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THE DIRAC EQUATION IN A NON-RIEMANNIAN MANIFOLD: II AN ANALYSIS USING AN INTERNAL LOCAL N-DIMENSIONAL SPACE OF THE YANG-MILLS TYPE

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Abstract

The geometrical properties of a flat tangent space-time local to the generalized manifold of the Einstein-Schroedinger non-symmetric theory, with an internal n-dimensional space with the $SU(n)$ symmetry group is developed here. As an application of the theory, it is then obtained a generalized Dirac equation where the electromagnetic and the Yang-Mills fields are included in a more complex field equation. When the 2-dimensional case is considered, the theory can be immediately interpreted through the algebra of quaternions, which, through the Hurwitz theorem, presupposes a generalization of the theory using the algebra of octonions.

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I. Introduction

A geometrical treatment of a gauge theory built to describe particles in the presence of gravitation, electromagnetism and Yang-Mills fields has been developed by some authors since Einstein's attempt to unify gravitation and electromagnetism in his (complex) non-symmetric theory, the so called Einstein-Schroedinger (ES) [1] theory. The Bonnor-Moffat-Boal (BMB) [2] theory was successful in obtaining a correct limit to the Einstein-Maxwell theory, and the Borchsenius [3] theory used the same principle to include the Yang-Mills field. Even though these theories have been criticized [4] and the "physical limit" has not been convincing, they are attractive from the point of view of a geometrical treatment of gravitation plus gauge theory. Also, they permit the extension to an octonionic theory, through a theorem of Hurwitz [5,6]. However, given the present status of actual unified theories, the use of such a theory is not yet clear, but at a minimum, it constitutes an attempt in making useful some mathematical tools such as algebra and symmetry properties in a (geometrical) unified theory on the curved space-time.

The main goal of the present work is to obtain the Dirac equation for a spin-1/2 particle placed locally to a curved space-time and in the presence of gravitation, electromagnetism and Yang-Mills fields, using the ES non-symmetrical theory (see ref.[11]). To achieve this it is necessary to introduce an n-dimensional internal space to the (complex) space-time manifold of the ES theory[1]. As we are interested in working with Yang-Mills fields we use here the internal space of the $n \times n$ matrices, with $SU(n)$ as the internal symmetry group, as in the Borchsenius theory [3]. Every object in this internal space can be expanded in terms of n^2 linearly independent matrices, $\{\tau_0, \tau_i, i = 1, 2, ..., (n^2-1)\}$, where $\tau_0 \equiv \mathbf{1}_{n\times n}$ and $\tau_i^\dagger = \tau_i$. The line element is defined on this extended manifold as:

$$ds^2 = \frac{1}{n} Tr(G_{\mu\nu} dx^\mu dx^\nu) \quad , \tag{1.1}$$

where,

$$G_{\mu\nu} = (G_{\mu\nu}{}^a{}_b(x)) \quad , \qquad a, b = 1, ..., n \tag{1.2}$$

is a matrix in the internal space such that,

$$\frac{1}{n} Tr G_{\mu\nu} = g_{\mu\nu} \quad , \tag{1.3}$$

[1]The notation used in this work is about the same as used in refs. [6] and [11].

with $g_{\mu\nu}$ being the metric of the ES non-symmetric theory. It is also imposed that:

$$G^{\dagger}_{\mu\nu} = G_{\nu\mu} \quad , \tag{1.4}$$

where the Hermitian conjugation operates on the internal matrix indices. There exists an inverse $G^{\mu\nu}$ such that:

$$G_{\mu\alpha}G^{\mu\nu} = G^{\nu\mu}G_{\alpha\mu} = \delta^{\nu}_{\alpha}\mathbf{1} \tag{1.5}$$

where the order of factors is important and where Eqs. (1.3) and (1.4) were used. The metric $G_{\mu\nu}$ can be written as :

$$G_{\mu\nu} = q_{\mu\nu 0}\tau_0 + q_{\mu\nu i}\tau_i \quad , \qquad i = 1, 2, ..., (n^2 - 1) \quad , \tag{1.6}$$

where, following conditions (1.3) and (1.4), $q_{\mu\nu_0}$ is the metric on the manifold of the ES or the BMB theory, which includes the electromagnetism through the Maxwell tensor $F_{\mu\nu}$:

$$q_{\mu\nu 0} = g_{\mu\nu} = g_{\underline{\mu\nu}} + ipF_{\underset{\vee}{\mu\nu}} \tag{1.7}$$

and $q_{\mu\nu_i}$ should be of the Yang-Mills type:

$$q_{\mu\nu i} = \frac{1}{2}\frac{i\epsilon p^2}{\hbar} f_{\underset{\vee}{\mu\nu}} \quad , \tag{1.8}$$

where ϵ is the elementary isotopic charge when $n = 2$. The constant p is defined such that in the limit $p \to 0$, the field equations and the geometrical properties of the Eisntein-Maxwell-Yang-Mills theory are obtained (see refs. [2] and [3]). Its value is given as $p = -\frac{2\hbar}{\epsilon}$, $|p| = 3.8 \times 10^{-32} cm$, $(c = G = 1)$.

The properties of covariant derivatives on the manifold for the ES non-symmetrical manifold, state that the space-time connection is such that

$$\Omega^{\mu}{}_{\alpha\nu} = \Omega^{*\mu}{}_{\nu\alpha} = \Omega^{\mu}{}_{\underline{\alpha\nu}} + iK^{\mu}{}_{\underset{\vee}{\alpha\nu}} \quad . \tag{1.9}$$

To obtain the field equations through a minimal action principle, we have also to define the Schroedinger connection:

$$\theta^{\rho}{}_{\mu\nu} = \Omega^{\rho}{}_{\mu\nu} - \frac{2}{ip}\delta^{\rho}_{\mu}A_{\nu} \quad , \tag{1.10}$$

where A_{ν} is the electromagnetic vectot potential and can be written in terms of the vector-torsion $\Omega^{\rho}{}_{\underset{\vee}{\rho\nu}}$ as:

$$A_{\nu} = -\frac{1}{3}(ip)\Omega^{\rho}{}_{\underset{\vee}{\rho\nu}} \quad . \tag{1.11}$$

Taking an internal vector $\psi^a = \psi^a(x)$, $a = 1...n$, the internal covariant derivative is given by:

$$\psi^a{}_{||\mu} = \psi^a{}_{,\mu} + \Gamma_\mu{}^a{}_b \psi^b \quad , \tag{1.12}$$

where Γ_μ is the internal connection. Γ_μ is taken here to be of the Yang-Mills form:

$$\Gamma_\mu = -\frac{i\epsilon}{\hbar}\vec{b}_\mu . \vec{\tau} \quad . \tag{1.13}$$

The internal curvature is then obtained through the difference:

$$\psi^a{}_{||\mu\nu} - \psi^a{}_{||\nu\mu} = P_{\mu\nu}{}^a{}_b \psi^b \quad , \tag{1.14}$$

where $P_{\mu\nu}$ is the internal curvature, given by:

$$P_{\mu\nu} = \Gamma_{\mu,\nu} - \Gamma_{\nu,\mu} - [\Gamma_\mu, \Gamma_\nu] \quad . \tag{1.15}$$

An object $K = (K^a{}_b)$ with two internal matrix indices transforms then, as:

$$K' = U^\dagger K U = U K U^\dagger \quad , \tag{1.16}$$

where, since the symmetry group is $SU(n)$, the transformation matrices U are unimodular matrices: $U^\dagger = U^{T*} = U^{-1}$, $detU = 1$.

A total covariant derivative of a space-time vector $V^\mu(x)$ can be obtained through the parallel transport of this vector on the extended space as:

$$V^\mu{}_{|\overset{\alpha}{+}}(x) = V^\mu{}_{,\alpha} + \Omega^\mu{}_{\rho\alpha} V^\rho + [\Gamma_\alpha, V^\mu] \quad . \tag{1.17}$$

A "total curvature" is then obtained through the difference:

$$V^\mu{}_{|\overset{\alpha\rho}{++}} - V^\mu{}_{|\overset{\alpha\rho}{++}} = \mathbf{R}^\mu{}_{\lambda\alpha\rho} V^\lambda - V^\mu P_{\alpha\rho} - 2V^{\overset{\mu}{+}}{}_{|\lambda} \Omega^\lambda_{\underset{\vee}{\alpha\rho}} \quad , \tag{1.18}$$

where $V^\mu(x)$ can be wirtten in terms of internal components as:

$$V^\mu(x) = v_0^\mu(x)\tau_0 + v_i^\mu(x)\tau_i \quad , \qquad i = 1, 2, 3 \quad .$$

The total curvature $\mathbf{R}^\mu{}_{\lambda\alpha\rho}$ gives the mixture of the space-time and internal curvatures:

$$\mathbf{R}^\mu{}_{\lambda\alpha\rho} = (\mathbf{\Gamma}^\mu{}_{\lambda\alpha,\rho} + \mathbf{\Gamma}^\mu{}_{\nu\rho}\mathbf{\Gamma}^\nu{}_{\lambda\alpha}) - (\mathbf{\Gamma}^\mu{}_{\lambda\rho,\alpha} + \mathbf{\Gamma}^\mu{}_{\nu\alpha}\mathbf{\Gamma}^\nu{}_{\lambda\rho})$$

$$= R^{\mu}{}_{\lambda\alpha\rho} + \delta^{\mu}_{\lambda} P_{\alpha\rho} \quad . \tag{1.19}$$

with

$$\mathbf{\Gamma}^{\rho}{}_{\nu\alpha} = \Omega^{\rho}{}_{\nu\alpha}\tau_0 + \delta^{\rho}_{\nu}\Gamma_{\alpha} \quad .$$

To obtain the field equations for the extended theory, we use the Palatini variational method. The action is:

$$\mathcal{A} = \int \mathcal{L} d^4x \quad ,$$

where the Lagrangian $\mathcal{L}$ is taken as:

$$\mathcal{L} = Tr(\mathcal{G}^{\mu\nu}\mathbf{R}_{\mu\nu} + \frac{1}{(ip)^2}\mathcal{G}^{\overset{\mu\nu}{\vee}} G_{\underset{\vee}{\mu\nu}}) \quad , \tag{1.20}$$

where $\mathbf{R}_{\mu\nu} = \mathbf{R}^{\rho}{}_{\mu\nu\rho}$ by (1.19), and,

$$\mathcal{G}^{\mu\nu} = wG^{\mu\nu} \quad , \tag{1.21}$$

$$w = [-\frac{1}{n}Tr(detG_{\mu\nu})]^{\frac{1}{2}}$$

The field equations obtained on this extended manifold are then [2]:

$$\mathcal{G}^{\overset{\mu\nu}{+-}}{}_{|\alpha} = \mathcal{G}^{\mu\nu}{}_{,\alpha} - \mathcal{G}^{\rho\nu}\theta^{\mu}{}_{\rho\alpha} - \mathcal{G}^{\mu\rho}\theta^{\nu}{}_{\alpha\rho} + \mathcal{G}^{\mu\nu}\theta^{\rho}_{\underline{\rho\alpha}} - [\Gamma_{\alpha}, \mathcal{G}^{\mu\nu}] = 0 \quad , \tag{1.22}$$

$$\mathcal{G}^{\overset{\mu\nu}{\vee}}{}_{,\alpha} = 0 \quad , \tag{1.23}$$

$$^{*}\mathbf{R}_{\underline{\mu\nu}}(\theta) = 0 \quad , \tag{1.24}$$

$$^{*}\mathbf{R}_{\underset{\vee}{\mu\nu},\sigma}(\theta) + {}^{*}\mathbf{R}_{\underset{\vee}{\sigma\mu},\nu}(\theta) + {}^{*}\mathbf{R}_{\underset{\vee}{\nu\sigma},\mu}(\theta) = 0. \tag{1.25}$$

Equation (1.25) is a consequence of the fact that:

$$^{*}\mathbf{R}_{\underset{\vee}{\mu\nu}}(\theta) = \frac{2}{3}(\Omega_{\mu,\nu} - \Omega_{\nu,\mu}) + \Gamma_{\nu,\mu} - \Gamma_{\mu,\nu} + [\Gamma_{\mu}, \Gamma_{\nu}] \quad . \tag{1.26}$$

In the above equations the argument θ in $\mathbf{R}_{\mu\nu}(\theta)$, means that the expression for the generalized Ricci tensor is written in terms of the Schroedinger connection. Also, in (1.25) and (1.26), we have:

$$^{*}\mathbf{R}_{\mu\nu}(\theta) = \mathbf{R}_{\mu\nu}(\theta) + I_{\mu\nu} \quad , \tag{1.27}$$

[2] In Eq. (1.22) the notation used for the covariant derivative of $\mathcal{G}^{\mu\nu}$ is the usual when it is given in terms of the Schroedinger connection $\theta^{\rho}{}_{\mu\nu}$.

where

$$I_{\mu\nu} = \frac{1}{(ip)^2}(G_{\mu\sigma}G^{\overset{\sigma\rho}{\vee}}G_{\rho\nu} + \frac{1}{2}G_{\mu\nu}G_{\sigma\rho}G^{\overset{\sigma\rho}{\vee}} + G_{\underset{\vee}{\mu\nu}}) \quad , \tag{1.28}$$

We proceed now to the next section where the properties of a tangent space on this extended manifold will be presented.

II. The n-dimensional complex tangent space

A local tangent space can be defined on this extended space-time manifold through a generalized correspondence principle [7]. It is also supposed that this tangent space has attached to it the same n-dimensional internal space.

Define $n \times n$ matrix vierbeins $E^a_\mu(x)$ such that:

$$G_{\mu\nu} = E^{\dagger a}_{\nu} E^b_\mu \eta_{ab} \quad , \tag{2.1}$$

Then, according to the correspondence principle [8] generalized to this case, the line element can be written in both spaces as:

$$ds^2 = \frac{1}{n}Tr(G_{\mu\nu}dx^\mu dx^\nu) = \frac{1}{n}Tr(\eta_{ab}d\mathbf{x}^{\dagger a}d\mathbf{x}^b) \quad , \tag{2.2}$$

$$d\mathbf{x}^a = E^a_\mu dx^\mu \qquad d\mathbf{x}^{\dagger a} = E^{\dagger a}_{\mu} dx^\mu$$

where, aiming toward a physical interpretation, the metric on the tangent space is taken with the structure of the Minkowski metric η_{ab} .

As there exists an inverse $G^{\mu\nu}$ such that (1.5) is true, we must have:

$$G^{\mu\nu} = E^{\dagger\mu}_{a} E^\nu_b \eta^{ab} \quad , \tag{2.3}$$

From this, we obtain the corresponding orthogonality relations for the matrix vierbeins:

$$E^b_\mu E^{\dagger\mu}_{c} = E^\mu_c E^{\dagger b}_{\mu} = \delta^b_c \tau_0 \quad ,$$

$$E^{\dagger a}_{\alpha} E^\nu_a = E^{\dagger\nu}_{a} E^a_\alpha = \delta^\nu_\alpha \tau_0 \quad . \tag{2.4}$$

The vierbeins can be developed through the internal basis, for example taking $E^a_\mu(x)$, as:

$$E^a_\mu = k^a_{\mu 0}(x)\tau_0 + k^a_{\mu i}(x)\tau_i \quad ,$$

and,

$$E^{\dagger a}_{\mu} = k^{*a}_{\mu 0}(x)\tau_0 + k^{*a}_{\mu i}(x)\tau_i \quad , \tag{2.5}$$

since $\tau_i^{\dagger} = \tau_i$.

The transformation law for vectors on the tangent space is defined as usual through the Lorentzian rotation matrices, $L^a{}_b$, such that $L^T \eta L = \eta$. Therefore, a more general transformation law for the matrix tangent vectors $E^a_\mu(x)$ shall be now:

$$E'^a_\mu(x) = L^a{}_b(x)(U(n)E^b_\mu(x)U^{\dagger}(n)) \quad . \tag{2.6}$$

We can define now, on this matrix tangent space, the operation of covariant differentiation, for example, on the vector $E = (E^\mu_a)$:

$$E^{\overset{\mu}{+}}_{a|\nu} = E^\mu_{a,\nu} + \Omega^\mu_{\rho\nu}E^\rho_a - \Lambda_\nu{}^c{}_a E^\mu_c + [\Gamma_\nu, E^\mu_a] \quad . \tag{2.7}$$

It is important to remember that the space-time connection, $\Omega^\mu{}_{\rho\nu}$, may include an (internal) complex connection related to the electromagnetic potential-vector A_ν, through the relations (1.9) and (1.10). Using the notation of ref. [6], it will be called here C_ν which, by (1.10), is given by:

$$C_\nu = -\frac{2}{ip}A_\nu = -\frac{ie}{\hbar}A_\nu \quad .$$

Therefore, the expressions corresponding to the field equation $G_{\underset{+-}{\mu\nu};|\alpha} = 0$ (and its inverse $G^{\overset{\mu\nu}{+-}}{}_{|\alpha} = 0$), for the matrix vierbeins, are as follows:

$$G_{\underset{+-}{\mu\nu};|\alpha} = 0 \quad \longleftrightarrow E^{\dagger a}_{\underset{-}{\mu};|\alpha} = (E^a_{\underset{+}{\mu};|\alpha})^{\dagger} = 0 \quad , \tag{2.8}$$

$$E^a_{\underset{+}{\mu};|\alpha} = E^a_{\mu,\alpha} - E^a_\rho \mathbf{\Gamma}^\rho{}_{\mu\alpha} + \mathbf{\Lambda}_\alpha{}^a{}_c E^c_\mu = 0 \quad ,$$

$$\mathbf{\Gamma}^\rho{}_{\mu\alpha} = \theta^\rho{}_{\mu\alpha}\tau_0 + \delta^\rho_\mu \Gamma_\alpha \quad , \qquad \Gamma_\alpha = -\frac{i\epsilon}{\hbar}\vec{b}_\alpha . \vec{\tau} = -\Gamma^{\dagger}_\alpha \quad ,$$

$$\mathbf{\Lambda}_\alpha{}^a{}_c = (\Lambda_\alpha{}^a{}_c + \delta^a_c C_\alpha)\tau_o + \delta^a_c \Gamma_\alpha \quad ;$$

$$G^{\mu\nu}_{\stackrel{+-}{|\alpha}} = 0 \longleftrightarrow E^{\dagger\stackrel{\mu}{+}}_{a|\alpha} = (E^{\stackrel{\mu}{-}}_{a|\alpha})^{\dagger} = 0 \quad , \tag{2.9}$$

$$E^{\stackrel{\mu}{-}}_{a|\alpha} = E^{\mu}_{a,\alpha} + E^{\rho}_{a}\Gamma^{\dagger\mu}{}_{\rho\alpha} - \Lambda^{\dagger}{}_{\alpha}{}^{c}{}_{a}E^{\mu}_{c} = 0 \quad ,$$

$$\Gamma^{\dagger\mu}{}_{\rho\alpha} = \theta^{\mu}{}_{\alpha\rho}\tau_0 - \delta^{\mu}_{\rho}\Gamma_{\alpha} \quad , \text{ because } \quad \theta^{*\mu}{}_{\rho\alpha} = \theta^{\mu}{}_{\alpha\rho} \quad ,$$

$$\Lambda^{\dagger}{}_{\alpha}{}^{a}{}_{b} = (\Lambda_{\alpha}{}^{a}{}_{b} - \delta^{a}_{b}C_{\alpha})\tau_0 \quad , \text{ because } C_{\alpha} = -\frac{ie}{\hbar}A_{\alpha} = -C^{*}_{\alpha} \quad .$$

From (2.8) and (2.9) we can obtain a new expression for Λ_{ν} in terms of the matrix vierbeins,

$$\Lambda_{\gamma}{}^{a}{}_{b} = E^{a}_{\mu}E^{\dagger\mu}_{b,\gamma} + E^{a}_{\mu}\Gamma^{\mu}{}_{\rho\gamma}E^{\dagger\rho}_{b} \quad ,$$

$$= E^{a}_{\mu}E^{\dagger\stackrel{\mu}{+}}_{b;\gamma} + E^{a}_{\mu}\Gamma_{\gamma}E^{\dagger\mu}_{b} \tag{2.10}$$

and

$$\Lambda_{\gamma}{}^{a}{}_{b} = -E^{a}_{\mu,\gamma}E^{\dagger\mu}_{b} + E^{a}_{\rho}\Gamma^{\rho}{}_{\mu\gamma}E^{\dagger\mu}_{b} \quad ,$$

$$= -E^{a}_{\stackrel{\mu}{+};\gamma}E^{\dagger\mu}_{b} + E^{a}_{\rho}\Gamma^{\rho}{}_{\mu\gamma}E^{\dagger\mu}_{b} \quad . \tag{2.11}$$

The tangent space-time connection Λ_{γ} can be then written in this theory as:

$$\Lambda_{\gamma}{}^{a}{}_{b} = Re\{\frac{1}{n}Tr[E^{a}_{\mu}E^{\dagger\stackrel{\mu}{+}}_{b;\gamma} + E^{a}_{\mu}\Gamma_{\gamma}E^{\dagger\mu}_{b}]\} \quad , \tag{2.12}$$

or

$$\Lambda_{\gamma}{}^{a}{}_{b} = Re\{\frac{1}{n}Tr[-E^{a}_{\stackrel{\mu}{+};\gamma}E^{\dagger\mu}_{b} + E^{a}_{\mu}\Gamma_{\gamma}E^{\dagger\mu}_{b}]\} \quad . \tag{2.13}$$

The expression that relates the curvatures in the curved and the tangent spaces is now:

$$E^{a}_{\rho}\mathbf{R}^{\rho}{}_{\mu\nu\gamma} - \mathbf{S}_{\nu\gamma}{}^{a}{}_{c}E^{c}_{\mu} = 0 \quad , \tag{2.14}$$

where $\mathbf{R}^{\rho}{}_{\mu\nu\gamma}$ is the total curvature (1.19), written with the "connections" $\mathbf{\Gamma}^{\rho}{}_{\mu\nu}$, and $\mathbf{S}_{\mu\nu}$ is the total curvature on the tangent space written with the "connections" Λ_{ν},

$$\mathbf{S}_{\nu\gamma}{}^{a}{}_{c} = (\mathbf{\Lambda}_{\nu,\gamma} - \mathbf{\Lambda}_{\gamma,\nu} - [\mathbf{\Lambda}_{\nu}, \mathbf{\Lambda}_{\gamma}])^{a}{}_{c} \quad ,$$

$$= [S_{\nu\gamma}{}^{a}{}_{c} + \delta^{a}_{c}(C_{\nu,\gamma} - C_{\gamma,\nu})]\tau_0 + \delta^{a}_{c}P_{\nu\gamma} \quad , \tag{2.15}$$

where $S_{\nu\gamma}$ is the curvature written with the tangent connection Λ_{ν}, and $P_{\nu\gamma}$ is the internal curvature written for the internal connection γ_{ν}. Also, the quantity $(C_{\nu,\gamma} - C_{\gamma,\nu})$ corresponds to the curvature of an internal (complex) space. It is here related to the electromagnetic tensor $F_{\nu\gamma}$.

III. The generalization of the Fock-Ivanenko coefficients.

We can obtain a new generalized set of Dirac equations when we extend the treatment from the curved space-time of General Relativity to this generalized matrix one. The anticommutation relations for the Dirac constant γ-matrices [9] are:

$$\{\gamma_a, \gamma_b\} = 2\eta_{ab}\mathbf{1}_4 \quad , \tag{3.1}$$

$$\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbf{1}_4 \quad , \tag{3.2}$$

Multiplying (3.1) by E_ν^{*a} and E_μ^b, and using (2.1), we obtain:

$$Tr\{\gamma_\mu, \dot{\gamma}_\nu\} = 2Tr(G_{\mu\nu})\mathbf{1}_4 = 2ng_{\mu\nu}\mathbf{1}_4 \quad , \tag{3.3}$$

where the Tr is taken on the n-dimensional matrix internal space, and where:

$$E_\mu^a\gamma_a = \gamma_\mu \quad ; \qquad E^{\dagger a}_{\ \mu}\gamma_a = \dot{\gamma}_\mu \quad , \tag{3.4}$$

In (3.3), $g_{\mu\nu}$ is the metric of the ES-non-symmetric theory, by (1.3).

Analogously, multiplying (3.2) by $E^{\dagger\mu}_{\ a}$ and E_b^ν and taking the Tr over the internal n-dimensional matrices, we obtain:

$$Tr\{\dot{\gamma}^\mu, \gamma^\nu\} = 2Tr(G^{\mu\nu})\mathbf{1}_4 = 2ng^{\mu\nu}\mathbf{1}_4 \quad , \tag{3.5}$$

where,

$$E_a^\mu\gamma^a = \gamma^\mu \quad ; \qquad E^{\dagger\mu}_{\ a}\gamma^a = \dot{\gamma}^\mu \quad . \tag{3.6}$$

and (2.3) was used. Considering the non-Riemannian manifold of the ES-Theory, the total covariant derivative of the new γ_μ is given by:

$$\gamma_{\underset{+}{\mu}|\nu} = \gamma_{\mu,\nu} - \Omega^\rho{}_{\mu\nu}\gamma_\rho + [\Delta_\nu, \gamma_\mu] + [\Gamma_\nu, \gamma_\mu] \quad , \tag{3.7}$$

where Δ_μ is the internal connection corresponding to the space of the generalized γ-matrices (or also, the Dirac wave functions space). Taking then (3.4) and (2.8), we have that:

$$\gamma_{\underset{+}{\mu}|\nu} = (E^a_{\underset{+}{\mu}}\gamma_a)_{|\nu} = (E^a_{\underset{+}{\mu}|\nu})\gamma_a = 0 \quad , \tag{3.8}$$

since γ_a is a constant matrix. In the same way, we obtain:

$$\dot{\gamma}_{\underset{-}{\mu}|\nu} = (E^{\dagger a}_{\underset{-}{\mu}}\gamma_a)_{|\nu} = (E^{\dagger a}_{\underset{-}{\mu}|\nu})\gamma_a = 0 \quad . \tag{3.9}$$

Expanding (3.8) and (3.9) we arrive at:

$$\gamma_{\underset{+}{\mu}|\nu} = \gamma_{\mu,\nu} - \theta^\rho{}_{\mu\nu}\gamma_\rho + C_\nu\gamma_\mu + [\Delta_\nu, \gamma_\mu] + [\Gamma_\mu, \gamma_\mu] = 0 \quad , \tag{3.10}$$

and,

$$\dot{\gamma}_{\underset{-}{\mu}|\nu} = \dot{\gamma}_{\mu,\nu} - \theta^\rho{}_{\nu\mu}\dot{\gamma}_\rho - C_\nu\dot{\gamma}_\mu + [\Delta_\nu, \dot{\gamma}_\mu] + [\Gamma_\nu, \dot{\gamma}_\mu] = 0 \quad . \tag{3.11}$$

Therefore, we can obtain an expression for Δ_ν:

$$\Delta_\nu = \frac{1}{4i}\Lambda^{ab}_\nu\sigma_{ab} \tag{3.12}$$

were Λ_ν is given in (2.12) or (2.13). This equation is similar to the corresponding one in General Relativity [10].

We can use then a Minimal Action Principle to obtain field equations for spin a 1/2 particle of mass m, where the wave function is $\psi(x)$, placed in a non-Riemannian manifold of the ES Theory, and also under the influence of an (n-dimensional) Yang-Mills field. The Action for this situation is:

$$A = \int \mathcal{L}\, d^4x \tag{3.13}$$

where the Lagrangian is given by:

$$\mathcal{L} = \sqrt{-w}\{\overline{\psi}\gamma^\mu[\vec{\partial}_\mu + \Delta_\mu + C_\mu + \Gamma_\mu]\psi + \overline{\psi}[\overleftarrow{\partial}_\mu + \Delta_\mu - C_\mu - \Gamma_\mu]\psi\dot{\gamma}^\mu - \mu\overline{\psi}\psi\} \quad , \tag{3.14}$$

where μ is the mass term, and the Tr is taken on the internal n-dimensional space. The function $\psi(x)$ is a complex object that locally transforms under the representation of Lorentz Group ($U(L)$), but also transforms under the (internal) $SU(n)$ group. The field equations obtained are:

$$\gamma^\mu[\vec{\partial}_\mu + \Delta_\mu + C_\mu + \Gamma_\mu]\psi - \mu\psi = 0 \quad , \tag{3.15}$$

and

$$-\overline{\psi}[\overleftarrow{\partial}_\mu + \Delta_\mu - C_\mu - \Gamma_\mu]\dot{\gamma}^\mu - \mu\overline{\psi} = 0 \quad , \tag{3.16}$$

where again, $\overline{\psi}(x) = \psi^\dagger(x)\gamma_0$. Also, we can find an expression for the "charge conjugate" wave function, ψ^c, which is:

$$\dot{\gamma}^\mu[\overrightarrow{\partial}_\mu + \Delta_\mu - C_\mu - \Gamma_\mu]\psi^c - \mu\psi^c = 0 \quad , \tag{3.17}$$

where $\psi^c = C\overline{\psi}^T$, and C is the charge conjugation matrix.

IV. Inclusion of Internal Mass Terms

We are going to analyse now the case of an extended mass-term where we suppose there is non-zero mass on the internal space, i.e., we will suppose that for each internal axis we have a different mass term.

The n^2-dimensional vierbein $E^\mu_a(x)$ can be written as in (2.5):

$$E^\mu_a(x) = k^\mu_a(x)_0\tau_0 + k^\mu_a(x)_i\tau_i \quad , \qquad i = 1, ..., n^2 - 1 \quad . \tag{4.1}$$

Suppose that the mass term μ is a matrix-like term:

$$\mu = \mu_0\tau_0 + \mu_i\tau_i \quad . \tag{4.2}$$

We also are going to assume here that:

$$k^\mu_{a0} = k^\mu_{a0R} + ik^\mu_{a0I} \quad , \tag{4.3}$$

$$\mu_0 = \mu_{0R} + i\mu_{0I} \quad , \tag{4.4}$$

and that k^μ_{ai} and μ_i are pure imaginary numbers. These hypotheses are consistent with the form of the metric defined in (1.6) and (1.7), and with the definition of the matrix-vierbeins in (2.5).

Define

$$k^\mu_{a0I} = p\lambda n^\mu_{a0} \quad , \qquad \mu_{0I} = p\lambda m_0 \quad , \tag{4.5}$$

and

$$k^\mu_{ai} = ip\lambda n^\mu_{ai} \quad , \qquad \mu_i = ip\lambda m_i \quad , \tag{4.6}$$

where now p is being considered a parameter, and λ is a constant with the value of the $1/p$, as in ref. [11]:

$$\lambda \sim \frac{e}{2\hbar} = 2.58 \times 10^{32} cm^{-1} \quad ,$$

where the maximal value for $|p|$ was taken, $|p| = \frac{-2\hbar}{e} = 3.8 \times 10^{-32} cm$, in the normalization used in ref. [3].

Placing the above quantities in the Dirac equation (2.29), we can expand it as:

$$[k^{\mu}_{a0R}\gamma^a \nabla_\mu \psi - \mu_{0R}\psi]\tau_0 + ip\lambda[n^{\mu}_0 \gamma^a \nabla_\mu \psi - m_0 \psi]\tau_0$$

$$+ ip\lambda[n^{\mu}_{ai}\gamma^a \nabla_\mu \psi - m_i \psi]\tau_i = 0 \quad , \tag{4.7}$$

where, $\nabla_\mu = \partial_\mu + \Delta_\mu + C_\mu + \Gamma_\mu$. In the limit of the parameter $p \to 0$, we should get the standard Dirac equation in the presence of gravitation, electromagnetism and Yang-Mills fields. Consequently, we can get n^2 other sets of Dirac equations when we take $n^{\mu}_{a0} = n^{\mu}_{ai} \sim h^{\mu}_a$, and $m_0 = m_i = \mu_{0R}$, for each i, and where h^{μ}_a and μ_{0R} are taken as the vierbeins and the mass term in the General Relativity theory.

Therefore, the above analysis results in some sort of "projections" of the Dirac equation on the internal space, which are due to the definition of more general vierbeins through (2.1). The value of the parameter p will determine then, the amplitude of those projections through (4.5) and (4.6).

V. Conclusion

Taking the complex manifold of the ES non-symmetrical theory, and adding to it an n^2-dimensional internal space, it is possible to develop a generalized theory that, in the case chosen here where we used the $SU(n)$ symmetry group, permitted us to include the $SU(n)$ Yang-Mills field. It is also possible to obtain the tangent space local to that extended manifold. Then, the corresponding generalized Dirac theory as well as the generalized Dirac field equation were developed. In the case of an extended mass-term where we assume there is non-zero mass on each internal axis, and defining the internal components of the vierbeins as well as the internal components of the mass term, as being proportional to the parameter p, we obtained n^2 other sets of Dirac equations which are some sort of projections of the standard Dirac equation on the internal space. The value of the parameter p determines in this theory, the amplitude of these projections through (4.5) and (4.6). In the limit of the parameter $p \to 0$, the standard situation of the General Relativity theory is obtained.

A question arises at this point: where would a theory like this be consistent with the real world? We could just say that this should happen in regions of the space-time with high intensity fields (gravitation, electromagnetism or Yang-Mills fields), and at distances of the order of the Planck length, where it would be reasonable to think of a non-zero p and the consequences of a more complex theory such as the one used in this work.

The present theory can be easily interpreted through a quaternionic theory in the case of $n = 2$. This will enable us to extend it to an octonionic theory, which would be convenient in this case, since we are using a complex non-symmetrical manifold. This is permitted by the theorem of Hurwitz. Thinking from this point of view, the gauge on the Dirac equation in a real manifold would just be one corresponding to the gravitational gauge. The electromagnetic gauge on the Dirac equation would be included when we consider the space-time manifold extended to a complex-manifold. The Yang-Mills gauge would then be included when we extend the manifold to the matrix-manifold, which is equivalent to the quaternionic manifold for the $SU(2)$ symmetry group. The next step would then be to extend the quaternionic theory to the octonionic one and determine to which gauge it corresponds. This is the goal proposed in the third part for the analysis of the Dirac equation in a non-Riemannian manifold.

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