On the Convergence of Asynchronous Parallel Pattern Search

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Overview of Proof

Part I

- Every process has a subsequence of the step lengths converging to zero

Part II

- All the processors share a common accumulation point which has s a zero gradient

Notation

Processes:	$\mathcal{P} = \{1, \ldots, p\}$
Search Directions:	$\mathcal{D} = \{d_1, \ldots, d_p\}$
Global Time Index:	$\mathcal{T} = \{0, 1, 2, \ldots\}$
Best Known Point:	x_i^t (at time t for process i)
Step Length Control:	Δ_i^t (at time t for process i)
Successful Time Steps:	$\mathcal{S}_i$ (for process i)
Contraction Time Steps:	$\mathcal{C}_i$ (for process i)

Rational Lattice

- The level set $\mathcal{L}(x^0) = \{x : f(x) \leq f(x^0)\}$ is bounded
- The search directions satisfy $\mathcal{D} = \{d_1, \ldots, d_p\} = \{Bc_1, \ldots, Bc_p\}$ where B is a non-singular $n \times n$ real matrix and $c_i \in \mathbb{Q}^n$
- The step length are updated as follows $\Delta_i^{t^+} = \Lambda^k \Delta_i^{t^0}$ where Λ is rational and k is a nonngative integer for expansion and a negative integer for contraction.

$$\Rightarrow \quad x_i^t = x^0 + \alpha B \sum_{j \in \mathcal{P}} \zeta_j(i, t, \Gamma)\, c_j$$

with α constant and $\zeta_j(i, t, \Gamma) \in \mathbb{Z}$

Reducing the Step Length

Goal: $\liminf_{t\to+\infty} \Delta_j^t = 0$ for all $j \in \mathcal{P}$.

Lemma: If $\mathcal{S}_i$ is finite for some $i \in \mathcal{P}$, then

$$\lim_{t\to+\infty} \Delta_i^t = 0.$$

Lemma: If $\mathcal{S}_i$ is finite for some i, then $\mathcal{S}_j$ is finite for all $j \in \mathcal{P}$.

$\Rightarrow$ *Finite case is trivial.*

Corollary: If $\mathcal{S}_i$ is infinite for some $i \in \mathcal{P}$, then $\mathcal{S}_j$ is infinite for all $j \in \mathcal{P}$.

One Step Length Goes To Zero

Lemma: Assume the level set $\mathcal{L}(x_0)$ is bounded. Suppose $\mathcal{S}_j$ is infinite for all $j \in \mathcal{P}$, then there exists $i \in \mathcal{P}$ such that

$$\liminf_{\substack{t\to+\infty \\ t\in\mathcal{S}_i}} \Delta_{\omega_i(t)}^{\tau_i(t)} = 0.$$

Finite rational lattice argument — Torczon, 1997.

Corollary: Suppose $\mathcal{S}_j$ is infinite for all $j \in \mathcal{P}$, then there exists $i \in \mathcal{P}$ such that

$$\liminf_{t\to+\infty} \Delta_i^t = 0.$$

All Step Lengths Go To Zero

We assume $\Delta_i^t \geq \Delta_{\min}$ for all $t \in \mathcal{S}_i$.
Let process i be such that

$$\liminf_{t \to +\infty} \Delta_i^t = 0,$$

Key: *The supremum of the time between successful iterates on process i goes to $+\infty$.*

So, on any other process j, the supremum of the number of contractions between successful iterates is also going to $+\infty$.

Theorem 1: $\liminf_{t \to +\infty} \Delta_j^t = 0 \quad \text{for all} \quad j \in \mathcal{P}.$

Accumulation Point

Goal: There exists $\hat{x} \ni \lim_{\substack{t \to +\infty \\ t \in \hat{\mathcal{C}}_j}} x_j^t = \hat{x}$ for all $j \in \mathcal{P}$

Lemma: Assume the set $\mathcal{L}(x_0)$ is bounded. Then there exists $\hat{x} \in \Re^n$ and $\hat{\mathcal{C}}_1 \subseteq \mathcal{C}_1$ such that

$$\lim_{\substack{t \to +\infty \\ t \in \hat{\mathcal{C}}_1}} \Delta_1^t = 0 \quad \text{and} \quad \lim_{\substack{t \to +\infty \\ t \in \hat{\mathcal{C}}_1}} x_1^t = \hat{x}.$$

Just use compactness of closure of $\mathcal{L}(x_0)$

$\Rightarrow$ We have an accumulation point for process 1.

Common Accumulation Point

Theorem 2: There exists $\hat{x}$ and, for each $j \in \mathcal{P}$, $\hat{\mathcal{C}}_j$ such that

$$\lim_{\substack{t \to +\infty \\ t \in \hat{\mathcal{C}}_j}} x_j^t = \hat{x} \text{ for all } j \in \mathcal{P}$$

Key: Forcing $\Delta_i^t \geq \Delta_{\min}$ for every success.

Then for each $\hat{t} \in \hat{\mathcal{C}}_1$ with $\hat{t} > t^*$, there is a corresponding time interval devoid of successful points on each of the other processes, so that they must all eventually accept $x_1^{\hat{t}}$ as the best known point and have a number of contractions.

Search Directions

The pattern must be chosen so that it nonnegatively spans $\Re^n$. *See Lewis and Torczon, 1996.*

Defn: A set of vectors $\{d_1, \ldots, d_p\}$ *nonnegatively spans* $\Re^n$ if any vector $x \in \Re^n$ can be written as

$$x = \alpha_1 d_1 + \cdots + \alpha_p d_p, \quad \alpha_i \geq 0 \quad \forall\, i.$$

That is, any vector can be written as a *nonnegative* linear combination of the basis vectors.

Fact: If $\{d_1, \ldots, d_p\}$ positively spans $\Re^n$, then $d_i^T v \geq 0$ for all $i \in \mathcal{P}$ iff $v = 0$.

Final Result

Theorem 3: Assume f is continuously differentiable. Then

$$\lim_{\substack{t\to+\infty \\ t\in\hat{\mathcal{C}}_i}} \nabla f(x_i^t) = 0.$$

Dates back to Wen-Ci, 1979...

For all $t \in \tau_i(\hat{\mathcal{C}}_i)$, $\exists\ \alpha_i^t \in [0,1]$ such that :

$$f(x_i^t) \leq f(x_i^t + \Delta_i^t d_i) = f(x_i^t) + \Delta_i^t \nabla f(x_i^t + \alpha_i^t \Delta_i^t d_i)^T d_i,$$

$$\Rightarrow 0 \leq \nabla f(x_i^t + \alpha_i^t \Delta_i^t d_i)^T d_i.$$

$$\Rightarrow 0 \leq \nabla f(\hat{x})^T d_i \text{ for all } \quad i \in \mathcal{P}.$$

Since the d-vectors form a positive basis, that implies that $\nabla f(\hat{x}) = 0$.

Summary

Fundamentals haven't changed.

Theorem 1: $\liminf_{t\to+\infty} \Delta_j^t = 0$ for all $j \in \mathcal{P}$.

Theorem 2: Assume $\mathcal{L}(x^0)$ is bounded. Then there exists $\hat{x}$ and, for each $j \in \mathcal{P}$, $\hat{\mathcal{C}}_j$ such that

$$\lim_{\substack{t\to+\infty \\ t\in\hat{\mathcal{C}}_j}} x_j^t = \hat{x} \quad \text{for all } \ j \in \mathcal{P}$$

Theorem 3: Assume f is continuously differentiable. Then

$$\lim_{\substack{t\to+\infty \\ t\in\hat{\mathcal{C}}_j}} \nabla f(x_j^t) = \nabla f(\hat{x}) = 0 \quad \text{for all } \ j \in \mathcal{P}.$$

Reference

K & T, *On the convergence of asynchronous parallel pattern search*, 2001. Submitted to SIOPT.

APPSPACK Software

- PVM/MPI/serial
- Unconstrained or bound constrained
- Lattice or sufficient decrease

Web Pages

http://csmr.ca.sandia.gov/~tgkolda/

http://www.cs.wm.edu/~va/research/