SC-MAG-616

Stress-Strain in Shells Under Mechanical, Thermal and Magnetic Loads.*

S. Caspi, K. Chow, G. Millos and C. Taylor

Lawrence Berkeley Laboratory
University Of California
Berkeley, CA 94720

March 23, 1998

* This was supported by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, High Energy Physics Division, U. S. Department of Energy, under Contract No. DE-AC03–76SF00098.

Abstract

In the design of a superconducting ECR source we need to manage the stress within solenoids that are subjected to mechanical, thermal and magnetic forces. In the past this problem has been solved[bc] and applied to nested cylindrical shells that are only subjected to an inner and outer pressure and a temperature change. The new solution takes care of possible Lorentz forces and permits the use of anisotropic properties. In the first part of this report we write down the differential equation for the displacement, solve it and obtain the displacement solution as well as the radial and hoop stress. We compare the results with those obtained by ANSYS. In part two we extend the single shell solution to a multi shell system that may include an interference fit between such shells.

Analysis

The expression of the strain in a plain stress problem including a temperature change is :

$$\varepsilon_r = \frac{\sigma_r}{E_r} - \nu_{rt}\frac{\sigma_t}{E_t} + \alpha_r \Delta T = \frac{1}{E_t}\left(m^2\sigma_r - \nu\sigma_t\right) + \alpha_r \Delta T$$

$$\varepsilon_t = \frac{\sigma_t}{E_t} - \nu_{tr}\frac{\sigma_r}{E_r} + \alpha_t \Delta T = \frac{1}{E_t}(\sigma_t - \nu\sigma_r) + \alpha_t \Delta T$$

$$\nu_{rt} = \nu$$

$$\nu_{tr} = \frac{E_r}{E_t}\nu = m^2\nu$$

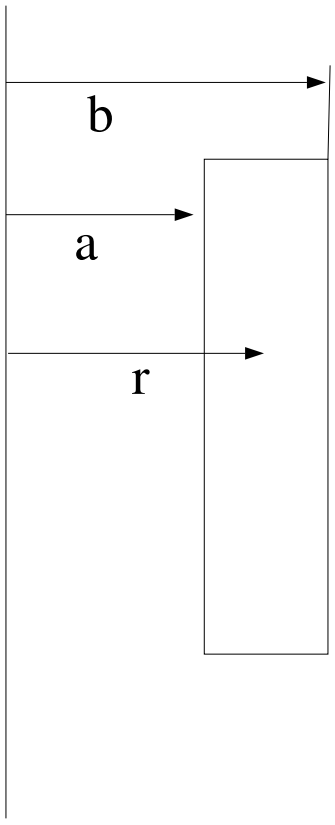

Figure 1 Coil subjected to Lorentz forces

[b] Meuser M5104, M5153, M5832

[c] "Stress in Cold Nested Cylinders" S. Caspi, M5537, June 1980.

We shall use the following nomenclature :

$$\begin{aligned}
\nu &= \text{ Poisson ratio} \\
E_t &= \text{ azimuthal Young's modulus} \\
E_r &= \text{ radial Young's modulus} \\
u &= \text{ displacement} \\
m^2 &= \frac{E_t}{E_r} \\
S &= \frac{E_t}{m^2 - \nu^2} \\
B_a &= \text{ axial magnetic field at inner radius a} \\
B_b &= \text{ axial magnetic field at outer radius b} \\
J_0 &= \text{ current density}
\end{aligned}$$

Solving for the stress :

$$\begin{aligned}
\sigma_r &= S[\varepsilon_r - \alpha_r \Delta T + \nu(\varepsilon_t - \alpha_t \Delta T)] \\
\sigma_t &= S\big[\nu(\varepsilon_r - \alpha_r \Delta T) + m^2(\varepsilon_t - \alpha_t \Delta T)\big]
\end{aligned}$$

From equilibrium (including Lorentz forces, B(r) is the axial field and J_0 the current density) :

$$\sigma_t - \sigma_r - r\frac{d\sigma_r}{dr} - BJ_0 r = 0$$

and the relation for the displacement u,

$$\varepsilon_r = \frac{du}{dr} \quad ; \qquad \varepsilon_t = \frac{u}{r}$$

we obtain the differential equation for the displacement :

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - m^2\frac{u}{r^2} = \frac{K}{r} + C_1 + C_2 r$$

where

$$\begin{aligned}
K &= \alpha_r \Delta T(1-\nu) + \alpha_t \Delta T\big(\nu - m^2\big) \\
C_1 &= -\frac{J_0}{S}\frac{B_a b - B_b a}{b-a} \\
C_1 &= \frac{J_0}{S}\frac{B_a - B_b}{b-a}
\end{aligned}$$

The general homogenous and particular solution to the D.E is :

$$u = \alpha_1 r^m + \alpha_2 r^{-m} + \frac{K}{1-m^2}r + \frac{C_1}{4-m^2}r^2 + \frac{C_2}{9-m^2}r^3 \qquad m \neq 1\ ,\ m \neq 2\ ,\ m \neq 3$$
$$u = \beta_1 r + \beta_2 r^{-1} + \frac{K}{2}r\ln\left(\frac{r}{b}\right) + \frac{C_1}{3}r^2 + \frac{C_2}{8}r^3 \qquad m = 1$$
$$u = \gamma_1 r^2 + \gamma_2 r^{-2} - \frac{K}{3}r + \frac{C_1}{4}r^2\ln\left(\frac{r}{b}\right) + \frac{C_2}{5}r^3 \qquad m = 2$$
$$u = \delta_1 r^3 + \delta_2 r^{-3} - \frac{K}{8}r - \frac{C_1}{5}r^2 + \frac{C_2}{6}r^3\ln\left(\frac{r}{b}\right) \qquad m = 3$$

where the α's, β's, and γ's are to be determined from the stress condition on the boundaries a and b.

We may wish to brake the solution into three distinct parts,

1. Mechanical part $u(p)$, where $\sigma_r|_{r=a} = P_i$ and $\sigma_r|_{r=b} = P_o$, by definition if P_i<0 (pressure into the body), the displacement u is positive (in the +r direction). If P_o<0 (pressure into the body), the displacement u is negative (in the — r direction).
2. Thermal part $u(t)$, where $\sigma_r|_{r=a} = 0$ and $\sigma_r|_{r=b} = 0$
3. Magnetic part $u(j_0)$, where $\sigma_r|_{r=a} = 0$ and $\sigma_r|_{r=b} = 0$

The total solution is the summation of the parts

$$u_{total} = u(p) + u(t) + u(j_o)$$
$$\sigma_{r_{total}} = \sigma_r(p) + \sigma_r(t) + \sigma_r(j_o)$$
$$\sigma_r(p) = S\left[\frac{du(p)}{dr} + \nu\frac{u(p)}{r}\right]$$
$$\sigma_r(t) = S\left[\frac{du(t)}{dr} + \nu\frac{u(t)}{r} - (\alpha_r\Delta T + \nu\alpha_t\Delta T)\right]$$
$$\sigma_r(j_o) = S\left[\frac{du(j_o)}{dr} + \nu\frac{u(j_o)}{r}\right]$$
$$\sigma_{t_{total}} = \sigma_t(p) + \sigma_t(t) + \sigma_t(j_o)$$
$$\sigma_t(p) = S\left[\nu\frac{du(p)}{dr} + m^2\frac{u(p)}{r}\right]$$
$$\sigma_t(t) = S\left[\nu\frac{du(t)}{dr} + m^2\frac{u(t)}{r} - \left(\nu\alpha_r\Delta T + m^2\alpha_t\Delta T\right)\right]$$
$$\sigma_t(j_o) = S\left[\nu\frac{du(j_o)}{dr} + m^2\frac{u(j_o)}{r}\right]$$
$$\sigma_r(p)|_{r=a} = P_i \quad \sigma_r(t)|_{r=a} = 0 \quad \sigma_r(j_o)|_{r=a} = 0$$
$$\sigma_r(p)|_{r=b} = P_o \quad \sigma_r(t)|_{r=b} = 0 \quad \sigma_r(j_o)|_{r=b} = 0$$

Mechanical Part

$$u(p) = \frac{b}{m+\nu}\frac{\left[\frac{P_o}{S} - \frac{P_i}{S}\left(\frac{a}{b}\right)^{m+1}\right]}{\left[1-\left(\frac{a}{b}\right)^{2m}\right]}\left(\frac{r}{b}\right)^m + \frac{a}{m-\nu}\frac{\left[\frac{P_o}{S}\left(\frac{a}{b}\right)^{m-1} - \frac{P_i}{S}\right]}{\left[1-\left(\frac{a}{b}\right)^{2m}\right]}\left(\frac{a}{r}\right)^m$$

$$\sigma_r(p) = \frac{1}{\left[1-\left(\frac{a}{b}\right)^{2m}\right]}\left\{\left[P_o - P_i\left(\frac{a}{b}\right)^{m+1}\right]\left(\frac{r}{b}\right)^{m-1} - \left[P_o\left(\frac{a}{b}\right)^{m-1} - P_i\right]\left(\frac{a}{r}\right)^{m+1}\right\}$$

$$\sigma_t(p) = \frac{m}{\left[1-\left(\frac{a}{b}\right)^{2m}\right]}\left\{\left[P_o - P_i\left(\frac{a}{b}\right)^{m+1}\right]\left(\frac{r}{b}\right)^{m-1} + \left[P_o\left(\frac{a}{b}\right)^{m-1} - P_i\right]\left(\frac{a}{r}\right)^{m+1}\right\}$$

Thermal Part $m \neq 1$

$$u(t) = \left\{\begin{array}{l} -\frac{(m-\nu)(\alpha_r\Delta T - \alpha_t\Delta T)}{1-m^2}\frac{\left[1-\left(\frac{a}{b}\right)^{m+1}\right]}{\left[1-\left(\frac{a}{b}\right)^{2m}\right]}b\left(\frac{r}{b}\right)^m + \frac{(m+\nu)(\alpha_r\Delta T - \alpha_t\Delta T)}{1-m^2}\frac{\left[1-\left(\frac{a}{b}\right)^{m-1}\right]}{\left[1-\left(\frac{a}{b}\right)^{2m}\right]}a\left(\frac{a}{r}\right)^m \\ +\frac{\alpha_r\Delta T(1-\nu) + \alpha_t\Delta T(\nu - m^2)}{1-m^2}r \end{array}\right\}$$

$$\sigma_r(t) = \frac{S(m^2-\nu^2)(\alpha_r\Delta T - \alpha_t\Delta T)}{1-m^2}\left\{1 - \frac{\left[1-\left(\frac{a}{b}\right)^{m+1}\right]\left(\frac{r}{b}\right)^{m-1} + \left[1-\left(\frac{a}{b}\right)^{m-1}\right]\left(\frac{a}{r}\right)^{m+1}}{\left[1-\left(\frac{a}{b}\right)^{2m}\right]}\right\}$$

$$\sigma_t(t) = \frac{S(m^2-\nu^2)(\alpha_r\Delta T - \alpha_t\Delta T)}{1-m^2}\left\{1 - m\frac{\left[1-\left(\frac{a}{b}\right)^{m+1}\right]\left(\frac{r}{b}\right)^{m-1} - \left[1-\left(\frac{a}{b}\right)^{m-1}\right]\left(\frac{a}{r}\right)^{m+1}}{\left[1-\left(\frac{a}{b}\right)^{2m}\right]}\right\}$$

Thermal Part $m = 1$

$$u(t) = \frac{(\alpha_r\Delta T + \alpha_t\Delta T)}{2}r + \frac{(1-\nu)(\alpha_r\Delta T - \alpha_t\Delta T)}{2}\left\{\frac{\ln\frac{a}{b}}{\left[1-\left(\frac{a}{b}\right)^2\right]}\left[\left(\frac{a}{b}\right)^2 r + \left(\frac{1+\nu}{1-\nu}\right)\frac{a^2}{r}\right] + r\ln\frac{r}{b}\right\}$$

$$\sigma_r(t) = \frac{S(1-\nu^2)(\alpha_r\Delta T - \alpha_t\Delta T)}{2\left[1-\left(\frac{a}{b}\right)^2\right]}\left\{\left(\frac{a}{b}\right)^2\ln\frac{a}{r} - \left(\frac{a}{r}\right)^2\ln\frac{a}{b} + \ln\frac{r}{b}\right\}$$

$$\sigma_t(t) = \frac{S(1-\nu^2)(\alpha_r\Delta T - \alpha_t\Delta T)}{2\left[1-\left(\frac{a}{b}\right)^2\right]}\left\{1 + \left(\frac{a}{b}\right)^2\left(\ln\frac{a}{r} - 1\right) + \left(\frac{a}{r}\right)^2\ln\frac{a}{b} + \ln\frac{r}{b}\right\}$$

Magnetic Part $m \neq 2 \quad m \neq 3$

$$
u(j_o) = \left\{ \begin{array}{l}
-\dfrac{1}{(m+\nu)\Big[1-\big(\frac{a}{b}\big)^{2m}\Big]}\left\{\dfrac{(2+\nu)C_1 b}{4-m^2}\left[1-\left(\dfrac{a}{b}\right)^{m+2}\right]+\dfrac{(3+\nu)C_2 b^2}{9-m^2}\left[1-\left(\dfrac{a}{b}\right)^{m+3}\right]\right\} b\left(\dfrac{r}{b}\right)^m \\
+\dfrac{1}{(m-\nu)\Big[1-\big(\frac{a}{b}\big)^{2m}\Big]}\left\{\dfrac{(2+\nu)C_1 a}{4-m^2}\left[1-\left(\dfrac{a}{b}\right)^{m-2}\right]+\dfrac{(3+\nu)C_2 a^2}{9-m^2}\left[1-\left(\dfrac{a}{b}\right)^{m-3}\right]\right\} a\left(\dfrac{a}{r}\right)^m \\
+\dfrac{C_1}{4-m^2}r^2+\dfrac{C_2}{9-m^2}r^3
\end{array} \right\}
$$

$$
\sigma_r(j_o) = \left\{ \begin{array}{l}
\dfrac{S}{\Big[1-\big(\frac{a}{b}\big)^{2m}\Big]}\left\{-\dfrac{(2+\nu)C_1 b}{4-m^2}\left[1-\left(\dfrac{a}{b}\right)^{m+2}\right]-\dfrac{(3+\nu)C_2 b^2}{9-m^2}\left[1-\left(\dfrac{a}{b}\right)^{m+3}\right]\right\}\left(\dfrac{r}{b}\right)^{m-1} \\
+\dfrac{S}{\Big[1-\big(\frac{a}{b}\big)^{2m}\Big]}\left\{-\dfrac{(2+\nu)C_1 a}{4-m^2}\left[1-\left(\dfrac{a}{b}\right)^{m-2}\right]-\dfrac{(3+\nu)C_2 a^2}{9-m^2}\left[1-\left(\dfrac{a}{b}\right)^{m-3}\right]\right\}\left(\dfrac{a}{r}\right)^{m+1} \\
+S\left[\dfrac{(2+\nu)C_1}{4-m^2}r+\dfrac{(3+\nu)C_2}{9-m^2}r^2\right]
\end{array} \right\}
$$

$$
\sigma_t(j_o) = \left\{ \begin{array}{l}
\dfrac{Sm}{\Big[1-\big(\frac{a}{b}\big)^{2m}\Big]}\left\{-\dfrac{(2+\nu)C_1 b}{4-m^2}\left[1-\left(\dfrac{a}{b}\right)^{m+2}\right]-\dfrac{(3+\nu)C_2 b^2}{9-m^2}\left[1-\left(\dfrac{a}{b}\right)^{m+3}\right]\right\}\left(\dfrac{r}{b}\right)^{m-1} \\
+\dfrac{Sm}{\Big[1-\big(\frac{a}{b}\big)^{2m}\Big]}\left\{\dfrac{(2+\nu)C_1 a}{4-m^2}\left[1-\left(\dfrac{a}{b}\right)^{m-2}\right]+\dfrac{(3+\nu)C_2 a^2}{9-m^2}\left[1-\left(\dfrac{a}{b}\right)^{m-3}\right]\right\}\left(\dfrac{a}{r}\right)^{m+1} \\
+S\left[\dfrac{(2\nu+m^2)C_1}{4-m^2}r+\dfrac{(3\nu+m^2)C_2}{9-m^2}r^2\right]
\end{array} \right\}
$$

Magnetic Part $m = 2$

$$u(j_o) = \left\{ \begin{array}{l} \frac{1}{(2+\nu)\left[1-\left(\frac{a}{b}\right)^4\right]} \left\{ \left(\frac{a}{b}\right)^4 \left[\frac{C_1}{4} + \frac{3C_1}{4}\ln\left(\frac{a}{b}\right) + \frac{4C_2}{5}a \right] - \frac{C_1}{4} - \frac{4C_2}{5}b \right\} b^2 \left(\frac{r}{b}\right)^2 \\ + \frac{1}{(2-\nu)\left[1-\left(\frac{a}{b}\right)^4\right]} \left[\frac{3C_1}{4}\ln\left(\frac{a}{b}\right) + \frac{4C_2}{5}(a-b) \right] a^2 \left(\frac{a}{r}\right)^2 + \frac{C_1}{4} r^2 \ln\left(\frac{r}{b}\right) + \frac{C_2}{5} r^3 \end{array} \right\}$$

$$\sigma_r(j_o) = S \left\{ \begin{array}{l} \frac{\left(\frac{a}{b}\right)^4 \left[\frac{C_1}{4} + \frac{3C_1}{4}\ln\left(\frac{a}{b}\right) + \frac{4C_2}{5}a\right] - \frac{C_1}{4} - \frac{4C_2}{5}b}{\left[1-\left(\frac{a}{b}\right)^4\right]} r - \frac{\frac{3C_1}{4}\ln\left(\frac{a}{b}\right) + \frac{4C_2}{5}(a-b)}{\left[1-\left(\frac{a}{b}\right)^4\right]} a \left(\frac{a}{r}\right)^3 \\ + \frac{3C_1}{4} r \ln\left(\frac{r}{b}\right) + \frac{C_1}{4} r + \frac{4C_2}{5} r^2 \end{array} \right\}$$

$$\sigma_t(j_o) = 2S \left\{ \begin{array}{l} \frac{\left(\frac{a}{b}\right)^4 \left[\frac{C_1}{4} + \frac{3C_1}{4}\ln\left(\frac{a}{b}\right) + \frac{4C_2}{5}a\right] - \frac{C_1}{4} - \frac{4C_2}{5}b}{\left[1-\left(\frac{a}{b}\right)^4\right]} r + \frac{\frac{3C_1}{4}\ln\left(\frac{a}{b}\right) + \frac{4C_2}{5}(a-b)}{\left[1-\left(\frac{a}{b}\right)^4\right]} a \left(\frac{a}{r}\right)^3 \\ + \frac{C_1(2+\nu)}{4} r \ln\left(\frac{r}{b}\right) + \frac{C_1\nu}{8} r + \frac{C_2(4+3\nu)}{10} r^2 \end{array} \right\}$$

We shall not include the solution for m=3

Example — a single shell

As a check we use a single shell subjected to a Lorentz force and compute the displacement, radial stress and azimuthal stress using Mathcad, a Fortran program, and the program ANSYS. The results are in good agreement (Fig. 2–4).

a=170 mm ; E_r = *20 GPa ; alfardT=0 ; Ba= 5.915 T*
b=244 mm ; E_t = *109 GPa ; alfatdT=0 ; Bb= -0.345 T*
ν = 0.2 ; J= 120 $\frac{A}{mm^2}$

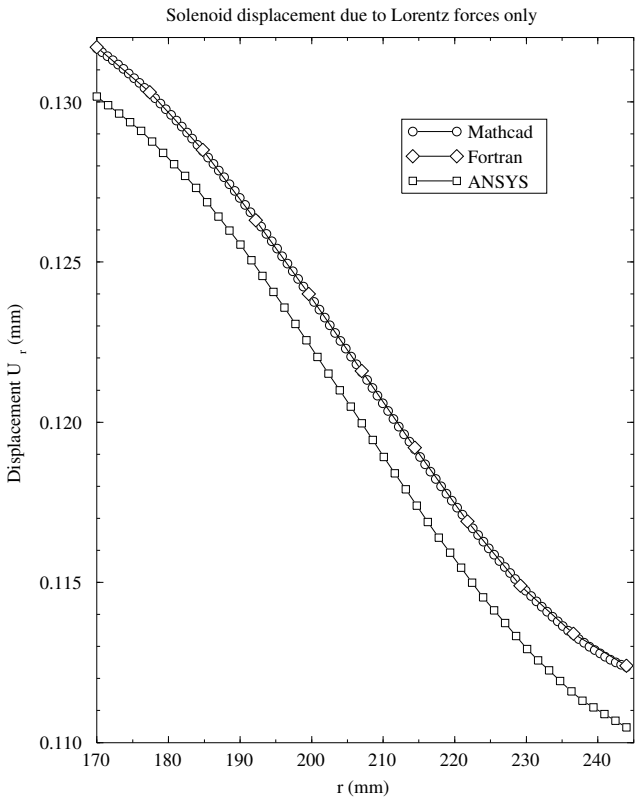

Figure 2 Displacement

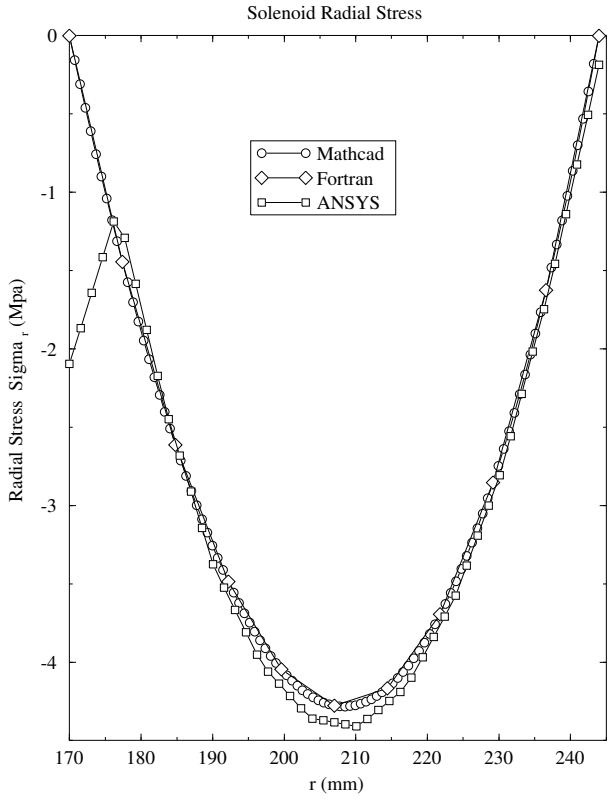

Figure 3 Radial Stress

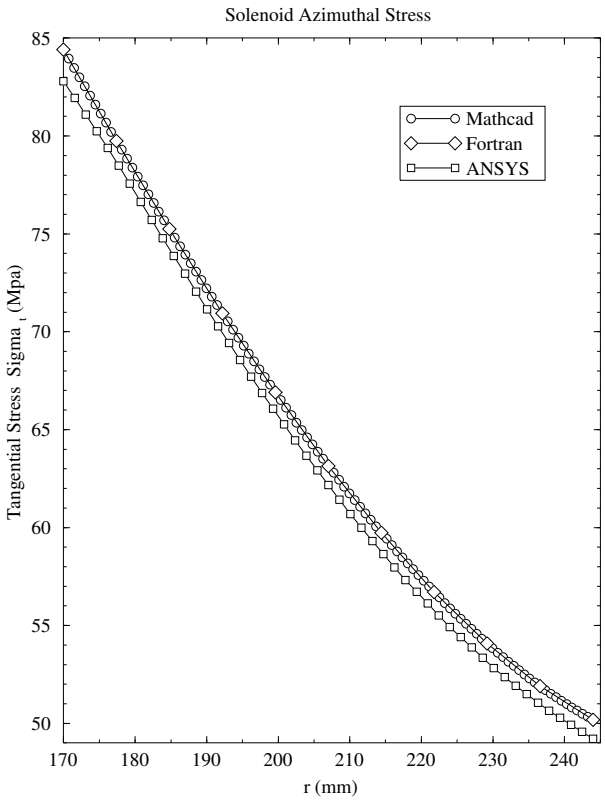

Figure 4 Azimuthal stress

Nested cylinders

We may extend our analysis to include a number of nested cylinders that are subjected to a magnetic field, a temperature change and may include an interference fit between shells.

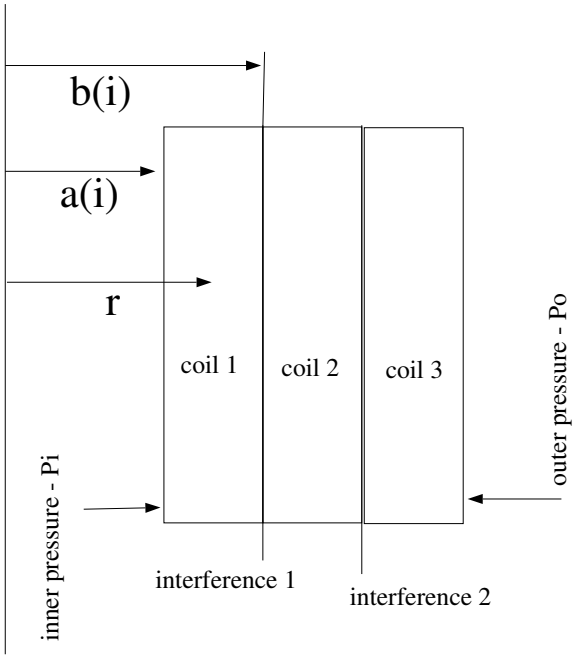

Figure 5 Nested coil subjected to Lorentz forces, interference and temperature change.

The displacements within each shell can be written in the general form : $u(P_i, P_o, r) = A(r)P_i + B(r)P_o + u_t + u_{jo}$. If an interference of size w(i) is included between coil i and coil i+1, the following relations are valid for the displacement at the shells interfaces,

$$\begin{gathered}
\Delta_{in} = u(P_{in}, P_{o1})|_{r=a_1} \\
u_2(P_{o1}, P_{o2})|_{r=a_2} - w(1) = u_1(P_{in}, P_{o1})|_{r=b_1} \\
u_3(P_{o2}, P_{o3})|_{r=a_3} - w(2) = u_2(P_{o1}, P_{o2})|_{r=b_2} \\
\cdots \\
\Delta_{out} = u(P_{oN}, P_{out})|_{r=b_N}
\end{gathered}$$

Where,

$$A(r) = \frac{-a}{S\left[1 - \left(\frac{a}{b}\right)^{2m}\right]}\left[\frac{1}{m+\nu}\left(\frac{a}{b}\right)^{m}\left(\frac{r}{b}\right)^{m} + \frac{1}{m-\nu}\left(\frac{a}{r}\right)^{m}\right]$$

$$B(r) = \frac{b}{S\left[1 - \left(\frac{a}{b}\right)^{2m}\right]}\left[\frac{1}{m+\nu}\left(\frac{r}{b}\right)^{m} + \frac{1}{m-\nu}\left(\frac{a}{b}\right)^{m}\left(\frac{a}{r}\right)^{m}\right]$$

The displacements of the above N nested cylinders can be expressed as N+1 equations with N+1 unknowns — the most inner and outer displacements and the radial stress between the shells interfaces. Such a set can be written in a form as shown on the next page, where the subscripts, in and out, refer to the inner and outer radius of a shell (i),

Example —3 nested shells

We have solved a problem of 3 nested shells and compared it with ANSYS,

	inner R (mm)	outer R (mm)	E_r (pa)	E_t (pa)	ν	$\alpha_r\Delta T$	$\alpha_t\Delta T$	B_a(T)	B_b(T)	J (A/mm2)
COIL 1	170.0	186.9	20.0e9	109.0e9	0.3	-0.00447	-0.00283	6.669	4.171	140.0
S.S wrap	186.9	190.84	195.0e9	195.0e9	0.294	-0.00281	-0.00281	4.171	4.285	0.0
COIL 2	190.84	235.15	20.0e9	109.0e9	0.3	-0.00447	-0.00283	4.285	-2.278	140.0

$$
\begin{pmatrix}
1 & -B(1)_{in} & 0 & 0 & 0 & 0 & 0 \\
0 & [-B(1)_{out} + A(2)_{in}] & B(2)_{in} & 0 & 0 & 0 & 0 \\
0 & -A(2)_{out} & [-B(2)_{out} + A(3)_{in}] & B(3)_{in} & 0 & 0 & 0 \\
0 & 0 & -A(3)_{out} & [-B(3)_{out} + A(4)_{in}] & B(4)_{in} & 0 & 0 \\
 & & & & \cdots & & \\
0 & 0 & 0 & -A(i-1)_{out} & [-B(i-1)_{out} + A(i)_{in}] & B(i)_{in} & 0 \\
 & & & & \cdots & & \\
0 & 0 & 0 & 0 & -A(N-1)_{out} & [-B(N-1)_{out} + A(N)_{in}] & 0 \\
0 & 0 & 0 & 0 & 0 & -A(N)_{out} & 1
\end{pmatrix} \times
$$

$$
\times \begin{pmatrix}
\Delta_{in} \\
P_o(1) \\
P_o(2) \\
P_o(3) \\
\cdots \\
P_o(i) \\
\cdots \\
P_o(N-1) \\
\Delta_{out}
\end{pmatrix} =
\begin{pmatrix}
A(1)_{in} P_{in} + u_t(1)_{in} + u_{j_o}(1)_{in} \\
w(1) + A(1)_{out} P_{in} + u_t(1)_{out} + u_{j_o}(1)_{out} - u_t(2)_{in} - u_{j_o}(2)_{in} \\
w(2) + u_t(2)_{out} + u_{j_o}(2)_{out} - u_t(3)_{in} - u_{j_o}(3)_{in} \\
w(3) + u_t(3)_{out} + u_{j_o}(3)_{out} - u_t(4)_{in} - u_{j_o}(4)_{in} \\
\cdots \\
w(i-1) + u_t(i-1)_{out} + u_{j_o}(i-1)_{out} - u_t(i)_{in} - u_{j_o}(i)_{in} \\
\cdots \\
w(N-1) - B(N)_{in} P_{out} + u_t(N-1)_{out} + u_{j_o}(N-1)_{out} - u_t(N)_{in} + u_{j_o}(N)_{in} \\
B(N)_{out} P_{out} + u_t(N)_{out} + u_{j_o}(N)_{out}
\end{pmatrix}
$$

for 2 cylinders only the above looks like :

$$
\begin{pmatrix}
1 & -B(1)_{in} & 0 \\
0 & [-B(1)_{out} + A(2)_{in}] & 0 \\
0 & -A(2)_{out} & 1
\end{pmatrix}
\begin{pmatrix}
\Delta_{in} \\
P_o(1) \\
\Delta_{out}
\end{pmatrix} =
\begin{pmatrix}
A(1)_{in} P_{in} + u_t(1)_{in} + u_{j_o}(1)_{in} \\
w(1) + A(1)_{out} P_{in} - B(2)_{in} P_{out} + u_t(1)_{out} + u_{j_o}(1)_{out} - u_t(2)_{in} - u_{j_o}(2)_{in} \\
B(2)_{out} P_{out} + u_t(2)_{out} + u_{j_o}(2)_{out}
\end{pmatrix}
$$

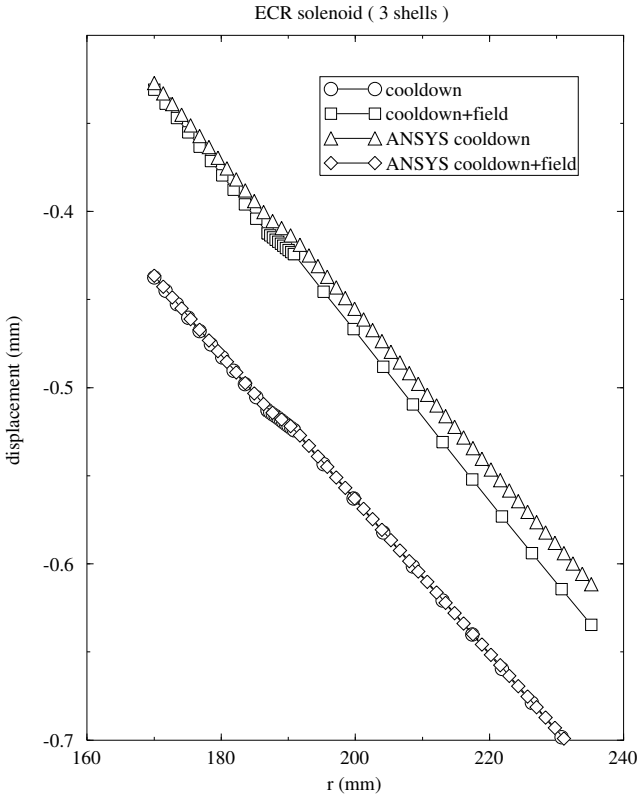

Figure 6 3 shell test displacement

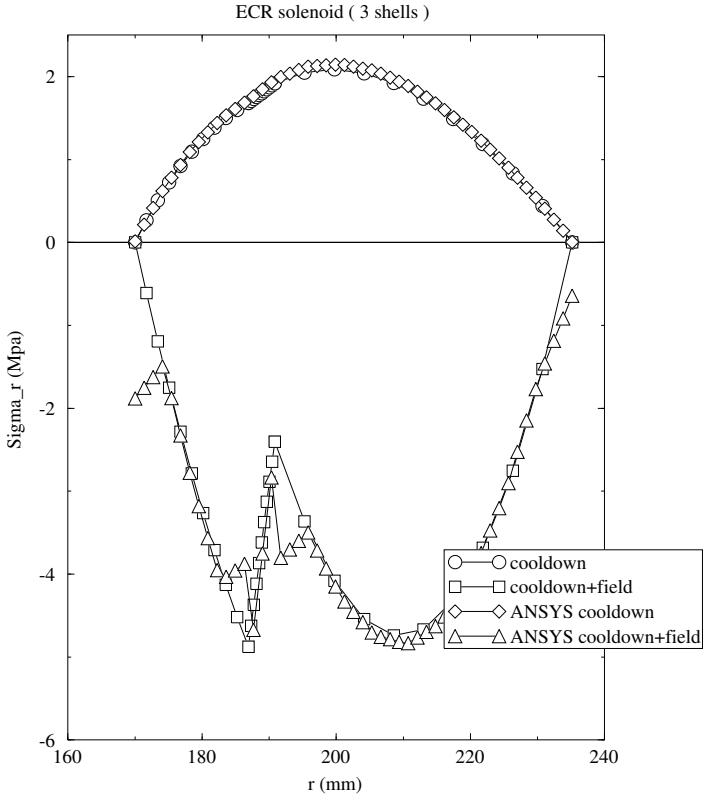

Figure 7 3 shell test — radial stress

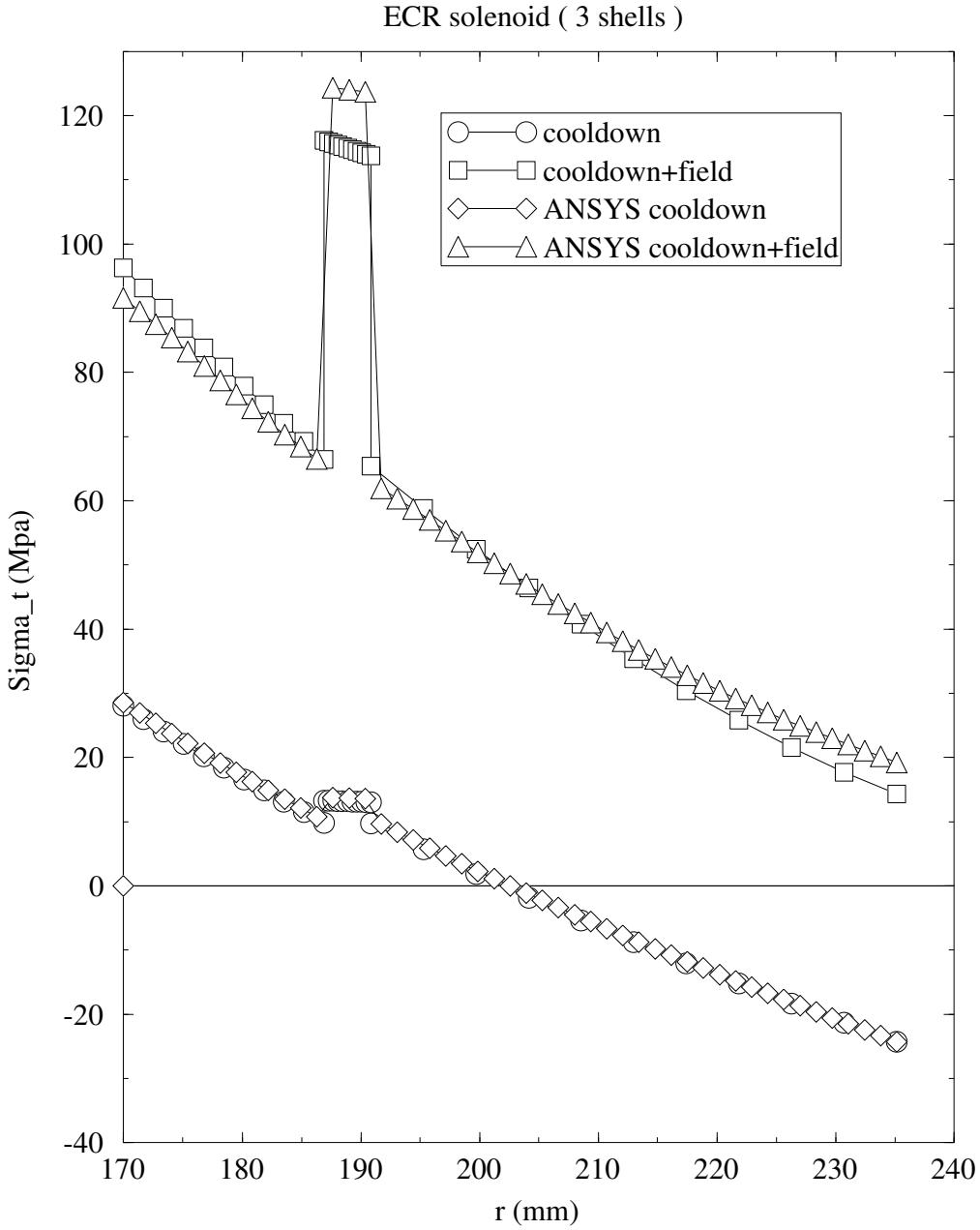

Figure 8 3 shell test — hoop stress