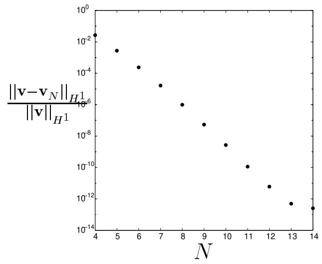
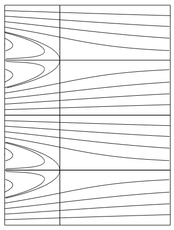
Kovasznay Example

(*Proc. Roy. Soc., 1948*)





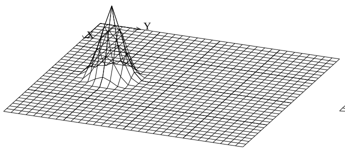
$$
\begin{aligned}
v_x &= 1 - e^{\lambda x}\cos 2\pi y \\
v_y &= \frac{\lambda}{2\pi} e^{\lambda x}\sin 2\pi y \\
\lambda &:= \frac{Re}{2} - \sqrt{\frac{Re^2}{4} + 4\pi^2}
\end{aligned}
$$

Nek5000 Anelastic Code Development

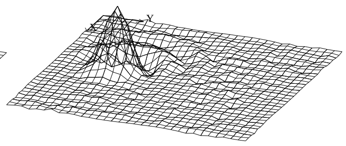
Background/History

- Outgrowth of Nekton 2.0 commercial code marketed by Fluent and developed by Rønquist, Ho, Fischer & Patera in 1986-1991, and modified by Fischer & Tufo in the '90s.
- Incompressible & unsteady Navier-Stokes with species transport.
- Variational-based spectral element formulation, similar to FEM.
- Elements are *locally* structured, *globally* unstructured.
- 2D, 3D, axisymmetric, with (some) moving mesh capability (Ho).
- Spectrally Accurate

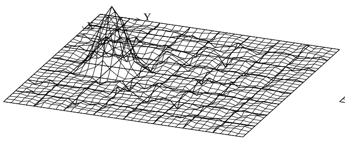
Numerical Dispersion vs. Order



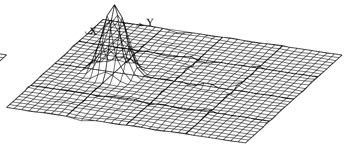
Initial Condition



$K_1 = 16,\ N = 2$



$K_1 = 8,\ N = 4$



$K_1 = 4,\ N = 8$

1D Convection Example

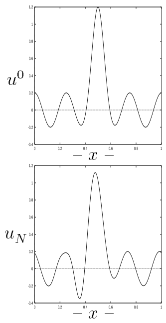
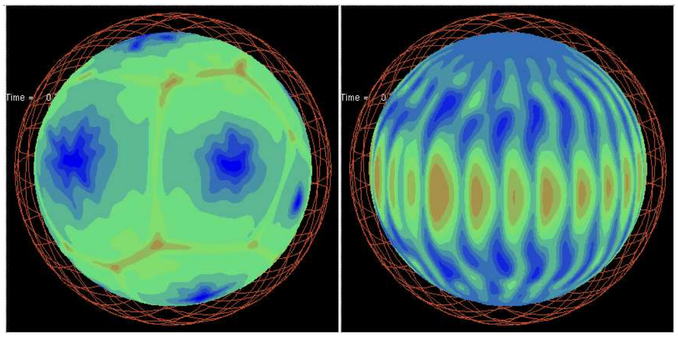


Table 1: Relative error at time $t = 10$, hyperbolic problem.

$\mathcal{N}$	Δt	$N=1$	$N=2$	$N=4$	$N=8$	$N=16$	$N=32$
64	.0025000	8.09e–1	1.29e–1	3.19e–2	2.91e–2	—	—
64	.0012500	8.13e–1	1.41e–1	1.27e–2	4.58e–3	4.11e–3	—
64	.0006250	8.13e–1	1.42e–1	1.22e–2	1.08e–3	5.43e–4	5.22e–4
64	.0003125	8.14e–1	1.43e–1	1.23e–2	8.27e–4	8.90e–5	6.55e–5
128	.0100000	—	—	—	—	—	—
128	.0012500	6.59e–1	1.42e–2	4.16e–3	4.10e–3	—	—
128	.0006250	6.60e–1	1.54e–2	6.14e–4	5.22e–4	5.22e–4	—
128	.0003125	6.60e–1	1.56e–2	2.00e–4	6.58e–5	6.55e–5	6.55e–5
256	.0025000	2.88e–1	—	—	—	—	—
256	.0012500	3.01e–1	4.66e–3	4.10e–3	—	—	—
256	.0006250	3.03e–1	1.51e–3	5.26e–4	5.22e–4	—	—
256	.0003125	3.03e–1	1.71e–3	7.02e–5	6.55e–5	6.55e–5	—

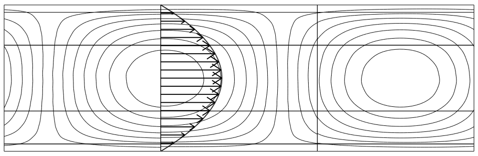
Convection in Deep Atmospheric Model

$Ra = 20,000$, increasing Ta



Error in Orr-Sommerfeld Growth Rates, $Re = 7500$

	$\Delta t = 0.003125$		$N = 17$	2nd Order		3rd Order	
N	$\alpha = 0.0$	$\alpha = 0.2$	Δt	$\alpha = 0.0$	$\alpha = 0.2$	$\alpha = 0.0$	$\alpha = 0.2$
7	0.23641	0.27450	0.20000	0.12621	0.12621	171.370	0.02066
9	0.00173	0.11929	0.10000	0.03465	0.03465	0.00267	0.00268
11	0.00455	0.01114	0.05000	0.00910	0.00911	161.134	0.00040
13	0.00004	0.00074	0.02500	0.00238	0.00238	1.04463	0.00012
15	0.00010	0.00017	0.01250	0.00065	0.00066	0.00008	0.00008



User Input for Nek5000 Transport Equations

$$\texttt{utrans}\frac{D\Psi}{Dt} = \nabla\cdot\texttt{udiff}\nabla\Psi + \texttt{qvol}$$

```
      subroutine uservp (ix,iy,iz,ieg)
      include 'SIZE'
      include 'TOTAL'
      include 'NEKUSE'
c
      udiff =x+y+time
      utrans=1.
c
      return
      end
c-----------------------------------------------------------------------
      subroutine userq  (ix,iy,iz,ieg)
      include 'SIZE'
      include 'TOTAL'
      include 'NEKUSE'
c
      qvol   = sin(time)
c
      return
      end
```

Standard Nek5000 Transport Equations

$$\begin{aligned}
\rho\frac{D\mathbf{u}}{Dt} + \nabla p &= \nabla\cdot\mu\left(\nabla\mathbf{u}+\nabla^T\mathbf{u}\right) + \mathbf{f}\\
\nabla\cdot\mathbf{u} &= 0\\
\rho c_p\frac{DT}{Dt} &= \nabla\cdot k\nabla T + q\\
\rho c_1\frac{D\Psi_1}{Dt} &= \nabla\cdot k_1\nabla\Psi_1 + q_1\\
&\vdots
\end{aligned}$$

- $\mathbf{f} = \mathbf{f}(t,\mathbf{x},\mathbf{u},T,\Psi_j,\ldots)$
- *Boussinesq*: $\mathbf{f} = \rho g\beta T\hat{\mathbf{z}}$, $\beta := -\frac{1}{\rho}\frac{\partial\rho}{\partial T}\Big|_p$, $Ra = \frac{g\beta\Delta T L^3\rho_0^2}{k\mu}$.
- In principle, ρ can also be a function (though no longer I.N.S.)
- μ, k, k_j can also be functions.

Anelastic Models — LF I

Momentum:

$$\rho_0 \frac{D\mathbf{u}}{Dt} + \nabla p_1 = \frac{1}{Re}\nabla \cdot \mathbf{\Pi} + \rho_1 g \hat{\mathbf{z}} \qquad (30)$$

$$\nabla \cdot \rho_0 \mathbf{u} = 0 \qquad (33)$$

Entropy transport:

$$\rho_0 T_0 \frac{Ds_1}{Dt} = \frac{1}{Pe}\nabla \cdot K \rho_0 T_0 \nabla s_1 \qquad (32)$$

E.O.S:

$$\longrightarrow \quad \rho_1(\rho_0, T_0, p_0, p_1)$$

Anelastic Models — *Simplified*

Lantz & Fan '99 – mixing length theory (*many* unspecified constants)

Background State:

$$\frac{dp_0}{dz} = -\rho_0 g$$

$$T_0 ds_0 = 0 \implies \frac{dT_0}{dz} = -\sigma \frac{\delta_p g}{c_p}$$

$$\delta_p := \frac{\partial \ln \rho}{\partial \ln T}$$

Actually use ($m = \frac{3}{2}$):

$$\rho_0 = \left[1 - \frac{z}{m+1}\right]^m \qquad (70)$$

$$T_0 = \left[1 - \frac{z}{m+1}\right] \qquad (71)$$

NOTE: ASH code does *not* assume isentropic background state.

Standard Unsteady Stokes Solve

$$\begin{bmatrix} H & -D^T \\ -D & 0 \end{bmatrix} \begin{pmatrix} \underline{\mathbf{u}} \\ \underline{p} \end{pmatrix}^n = \begin{pmatrix} \underline{\mathbf{f}} \\ \underline{0} \end{pmatrix}$$

$H = \frac{\rho_0}{\Delta t} B + \mu A$, A – Laplacian, B – diagonal mass matrix $\underline{f}$ – nonlinear terms, etc.

Time splitting of discrete system *(Perot, JCP'93)*

$$\begin{bmatrix} H & -D^T \\ -D & 0 \end{bmatrix} \begin{pmatrix} \underline{\mathbf{u}} \\ \delta \underline{p} \end{pmatrix}^n = \begin{pmatrix} \underline{\mathbf{f}} + D^T \underline{p}^{n-1} \\ \underline{0} \end{pmatrix}$$

$$\begin{bmatrix} H & -HQD^T \\ -D & 0 \end{bmatrix} \begin{pmatrix} \underline{\mathbf{u}} \\ \delta \underline{p} \end{pmatrix}^n = \begin{pmatrix} \underline{\mathbf{f}} + D^T \underline{p}^{n-1} + (I - HQ) D^T \delta \underline{p} \\ \underline{0} \end{pmatrix}$$

Choose $Q = \left(\frac{\rho_0}{\Delta t} B\right)^{-1} \implies (I - HQ) = -\Delta t \, \nu \, AB^{-1}$.

Anelastic Models — LF II (M.L.T. + Ideal Gas)

Momentum:

$$\rho_0 \frac{D\mathbf{u}}{Dt} + \rho_0 \nabla \phi = \frac{1}{Re} \nabla \cdot \mathbf{\Pi} + \rho_0 g \frac{\delta_p}{c_p} s_1 \hat{\mathbf{z}} \qquad (45)$$

$$\nabla \cdot \rho_0 \mathbf{u} = 0 \qquad (33)$$

$$\phi := \frac{p_1}{\rho_0}$$

...plus entropy (s_1) transport (32).

- Very much like Boussinesq (from E.O.S. perspective).
- Strange multiplier in front of pressure term.

LF-I Unsteady Stokes Solve

$$\begin{bmatrix} H & -D^T \\ -D\rho_0 & 0 \end{bmatrix} \begin{pmatrix} \underline{\mathbf{u}} \\ \underline{p} \end{pmatrix}^n = \begin{pmatrix} \underline{\mathbf{f}} \\ \underline{0} \end{pmatrix}$$

Splitting + Uzawa decoupling:

$$\begin{bmatrix} H & -HQD^T \\ 0 & E \end{bmatrix} \begin{pmatrix} \underline{\mathbf{u}} \\ \delta\underline{p} \end{pmatrix}^n = \begin{pmatrix} \underline{\mathbf{f}} + D^T \underline{p}^{n-1} \\ \underline{g} \end{pmatrix}$$

$E := D\rho_0 \left(\frac{\rho_0}{\Delta t} B\right)^{-1} D^T = \frac{1}{\Delta t} D B^{-1} D^T$.

E is a consistent Poisson operator.

Formal Block-Gaussian Elimination (Uzawa decoupling):

$$\begin{bmatrix} H & -HQD^T \\ 0 & E \end{bmatrix} \begin{pmatrix} \underline{\mathbf{u}} \\ \delta\underline{p} \end{pmatrix}^n = \begin{pmatrix} \underline{\mathbf{f}} + D^T \underline{p}^{n-1} \\ \underline{g} \end{pmatrix}$$

$E := D \left(\frac{\rho_0}{\Delta t} B\right)^{-1} D^T$ — SPD Schur complement for pressure.

User Input for LF-II Nek5000 Transport Equations

```
      subroutine uservp (ix,iy,iz,ieg)
      include 'SIZE'
      include 'TOTAL'
      include 'NEKUSE'
c
      common /lantzfanc/ g,xi,gamma,cp,m
      real               g,xi,gamma,cp,m
      common /lantzfana/ rho1(lx1,ly1,lz1,lelt)
     $                 , rho0(lx1,ly1,lz1,lelt)
     $                 , T0  (lx1,ly1,lz1,lelt)
c
      visc = param(2)
      udiff =visc
      utrans=1.
      if (ifield.eq.2) then     ! Entropy transport
         ie      = gllel(ieg)
         conduct = param(8)
         utrans  = rho0(ix,iy,iz,ie)*t0(ix,iy,iz,ie)
         udiff   = conduct*rho0(ix,iy,iz,ie)*t0(ix,iy,iz,ie)  ! lack of better
      endif
c
      return
      end
```

LF-II Unsteady Stokes Solve

$$\begin{bmatrix} H & -\rho_0 D^T \\ -D\rho_0 & 0 \end{bmatrix} \begin{pmatrix} \underline{\mathbf{u}} \\ \underline{\phi} \end{pmatrix}^n = \begin{pmatrix} \underline{\mathbf{f}} \\ \underline{0} \end{pmatrix}$$

Splitting + Uzawa decoupling:

$$\begin{bmatrix} H & -HQD^T \\ 0 & E \end{bmatrix} \begin{pmatrix} \underline{\mathbf{u}} \\ \delta\underline{\phi} \end{pmatrix}^n = \begin{pmatrix} \underline{\mathbf{f}} + D^T \underline{p}^{n-1} \\ \underline{g} \end{pmatrix}$$

$$E := D\rho_0 \left(\frac{1}{\Delta t} B\right)^{-1} D^T = \frac{1}{\Delta t} D\rho_0 B^{-1} D^T.$$

Summary:

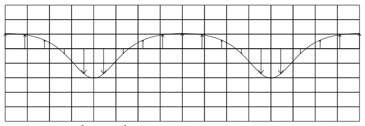
Std: $E = \frac{1}{\Delta t} D\rho_0^{-1} B^{-1} D^T$

LF-I: $E = \frac{1}{\Delta t} D B^{-1} D^T$

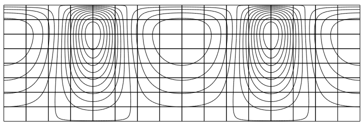
LF-II: $E = \frac{1}{\Delta t} D\rho_0 B^{-1} D^T$

A Test Problem

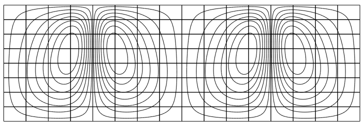
$Ra \approx 1000$, $h = 2H$



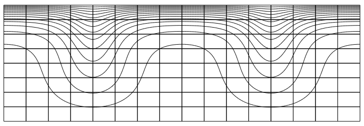
Vertical Velocity



Divergence



Vorticity



Entropy

```
      subroutine userf  (ix,iy,iz,ieg)
      include 'SIZE'
      include 'TOTAL'
      include 'NEKUSE'
c
      common /lantzfanc/ g,xi,gamma,cp,m
      real               g,xi,gamma,cp,m
c
      ie = gllel(ieg)
      if (if3d) then
         ffx = 0.0
         ffy = 0.0
         ffz =  g*t(ix,iy,iz,ie,1)/cp
      elseif (ifaxis) then
         ffx = -g*t(ix,iy,iz,ie,1)/cp
         ffy = 0.0
         ffz = 0.0
      else
         ffx = 0.0
         ffy =  g*t(ix,iy,iz,ie,1)/cp
         ffz = 0.0
      endif
c
      return
      end
```