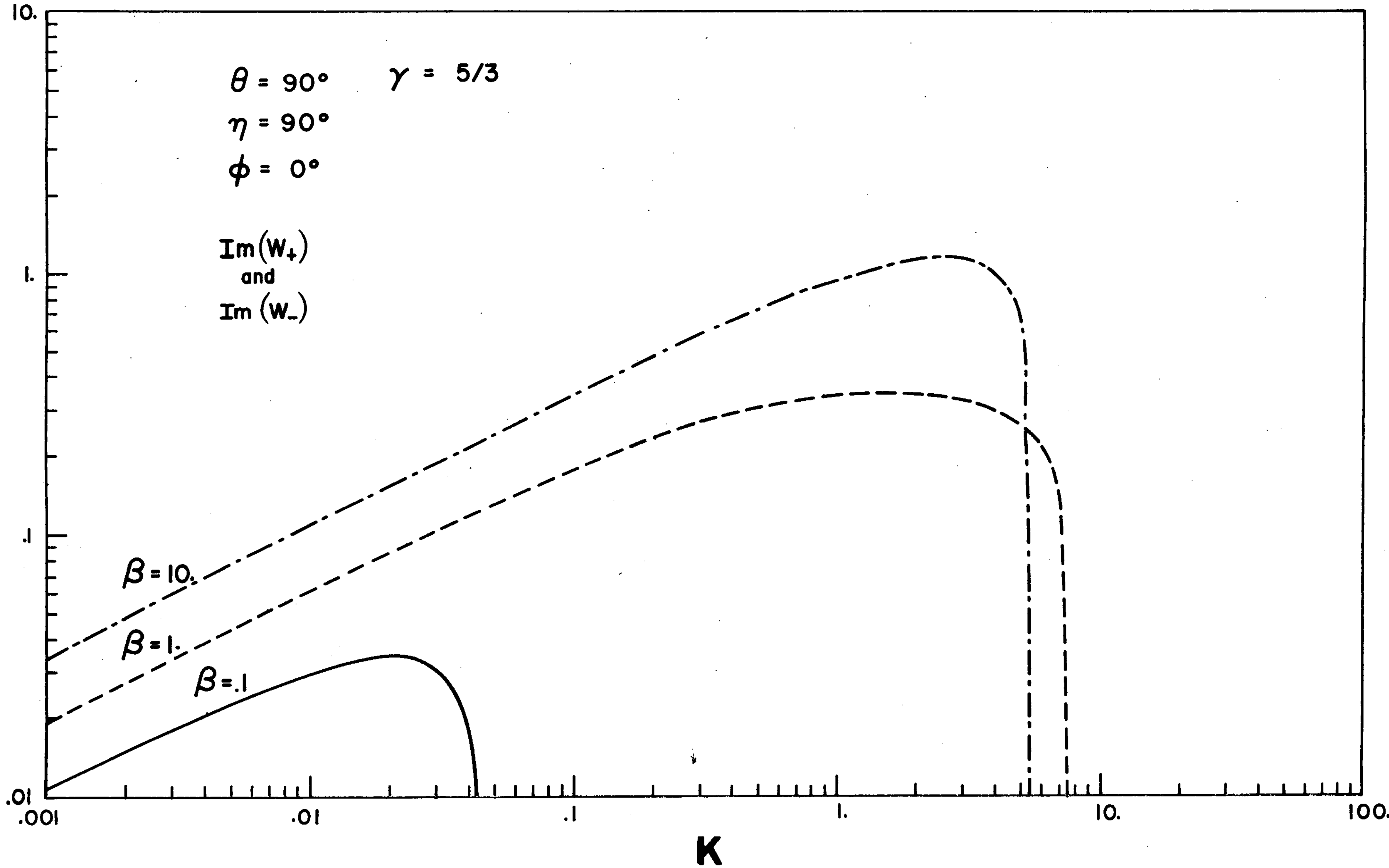


Figure 4.

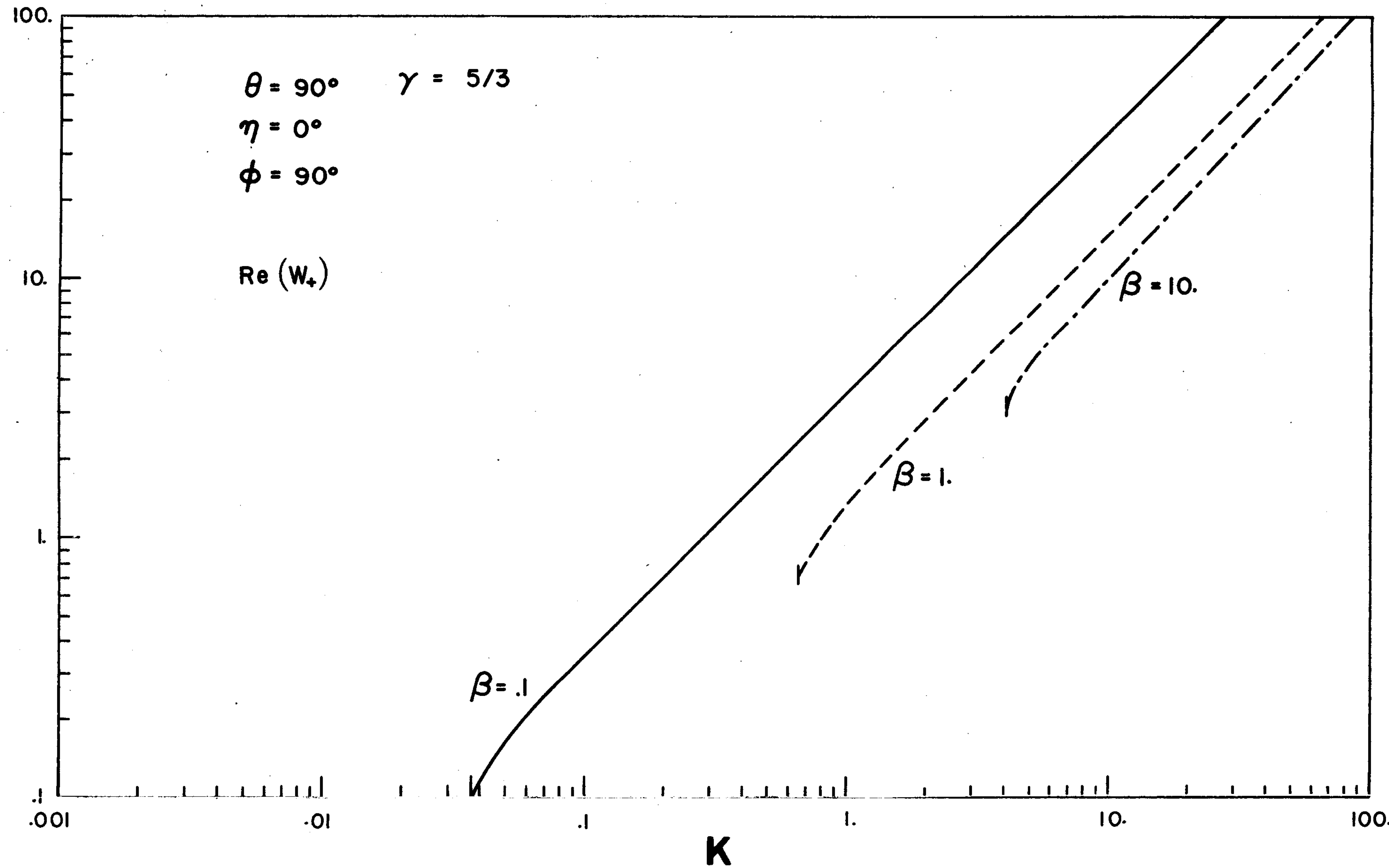


Figure 5.

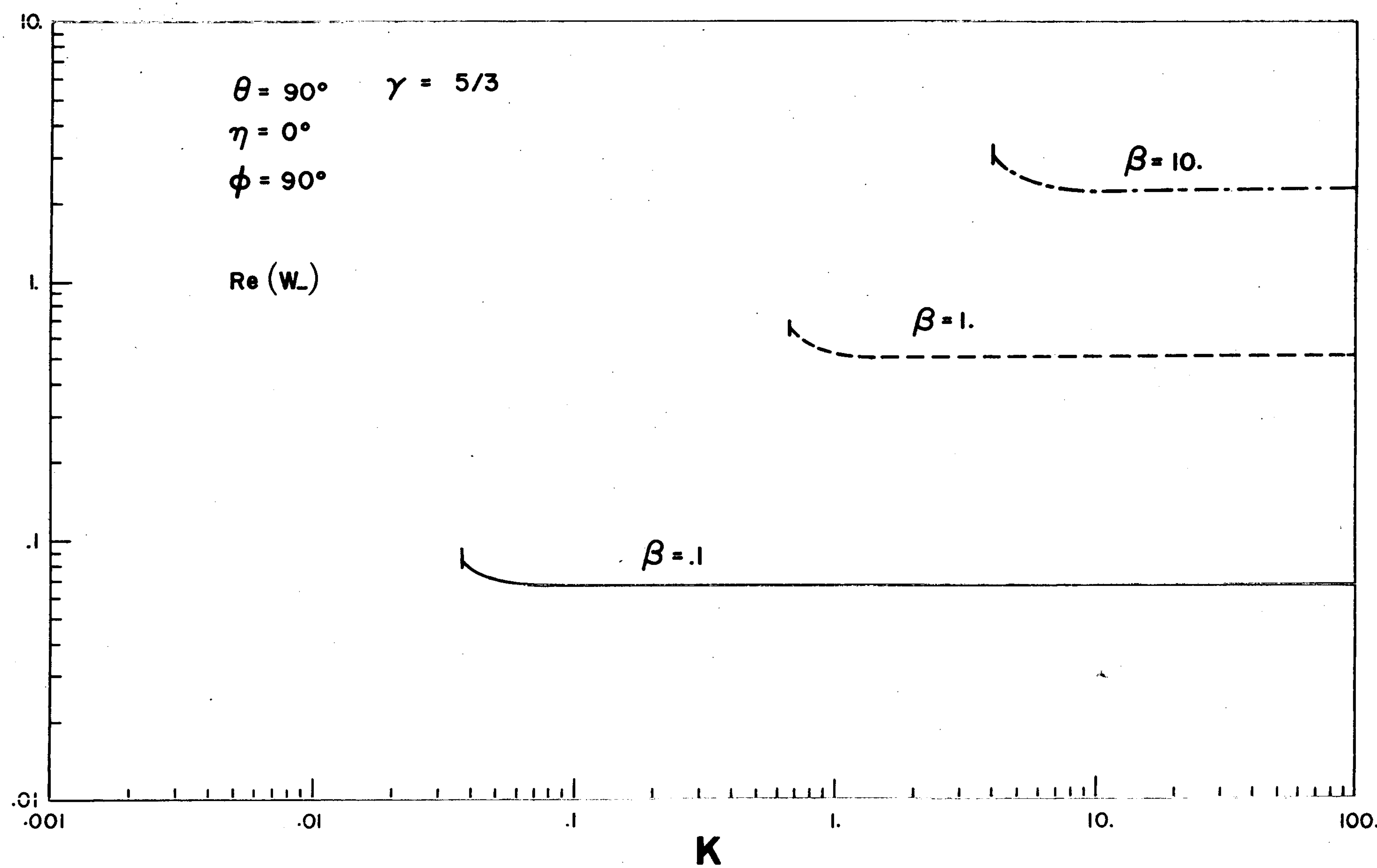


Figure 6.

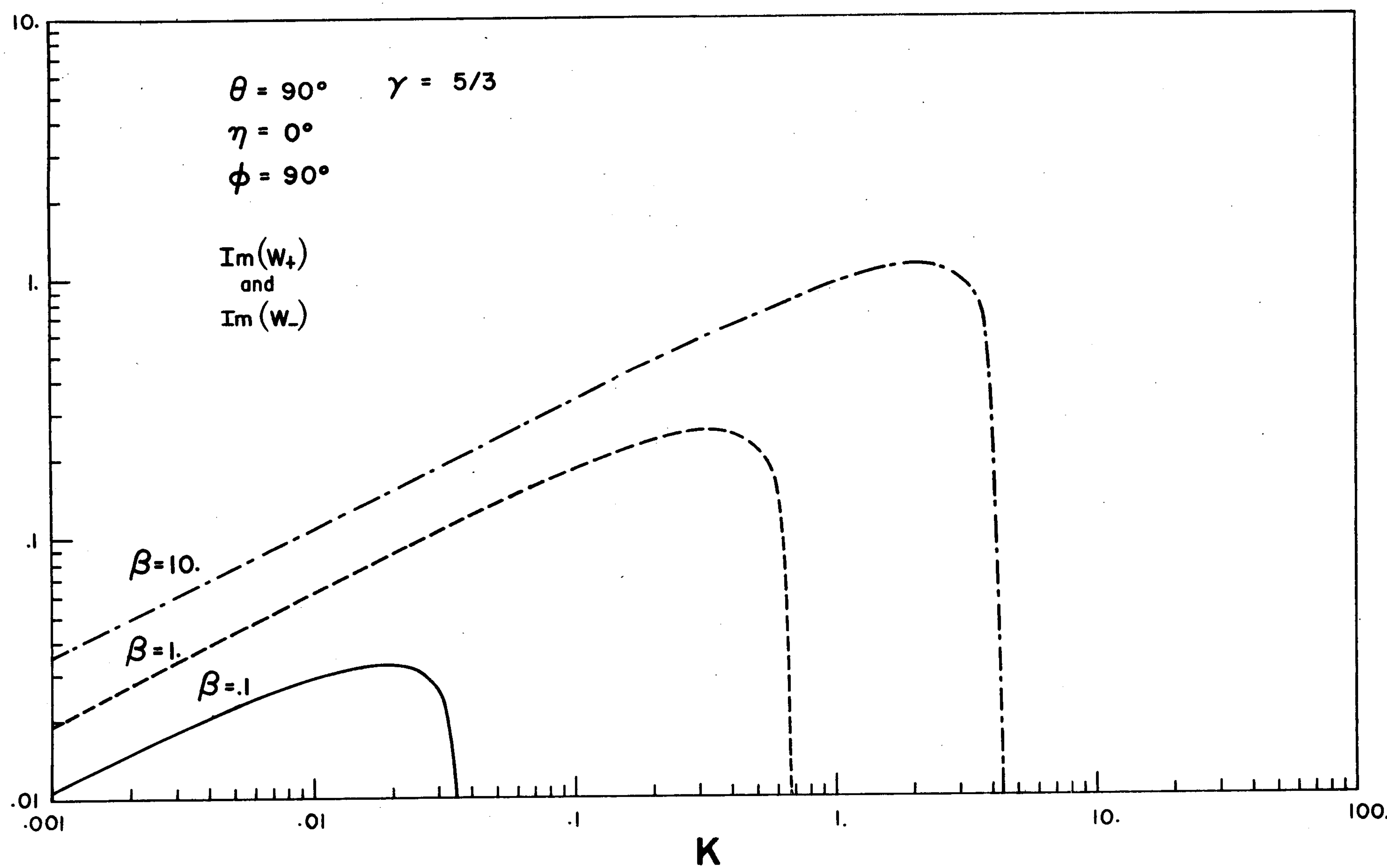


Figure 7.

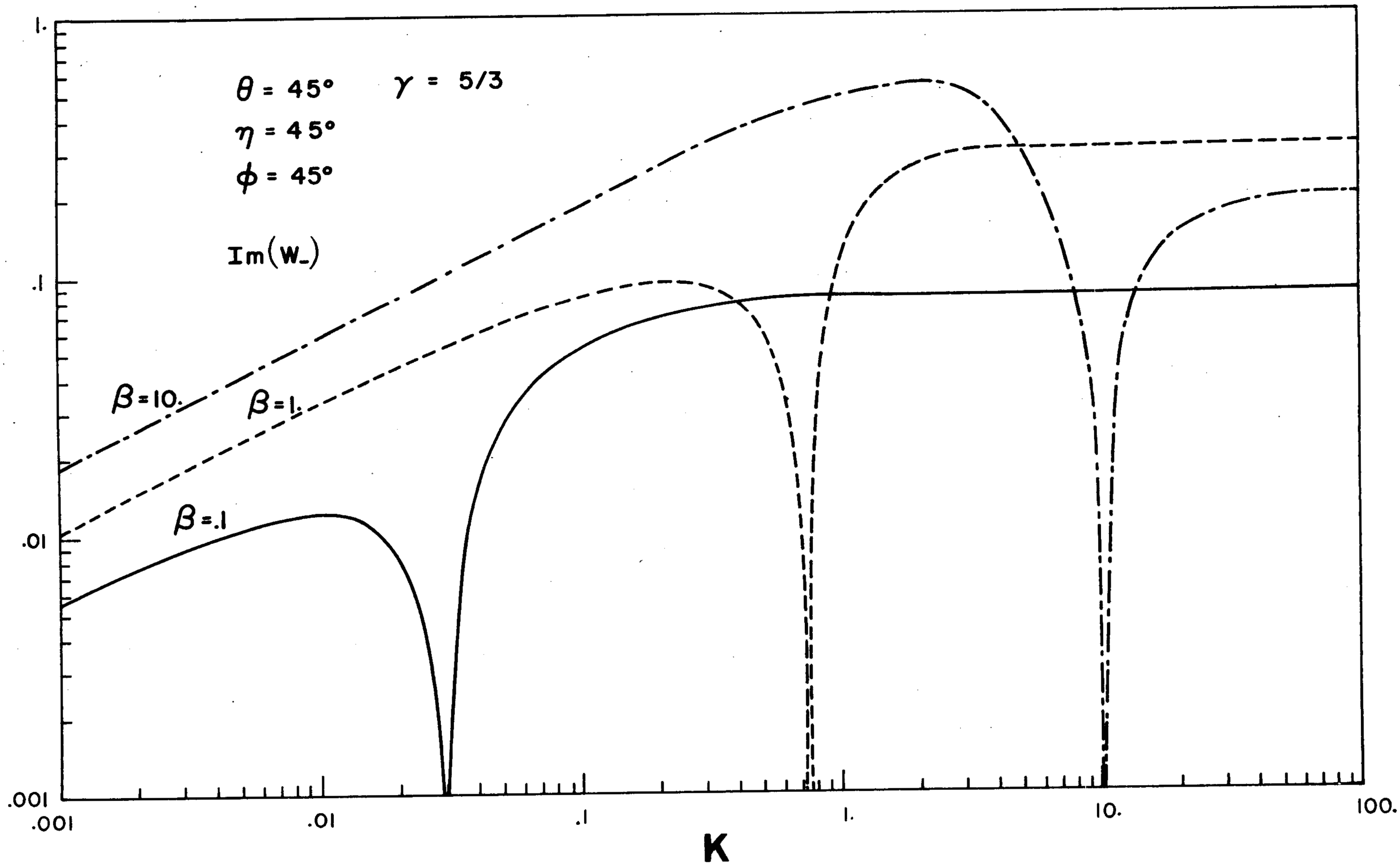


Figure 8.