Four Memos for the Next Millennium: A Brief Future of Particle Physics

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Solution to Exercise 2
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2. (a) Using the evolution equations, we may form the combination

$$(8/3) \cdot 1/\alpha_3(M_Z^2) - 1/\alpha(M_Z^2) = \frac{11}{2\pi} \ln \frac{M_Z^2}{u^2}$$

[*Gauge Theories* (9.2.54)] , which does not depend on the number of fermion generations. This gives us

$$u = M_Z \exp \left[\frac{\pi}{11} \left(1/\alpha(M_Z^2) - \left(\frac{8}{3}\right) \cdot 1/\alpha_3(M_Z^2) \right) \right] .$$

With $M_Z = 91.1888$ GeV/c^2, $1/\alpha(M_Z^2) = 129.08$, and $1/\alpha_3(M_Z^2) = 0.116$, we compute

$$u = 1.315 \times 10^{15} \text{ GeV} .$$

The $SU(5)$ coupling constant at the unification scale may be evaluated using [*Gauge Theories* (9.2.42,43)] as

$$\begin{aligned} 1/\alpha_u &= 1/\alpha_s(M_Z^2) + \frac{7}{4\pi} \ln \frac{u^2}{M_Z^2} \\ &= 42.377 . \end{aligned}$$

(b) The weak mixing parameter at the unification scale is

$$x_W(M_Z^2) = \frac{3}{8} + \frac{55\alpha(M_Z^2)}{48\pi} \ln \frac{M_Z^2}{u^2}$$

[*Gauge Theories* (9.2.62)] , so that

$$x_W(M_Z^2) = 0.2038 .$$

This is close to, but not equal to, the value $x_W = 0.2307$ we found from M_Z and $\alpha(M_Z^2)$ in Exercise 1.

(c) Inserting the measured value of x_W into the connection

$$1/\alpha_2(M_Z^2) = x_W(M_Z^2) \cdot 1/\alpha(M_Z^2)$$

[*Gauge Theories* (9.2.60)] leads to

$$1/\alpha_2(M_Z^2) = 29.779 .$$

Then using

$$1/\alpha_Y(M_Z^2) = 1/\alpha(M_Z^2) - 1/\alpha_2(M_Z^2)$$

[*Gauge Theories* (9.2.52)] , we compute

$$1/\alpha_Y(M_Z^2) = 99.301$$

and [*Gauge Theories* (9.2.49)]

$$1/\alpha_1(M_Z^2) = \frac{3}{5} \cdot 1/\alpha_Y(M_Z^2) = 59.581 .$$

We plot the coupling constants as functions of the energy scale Q using the evolution equations

$$1/\alpha_3(Q^2) = 1/\alpha_3(M_Z^2) + \frac{7}{4\pi}\ln\frac{Q^2}{M_Z^2}$$

[*Gauge Theories* (9.2.42,43)] ,

$$1/\alpha_2(Q^2) = 1/\alpha_2(M_Z^2) + \frac{5}{6\pi}\ln\frac{Q^2}{M_Z^2}$$

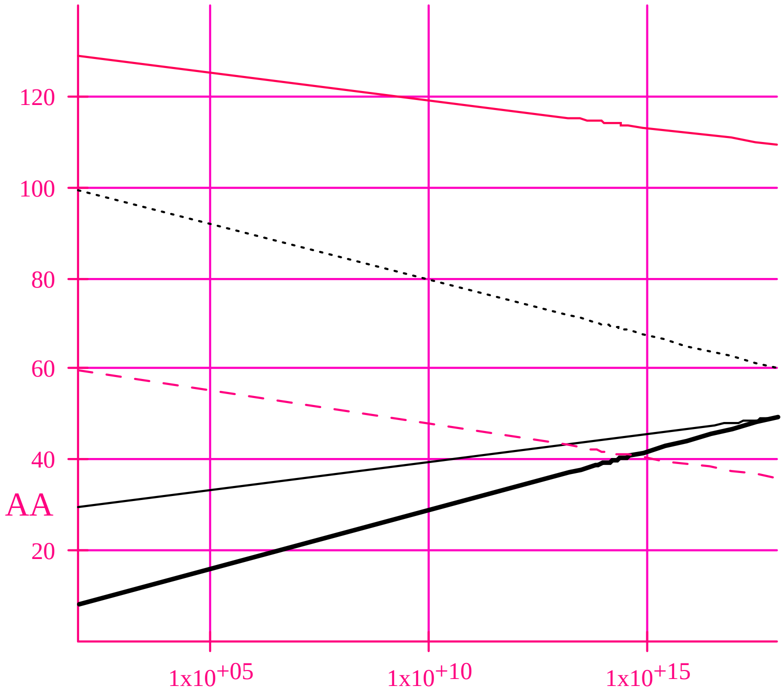
[*Gauge Theories* (9.2.45,46)] ,

$$1/\alpha_Y(Q^2) = 1/\alpha_Y(M_Z^2) - \frac{5}{3\pi}\ln\frac{Q^2}{M_Z^2}$$

[*Gauge Theories* (9.2.49)] , and

$$1/\alpha(Q^2) = 1/\alpha_Y(Q^2) + 1/\alpha_2(Q^2)$$

[*Gauge Theories* (9.2.53)] . The following plot shows $1/\alpha$, $1/\alpha_Y$, $1/\alpha_1 = \frac{3}{5}\cdot 1/\alpha_Y$, $1/\alpha_2$, and $1/\alpha_3$ (from top to bottom at low energy).



The three couplings $1/\alpha_1$, $1/\alpha_2$, and $1/\alpha_3$ do not meet at a single point. This is equivalent to the statement that the $SU(5)$ prediction for the weak mixing parameter does not agree with experiment.

References

C. Quigg, *Gauge Theories of the Strong, Weak, and Electromagnetic Interactions.*
G. G. Ross, *Grand Unified Theories* (Benjamin / Cummings, Menlo Park, CA, 1985)

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The exercises to accompany these lectures will be available on World Wide Web. Point your web client to `http://chrisq.fnal.gov/Memos/Exercises.html`.