Atmospheric and Oceanic Sciences Program
Princeton University

Course: AOS 572 Atmospheric and Oceanic Waves Spring 2001.

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1) Introduction to Mountain Effects

Large mountain regions with heights extending over a considerable portion of the earth's atmosphere have a profound effect on the atmospheric circulation. It is commonly accepted that mountains significantly affect the weather in many parts of the world, and the large-scale distribution of orography has important effects on the atmospheric general circulation and hence on the regional and global climate. Air flow in the vicinity of large mountain barriers creates many unique weather anomalies of varied space and time scales. At the climatic end of the spectrum is the large deflection of storm tracks due to the production of blocking highs by mountain ranges. The role of mountains in maintaining extensive midlatitude arid regions has been suggested by a several studies such as Broccoli and Manabe (1992). They concluded that the large mountain chains produce stationary waves and that the dry regions occur upstream of the trough of these waves (shown in Fig A).

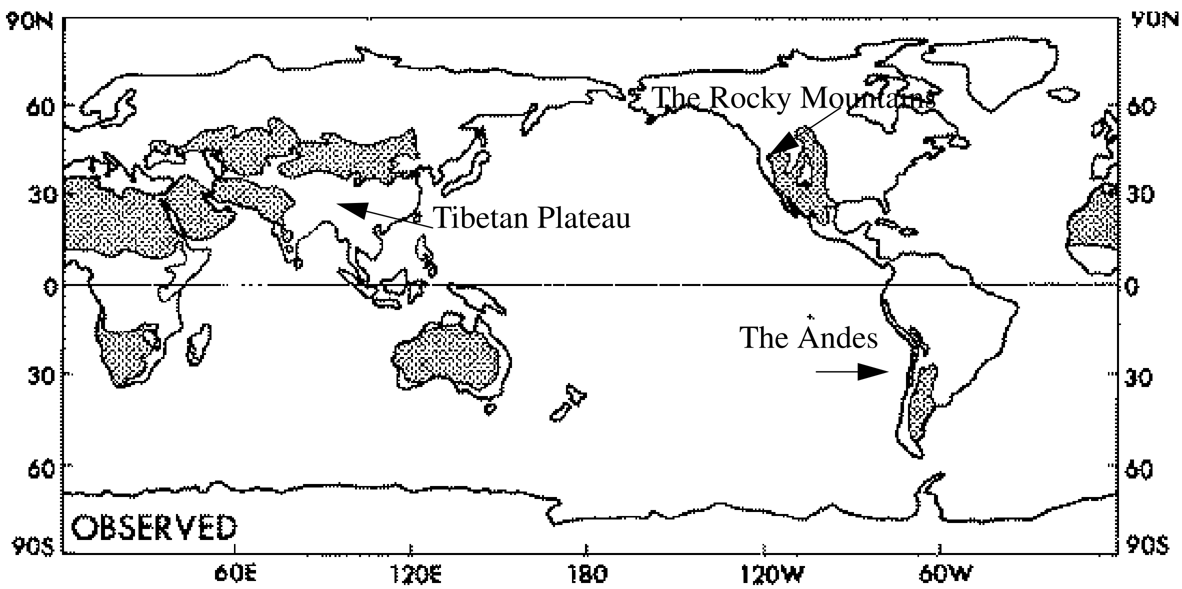


Fig A. Observed distribution of arid and semiarid climates according to the Koppen climate classification (after Oliver 1973).

On the shorter time scales (few days) regional mountain chains are an enhanced source of synoptic disturbances. Lee-cyclogenesis has attracted a great deal of recent attention in particular on the lee of the Alps. One typical lee-cyclogenesis event occurred on 3-6 March 1982 during the Alpex special observing period. In this case, a deepening upper level trough approached the European continent and the Alpine massif. A distinct cutoff low formed at 300 hPa directly south of the Alps within 24 hours, as shown in Fig B.

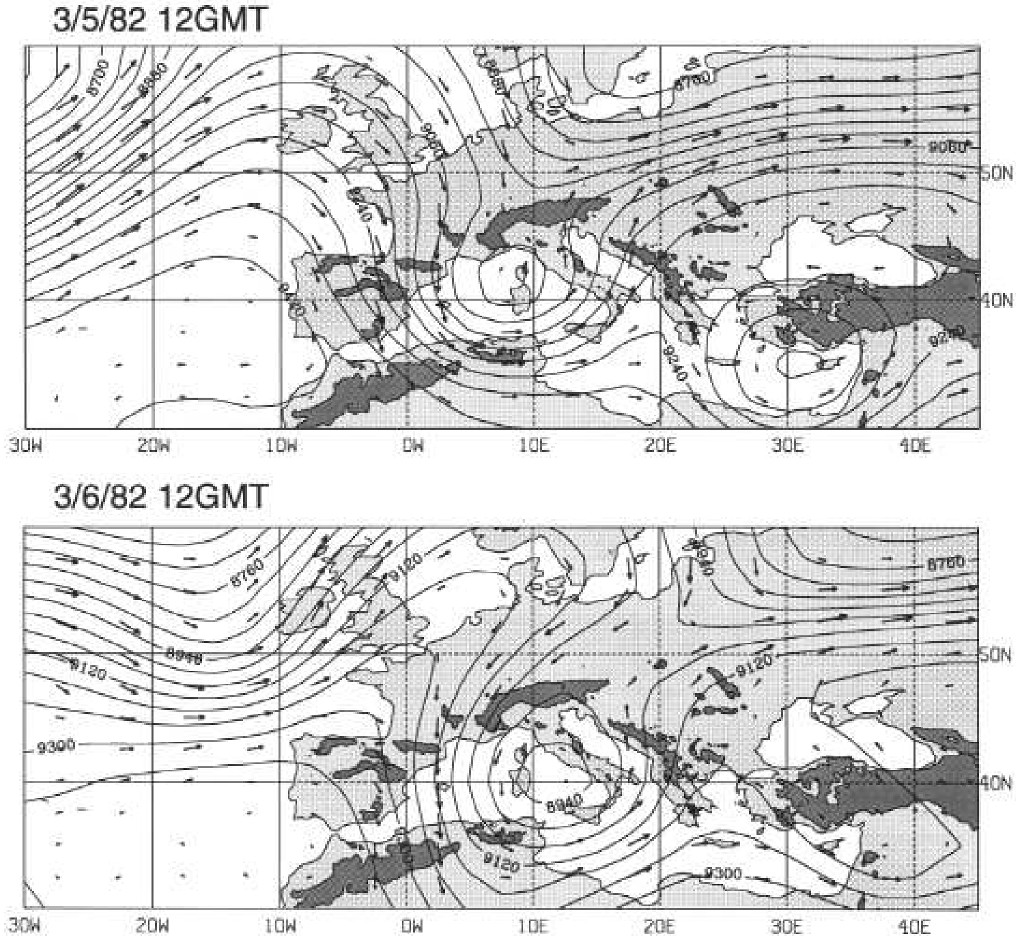


Fig B. Development of the 5 March 1982 Alpine cyclone in the 300-hPa geopotential height and velocity fields for the selected times. The contour interval is 60 dam. (after Orlanski and Gross 1994).

Very intense downslope winds are generated due to mountains in only a few tens of kilometers scales as well. Every few years the eastern slope of the Colorado Front Range (part of the Rocky Mountains) experiences a damaging windstorm, with peak gusts as high as 60 m/s. Similar winds are also observed along the lee slopes of many other mountain barriers. The local names for these winds include tha Alpine foehn, the Rocky Mountain chinook, the Yugoslavian bora and the Argentine zonda. A very authoritative review on the subject was written by Ronald Smith (1979)[1]. In his book a considerable attention was devoted to short scale mountain waves and their impact on the weather.

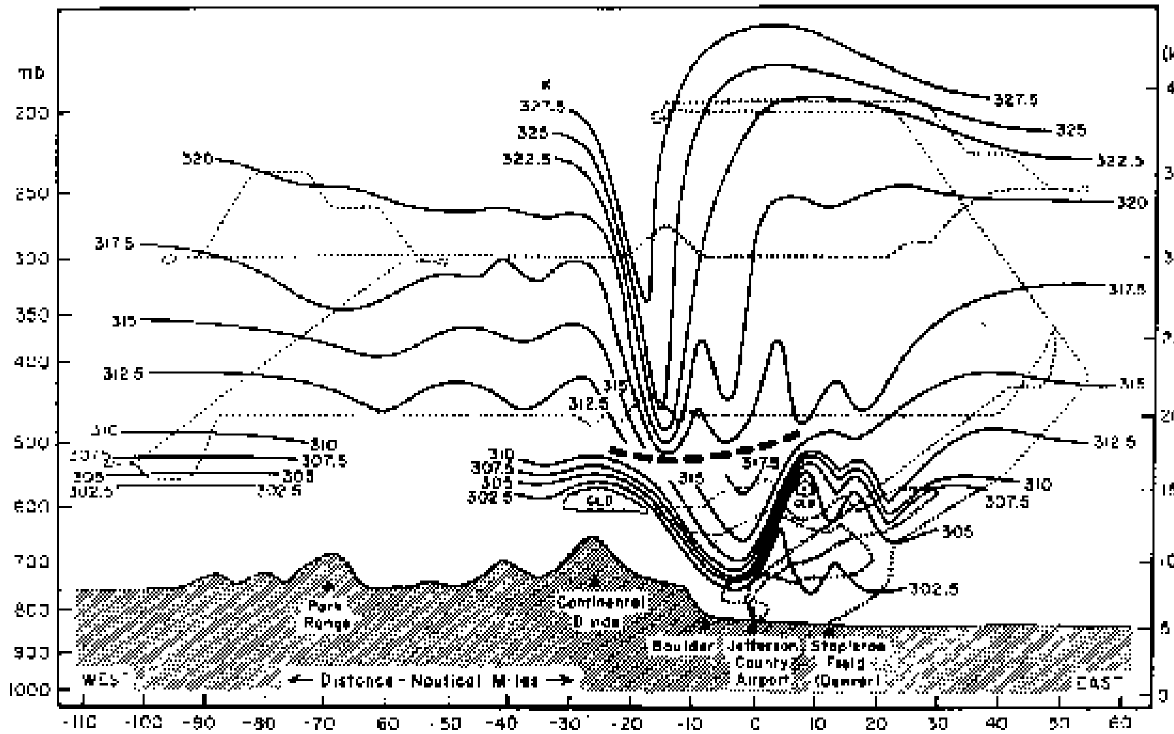


Fig. C Cross-section of the potential temperature field (K) along an east-west line through Boulder, as obtained from research aircraft on 11 January, during a downslope windstorm in Boulder. To the extent that the flow is steady and adiabatic, these isentropes are good indicators of the streamlines of air motion. Note that while the predicted vertically propagating nature of the disturbance is evident from its great vertical extent and from its tilted phase lines, the amplitude is much larger than predicted from linear theory (see Fig 3). (From Lilly and Zipser 1972, Smith article)

A disturbance is created when stably stratified air is forced to rise or deflected by a topographic barrier. The energy associated with the disturbance is carried away from the mountain by waves. The wavelength of those waves could be as small as few tens of kilometers, (known as mountain waves) or extend to the planetary scale as quasi-stationary Rossby waves. In this note however more attention will be given to the larger synoptic and planetary response. Let us start then by defining some parameters important for describing the different regimes.

1. “The influence of Mountain on the Atmosphere” Voidances in Geophysics, Vol 21.

1.1 Meso-scale analysis

To illustrate how such dimensionless parameters can be derived, let us consider a stratified fluid with stratification frequency N flowing horizontally at a speed U and encountering an obstacle of width L and height h_m (Fig 1). We can think of a wind in the lower atmosphere blowing over a mountain range.

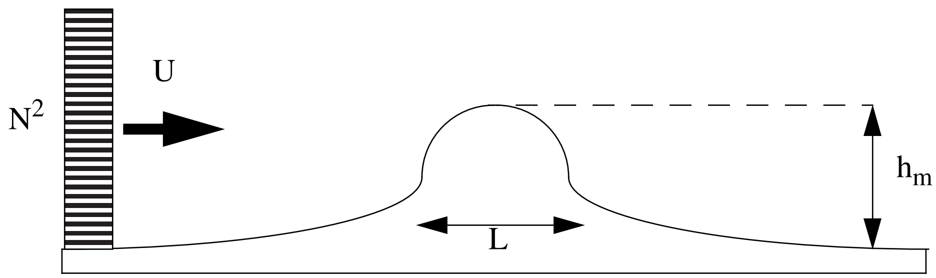


Fig 1. Situation in which a stratified fluid encounters an obstacle, forcing some fluid parcels to move vertically against gravity.

The time passed in the vicinity of the obstacle is approximately the time spent by a fluid parcel to cover the horizontal distance L at a speed of U, that is $\tau = L/U$ (advective time scale). This time is crucial for determining the role of the earth's rotation and the stratification. For instance, if $\tau > 1/2 * \Omega^{-1}$ (where Ω is the local value of the earth rotation) a time longer than a pendulum day, the flow will strongly feel the influence of rotation and the response will obey the quasi-geostrophic balances. However, if $\tau < 1/2 * \Omega^{-1}$ we could consider the flow to be in a non-rotating atmosphere and only stratification will constrain the fluid. Let us start with this limit $\tau < 1/2 * \Omega^{-1}$.

The vertical displacement close to the topography should be h_m since the vertical velocity $W = Uh_x$ at the lower boundary. Due to the effect of stratification $\Theta(z)$ the displacement cause potential temperature perturbations on the order:

$$\Delta\Theta = \frac{d\Theta}{dz} h_m = \frac{\Theta_0 N^2}{g} h_m = \theta' \qquad \text{Eq. 1.1}$$

Where $\Theta(z)$ is the fluid potential temperature upstream and $N^2 = g/\Theta_0 d\Theta(z)/dz$ The temperature variation gives rise to pressure disturbances that scale via the hydrostatic balance, as:

$$c_p\Theta_0\Pi=g\theta'/\Theta_0 H=BH$$

where H is the internal scale Eq. 1.2 and $B=g\theta'/\Theta_0$ is the buoyancy

The internal scale H can be simply derived by considering the balance of forces in the horizontal, the pressure gradient force must be balanced by the acceleration of the particle. Using the hydrostatic balance

$$U\frac{\partial u}{\partial x} = \frac{c_p\Theta_0\Pi}{L} = \frac{BH}{L} \qquad \text{Eq. 1.3}$$

since particles conserve B for an adiabatic process. The horizontal advection of B is balanced by the vertical advection given the relation $B/L=N^2W/U$. The continuity equation allows to scale the term u_x as:

$$u_x=W/H \qquad \text{Eq. 1.4}$$

Note that u_x represents the horizontal divergence and cannot be assumed to be U/L; u_x depends on the amplitude of the perturbation u velocity, which is usually much smaller than U.

Replacing Eq. 1.4 in Eq. 1.3 and using Eq. 1.2. the internal scale is given by

$$H=U/N \qquad \text{Eq. 1.5}$$

The vertical wave-length for mountain waves is given by H. It is easy to verify that:

$$h_m/H=Nh_m/U=Fr$$

$$Fr=Nh_m/U$$

Called the Froude Number Fr, is a measure of the stratification (note that in some work the Froude number is U/Nh_m). It is easy to see that the vertical gradient of buoyancy $Bz=B/H=N^2h_m/H$, $Bz=N^2Fr$. When the stratification due to the disturbance is equal to or larger than the basic stratification convection and instabilities could be generated. This condition is achieved when the Fr>1.

2.) The Anelastic system.

The momentum equation in the anelastic system is:

$$\frac{\partial}{\partial t}\vec{v} + \vec{v}\cdot\nabla\vec{v} + w\frac{\partial}{\partial z}\vec{v} + f\vec{k}\times\vec{v} = -c_p\Theta_0\nabla\pi + Diss\ . \qquad \text{Eq. 2.1}$$

v is the horizontal velocity vector with components (u,v) and w the vertical component of the velocity; Θ_0 is the reference potential temperature and π ισ the exner pressure $(p/p_0)^k$; cp is the specific heat constant and Diss represent the dissipative forces.

The other equations in the anelastic system are the modified continuity equation:

$$\nabla \bullet V + 1/\rho(\rho w)_z = 0 \qquad \text{Eq 2.2}$$

equivalent to $\nabla \bullet V + w_z = 0$ for the incompressible case and the thermodynamic equation:

$$\frac{d\theta}{dt} + w\frac{d\theta}{dz} + w\Gamma(z) = 0 \qquad \text{Eq. 2.3}$$

Remember that $\Gamma(z) = d\Theta(z)/dz$ is the lapse rate of the state at rest and θ is the deviation potential temperature of that state. This equation is the parallel of the density equation in the incompressible system where θ has the same role as the density ρ.

The system is complete with hydrostatic balance

$$cp\Theta_0\pi_z = g\theta/\Theta_0 \qquad \text{Eq. 2.4}$$

that replaces the vertical momentum equation. This assumption is valid in so far as $h_m/L \ll 1$.

2. 1 Gravity Waves

We will assume a simplified atmospheric state no rotation f=0 and the amplitude of the disturbance to be small compared with the basic state variables. As in Fig 1, let us assume a basic state in which $U=U_0$ and $N=N_0$ and ρ_0=const. This is justified if the internal scale is smaller than the scale height (33km). The linearized anelastic equations take the form;

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} = -\frac{\partial P}{\partial x} \qquad \text{Eq. 2.5}$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} = -\frac{\partial P}{\partial y} \qquad \text{Eq. 2.6}$$

$$\frac{\partial \theta}{\partial t} + U\frac{\partial \theta}{\partial x} + w\Gamma = 0 \qquad \text{Eq. 2.7}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{Eq. 2.8}$$

where $P=cp\Theta_0\pi$. Because all the coefficients in the preceding linear equations are constant, we have a wave solution of the form

$$e^{(lx+ky+mz-\omega t)}$$

Replacing the solution in the system of equations 2.5-2.8 for a non-trivial solution requires that the frequency ω be given by:

$$(\omega - lU)^2 = N^2\frac{(k^2 + l^2)}{m^2} \qquad \text{Eq. 2.9}$$

Without the hydrostatic approximation ω is given by:

$$(\omega - lU)^2 = N^2\frac{(k^2 + l^2)}{k^2 + l^2 + m^2} \qquad \text{Eq. 2.9}$$

where the terms k, l, m are the wavenumbers and N is the Brunt-Vaisala frequency previously defined (Eq. 1.1). It is easy to see that ω depends on N and the angle of the k wavenumber vector to the horizontal plane[1]. It is also possible to show that:

$$Cph = \frac{\omega\vec{K}}{|K|^2} \qquad \text{Eq. 2.10}$$

the phase velocity is parallel to the wavenumber vector and

$$Cg = \nabla\omega\perp\vec{k} \qquad \text{Eq. 2.11}$$

The group velocity is perpendicular to the wavenumber vector, this is a consequence of Eq 2.8. Replacing the solution in the divergence equation, it is simple to show that the velocity perturbation is also perpendicular to the wavenumber vector and so the direction of the energy flux.

2.2 Lee Wave solution.

Gravity waves can be generated by a number of processes, convection, orographic forcing etc. For orographic forcing, let us assume a basic state asd discussed before. The wind is blowing over a corrugate topography that only varies in x; then the y variation can be eliminated (k=0) in this problem. The steady solution will requires that w=0 and the dispersion relation 2.9 became:

$$m^2=N^2/U^2-l^2$$

Only vertically propagating waves are possible if N/U>l and for N/U>>l, the hydrostatic limit, m=N/U as was shown in our previous analysis. The wave structure in the framework fixed to the earth (Fig 2) is steady and all density surfaces undulate like the terrain, with no vertical attenuation but with an upwind phase tilt with height[1].

1. To show that write $l=K\cos\alpha s\cos\nu, k=K\cos\alpha\sin\nu\ n=K\sin\alpha$
1. The figure was copy from the book “Introduction to Geophysical Fuid Dynamics” by Benoit Cushman-Roisin

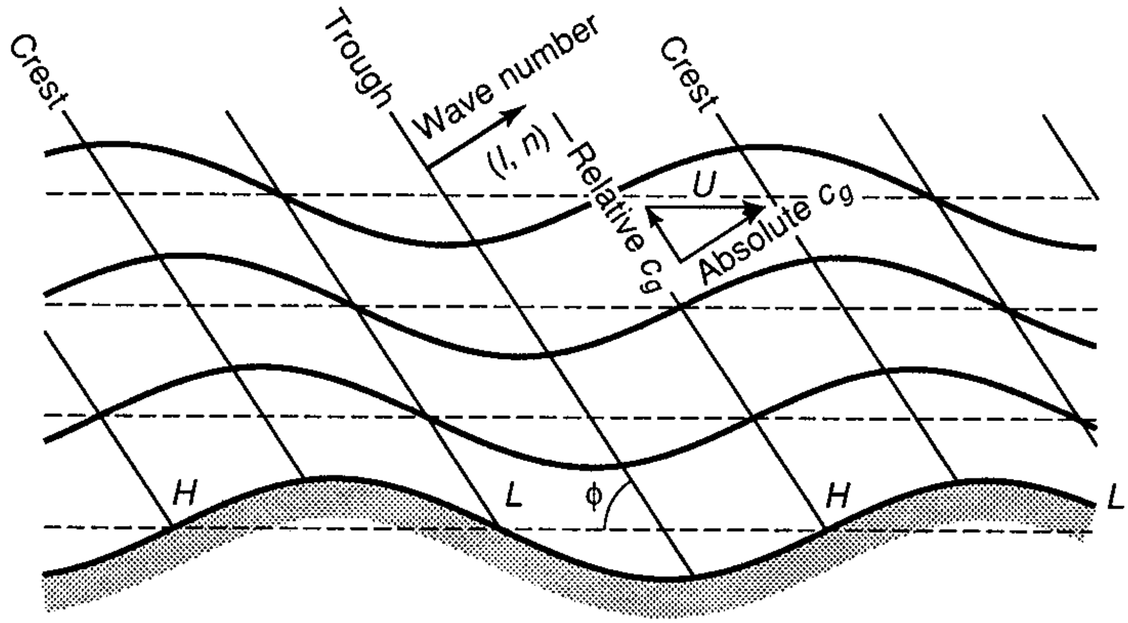


Fig 2 **Structure of the mountain wave in the case of strong stratification or long wavelength ($N > lU$). Note the absence of vertical attenuation and the presence of a phase shift with height. The group velocity with respect to the ground is oriented upward and downwind. The pressure distribution, with highs on wind-facing slopes and lows on flanks in the wind's shadow, exerts a drag on the moving air mass.**

2.4 Numerical simulation.

The importance of severe weather on the lee of mountains like downslope windstorms have been discussed by Simth (1979), Durran (1990)[1] and others. Durran showed that the development of the windstorms including wave breaking in the upper troposphere was triggered by formation of a hydraulic-jump-like disturbance in an elevated inversion near the mountaintop level (as shown in Fig C). He has performed non-linear simulations of the flow over an isolated mountain using a two dimensional model with an upstream sounding and successfuly predicted severe windstorm conditions. A solution of the model for lee-wave conditions is shown in Fig 3.

1. Book on Atmopsheric Processes over complex terrain W. Blumen Editor Meteorological Monographs AMS.

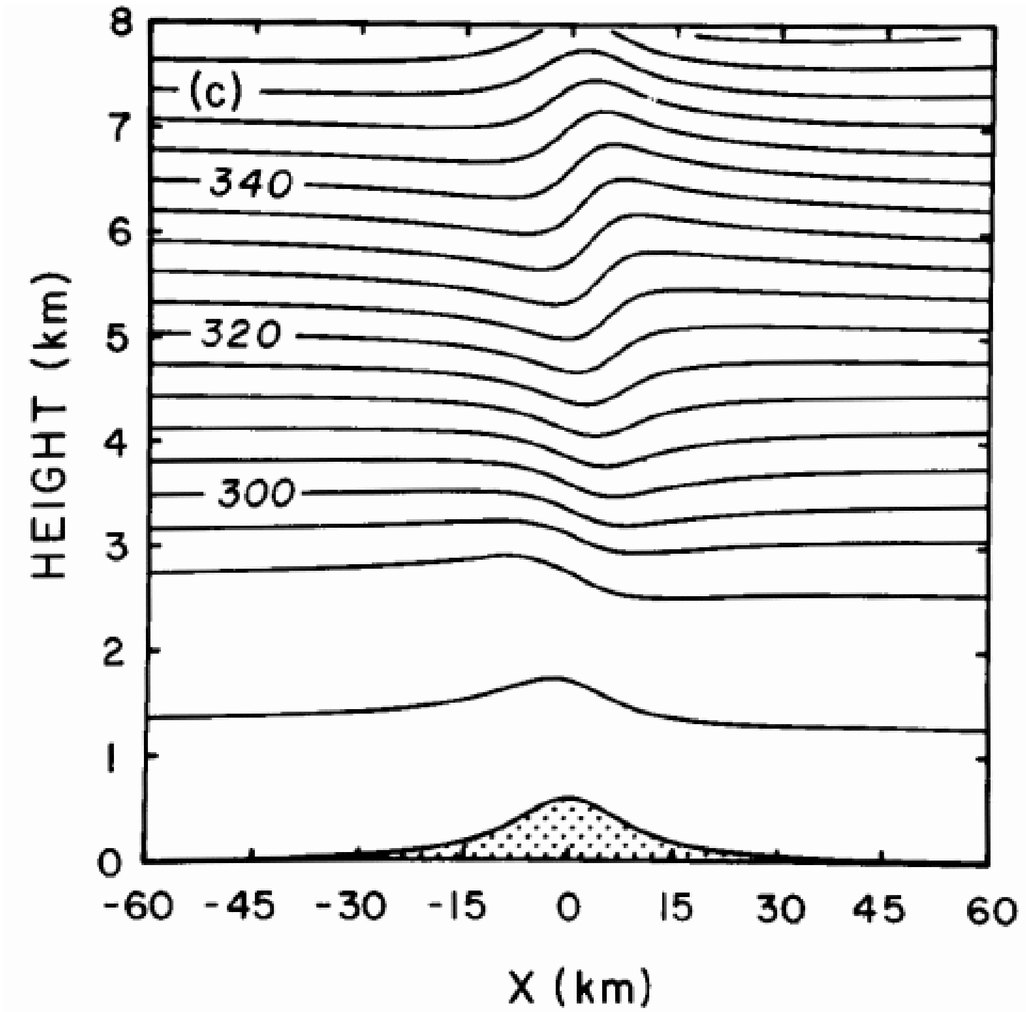


Fig 3. Isentropes in a two stratified layers atmosphere flowing over an isolated mountain at a non-dimensional time ut/a=20. The figure shows a solution with a basic flow that has a change in stratification at 3km. Nl=0.5Nu.

A detailed look at the effects of nonlinearity on the drag and surface wind was done by Durran (1990). His conclusion is that the reflection of the waves by a change in stratification can lead to a similar behavior as described for the hydraulic jump. The drag

$$D = \int_{-\infty}^{\infty} p \frac{\partial h}{\partial x} dx$$

For linear waves the drag $D=\pi\rho N_L u h_m^2/4$.

and for the nonlinear solution shown inFig3 is shown in Fig4.

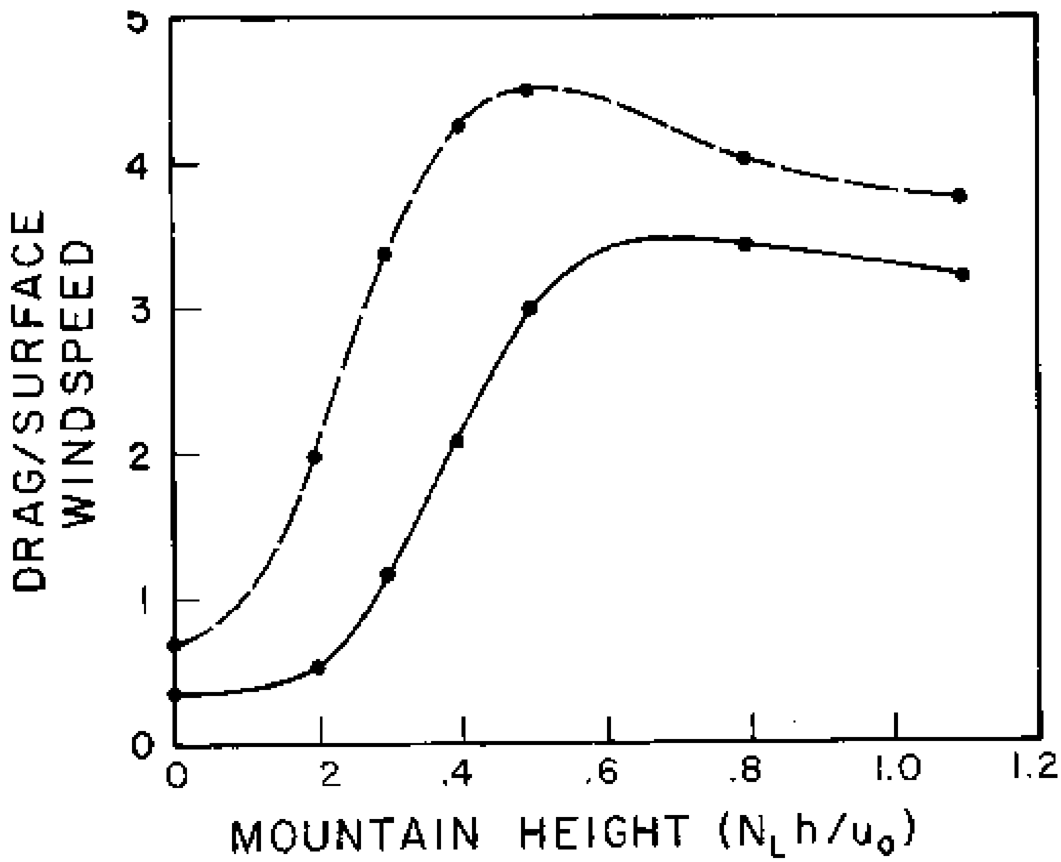


Fig 4. Pressure drag (solid) and maximum surface wind speed perturbation (dashed) as a function of Froude Number. The drag is normalized by the linear drag.

3) Quasi-Geostrophic Response

In general, non-linear dynamics are very complicated. However, the systems are extremely simplified if a time scale selection is considered. The dynamics for periods longer than a 24hs are substantially different than those for shorter time scales. In particular, τf larger or smaller than 1 (τ is the time scale of the transient response equal to the inverse of the frequency ω^{-1},and f the local coriolis parameter) determines two different regimes: $\tau f >> 1$ the earth rotation is important and the circulation is mainly horizontal and in geostrophic balance. On the other hand, for $\tau f < 1$ the flow is vertical and primarily influenced by stratification as discussed in the previous sections. Let us now consider the limit $\tau f >> 1$.

3.1 Scale -analysis:

To derive the quasi-geostrophic system we will make use of the anelastic system Eq. 2.1-2.4. We assume the following scales:

t=T*tn, x,y=L(xn,yn), z=H*zn, v=U*vn, w=W*wn, f=Ffn

Where "n" indicates the non-dimensional variables and which will be dropped in the following equations.

Let us assume that the dynamic variables are deviations of a state at rest v,w=0 and πr(z) is in hydrostatic balance with a basic potential temperature Θr(z)=Θ_0+Θ(z)=Θ_0+Γz. Furthermore, assume for the moment Diss=0. The non dimensional momentum eq is given by:

$$\left(\frac{U}{T}\right)\frac{\partial}{\partial t}\vec{v} + \left(U\frac{U}{L}\right)\vec{v}\cdot\nabla\vec{v} + \left(\frac{WU}{H}\right)w\frac{\partial}{\partial z}\vec{v} + (UF)f\vec{k}\times\vec{v} = -\left(\frac{c_p\Theta_0\Pi}{L}\right)\nabla\pi \qquad \text{Eq. 3.1}$$

Only the terms in parenthesis contain dimensional variables. Note that since π is the deviations from the πr, Π is a non dimensional amplitude such as Π/L that characterizes the exner pressure gradients. The atmosphere and the ocean for Tf>>1 are in geostrophic balance; the last term on the LHS is equal to the pressure gradient forces. Dividing all the terms by UF, we have the coriolis force of order unity the first term on the LHS is small because FT>>1. The advective terms (second and third of LHS) are small only if the Rossby number (U/LF)=Ro <<1. Since the pressure forces should balance the coriolis force

$$\Pi=(UFL/cp\Theta_0) \qquad \text{Eq. 3.2}$$

Characteristic thermodynamic and mid-latitude atmospheric values are:

H=10km, F=0.0001s^{-1} U=10m/s, L=1000km, Θ_0=300K, cp=1004 J/(K kg)=1004$(m/s)^2K^{-1}$
Γ=6K/km

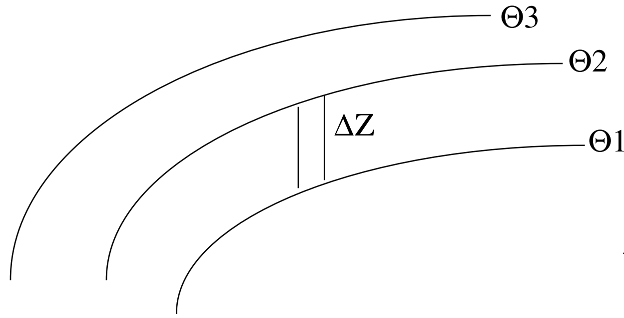
$$\Pi=(UFL/cp\Theta_0)\sim 1/300 \qquad \text{Eq 3.3}$$

The selection of the time scale T is rather arbitrary and will depend on the phenomena to be describe. For the planetary scale, a relevant time scale is the advective time scale T=L/U.

$$R_0\left(\frac{\partial}{\partial t}\vec{v} + \vec{v}\cdot\nabla\vec{v}\right) + \left(\frac{W}{HF}\right)w\frac{\partial}{\partial z}\vec{v} + f\vec{k}\times\vec{v} = -\nabla\pi \qquad \text{Eq. 3.4}$$

Since TF>>1, FL/U>>1 or (U/LF)=Ro<<1

To chose the scale of W, we should use the Potential Vorticity conservation argument.



$$\frac{d}{dt}\left(\frac{\zeta + f}{\Delta Z}\right) = 0$$

where ζ is the vertical component of the relative vorticity (v_x-u_y)

$$\frac{d\zeta}{dt} = -\frac{f}{\Delta z}\frac{d}{dt}(\Delta z) \qquad \text{Eq. 3.5}$$

since ζ scales as U/L and d/dt as U/L, further more $d\Delta z/dt$ scales as W. Then

$$W = R_0\,(H/L)U \text{ and } (W/HF) = R_0^2 \qquad \text{Eq. 3.6}$$

Finally if $f=f_0$, the main balance of the expression above is the **geostrophic relation.** The exner

$$R_0\left(\frac{\partial}{\partial t}\vec{v} + \vec{v}\cdot\nabla\vec{v} + R_0 w\frac{\partial}{\partial z}\vec{v}\right) + f\vec{k}\times\vec{v} = -\nabla\pi \qquad \text{Eq 3.7}$$

pressure π is proportional to a stream function ϕ such as $V=K\mathrm{x}\nabla\,\phi$ and $\phi=\pi/f_0$. The other equations in the anelastic system are the modified continuity equation:

$\nabla\bullet\, V+1/\rho(\rho w)_z=0$ equivalent to $\nabla\bullet\, V+ w_z=0$ for the incompressible case

and the thermodynamic equation:

$$\frac{d\theta}{dt} + w\frac{d\theta}{dz} + w\Gamma = 0 \qquad \text{Eq 3.8}$$

Remember the Γ is the lapse rate of the state at rest and θ is the deviation potential temperature of that state. This equation is the parallel of the density equation in the incompressible system where θ has the same role as the density ρ. The scale for θ is (L/U)W Γ.

Recalling W= R_0(H/L)U, substituting for q in B=g θ /Θ_0 (the buoyancy) and since N, the Brunt-Vaisala frequency of the atmosphere at rest, is defined as N=(g $\Gamma/\Theta_0)^{1/2}$, B scales as:

$$B=R_0N^2H \qquad \text{Eq 3.9}$$

The Buoyancy is R_0 smaller than the amplitude of the buoyancy N^2H of the state at rest. Furthermore, at these scales the atmosphere and oceans are in ***hydrostatic and geostrophic*** balance. Geostrophic balance has been achieved by requiring R_0 <<1, where as for hydrostatic balance.

$$cp\Theta_0\Pi/Hd\pi/dz=Bb \text{ or the scale } cp\Theta_0\Pi/H=B$$

Since Π=(UFL/cpΘ_0) the relation NH/FL=1 should be satisfied. Eq. 3.10

The length scale L satisfying this relation is often called the Rossby radius of deformation or conversely H Rossby depth of penetration. If surface forcing is acting over the ocean with a given length L the maximum depth that can be achieve in a geostrophic balance flow is H. Similarly if there is a body force with a height H acting in the interior of the fluid, the horizontal extent will be L.

The $(NH/FL)^2$ is also known as the Burger number that is defined as R_0^2Ri. The Richardson number

$$Ri=N^2/Uz^2, \text{ is a ratio of buoyancy and vertical shear}$$

The scale is $(NH/U)^2$ and when multiplied by the $R_0^2 = (U/LF)^2$ gives $(NH/FL)^2$. This ratio should be the order of unity. For small Ro<<1, Ri should be large>>1. For these scales in the atmosphere Ro$\cong$ 0.1 and for the ocean Ro$\cong$ 0.01 whereas the Ri$\cong$100.

Ro<<1 and $(NH/FL)^2 \cong 1$ are the basic scaling for the ***quasi-geostrophic approximation.***

3.2 Quasi Geostrophic Equations

Expanding the variables in power of Ro:

$\mathbf{v}=\mathbf{v}o+Rov_1+Ro^2\mathbf{v}_2+O(Ro^3)$

$w=Ro(H/L)(wo+Row_1+Ro^2w_2+O(Ro^3))$

$\pi=\pi_r(z) + \Pi(\pi_0+Ro\pi_1+Ro^2\pi_2+O(Ro^3))$

$b=b_r(z) + N^2HRo(b_0+Rob_1+Ro^2b_2+ O(Ro^3))$

$f=f_0+Ro\beta(y-y_0)+O(Ro^2)$

$N^2(z)=N^{2}*N(z)^2$

Substituting in the equations of motion:

$$R_0\left(\frac{\partial}{\partial t}\vec{v} + \vec{v}\cdot\nabla\vec{v} + R_o w\frac{\partial}{\partial z}\vec{v}\right) + f\vec{k}\times\vec{v} = -\nabla\pi$$ Eq. 3.11

$p_z=b$ Eq. 3.12

$\nabla\bullet V+1/\rho(\rho w)_z=0$ Eq 3.13

and

$$\frac{db}{dt} + w\frac{db}{dz} + wN^2 = 0$$ Eq 3.14

The O(1)

$f_0kxv_0=-\nabla\pi_0$ geostrophic balance

$\nabla\bullet v_0=0$ non divergent

$p_0z=b_0$ hydrostatic

wo=-(1/N(z)2)dbo/dt vertical velocity is determine by heat eq.

$\phi o=\pi_0/f_0$ where ϕo is the geostrophic stream function

$v_0=kx\nabla\phi o$

$b_0=f_0\phi oz$ In the quasi geostrophic system all the variables can be derived from ϕo to an order of O(Ro).

$$\left(\frac{\partial}{\partial t}\vec{v}_0 + \vec{v}_0 \cdot \nabla\vec{v}_0\right) + f_0\vec{k} \times \vec{v}_1 + \beta(y - y_0)\vec{k} \times \vec{v}_0 = -\nabla\pi_1 \qquad \text{Eq 3.15}$$

$$\nabla \bullet v_1 = -1/\rho(\rho w_0)_z \qquad \text{Eq 3.16}$$

taking the curl of the first order momentum equation and because $\mathbf{k} \bullet \nabla \times v_0 = \zeta o = \nabla^2 \phi o$.

$$\left(\frac{\partial}{\partial t}\zeta_0 + \vec{v}_0 \cdot \nabla\zeta_0\right) + f_0\nabla\vec{v}_1 + \beta v_0 = 0 \qquad \text{Eq. 3.17}$$

The changes in the vertical component of the relative vorticity are accomplished by advecting planetary vorticity and stretching by convergence.

Combining the vorticity equation and the thermodynamic equations we can derive the conservation of *pseudopotential vorticity or quasigeostrophic potential vorticity Q*

3.3 Quasigeostrophic Potential Vorticity

Since the horizontal divergence $\nabla \bullet v_1 = -1/\rho(\rho w_0)_z$ from the potential temperature eq.

$$wo = -(1/N(z)2)dbo/dt$$

the advection by the geostrophic flow $v_0 \nabla bo$ can be expressed as the Jacobian $J(\phi, bo)$ and remembering that $b_0 = f_0 \phi_{oz}$

$$wo = -(f_0/N(z)2)(\phi o_z t + J(\phi, \phi o_z))$$

multiplying by ρ and differentiating by z the divergence is given by:

$$\nabla \bullet v_1 = 1/\rho(f_0\rho/N(z)2)(\phi_{oz}t + J(\phi, \phi_{oz}))_Z$$

The divergence of the first order velocity is given by the thermodynamic equation to zero order and can be replaced in the relative vorticity equation, finally the quasigeostrophic Potential Vortic-

$$\left(\frac{\partial}{\partial t}Q_0 + \vec{v}_0 \cdot \nabla Q_0\right) = \frac{\partial}{\partial t}Q_0 + J(\phi_0, Q_0) = 0 \qquad \text{Eq 3.18}$$

ity Q_0 is given by:

$$Q_0 = \nabla^2\phi_o + \beta(y-y_0) + f_0 2/\rho(\rho/N(z)2\phi_{oz})_Z \qquad \text{Eq 3.19}$$

Q_0 is only conserved in a quasigeostrophic system.

The conservation of Q and the relation of the vertical velocity at the boundaries $wo = -(f_0/N(z)2)(\phi_{ozt} + J(\phi, \phi_{oz}))$ solely defines the quasigeostrophic system.

3.4 Orographic Forcing (Brian Gross)

3.4.1 Steady State

The dimensional potential vorticity equation for steady motion in a quasigeostrophic system characterized by uniform potential vorticity is

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$$\nabla^2\varphi_0 + \frac{f_0^2}{\bar{\rho}}\frac{\partial}{\partial z}\left(\frac{\bar{\rho}}{\bar{N}^2}\frac{\partial\varphi_0}{\partial z}\right) = Q_0 = \text{constant} \qquad \text{Eq. 3.20}$$

to zeroth order in the Rossby number. If we make the following assumptions and approximations:

- *f*-plane (β=0)
- uniform N
- Boussinesq approximation ($\bar{\rho} = \text{constant}$ in (1))
- Q_0=0 (zero potential vorticity)
- $H=fL/N$ (vertical scale is the Rossby depth)

then the nondimensional potential vorticity equation becomes Laplace's equation

$$\nabla^2\varphi_0 + \frac{\partial^2\varphi_0}{\partial z^2} = 0 \qquad \text{Eq. 3.21}$$

Boundary conditions

Flow over terrain of height h_m that satisfies the free-slip condition at the boundary must also satisfy the (nondimensional) kinematic condition

$$w_0 = Fr[\boldsymbol{v}_0 \bullet \nabla h(x, y)] \qquad \text{Eq. 3.22}$$

at z=0, to zeroth order in the Rossby number. Here,

$$Fr = \frac{Nh_m}{U} \qquad \text{Eq. 3.23}$$

is the Froude number as before, which for quasigeostrophic flow must be small according to the scaling used to derive the quasigeostrophic system. However, for steady flow the potential temperature equation at z=0 is

$$\boldsymbol{v}_0 \bullet \nabla b_0 + w_0 = 0 \qquad \text{Eq. 3.24}$$

so that

$$\boldsymbol{v}_0 \bullet \nabla[b_0 + Frh] = J(\varphi_0, b_0 + Frh) = 0 \qquad \text{Eq. 3.25.}$$

The solutions that will be discussed here will satisfy (3.25) by specifying isentropic terrain for which

$$b_0 = -Frh(x, y) \qquad \text{Eq 3.26}$$

at z=0. This represents a cold perturbation along the topography. An example is shown in Fig. 5. Note that with this scaling, the only nondimensional parameter appearing in the problem is the

Froude number *Fr*.

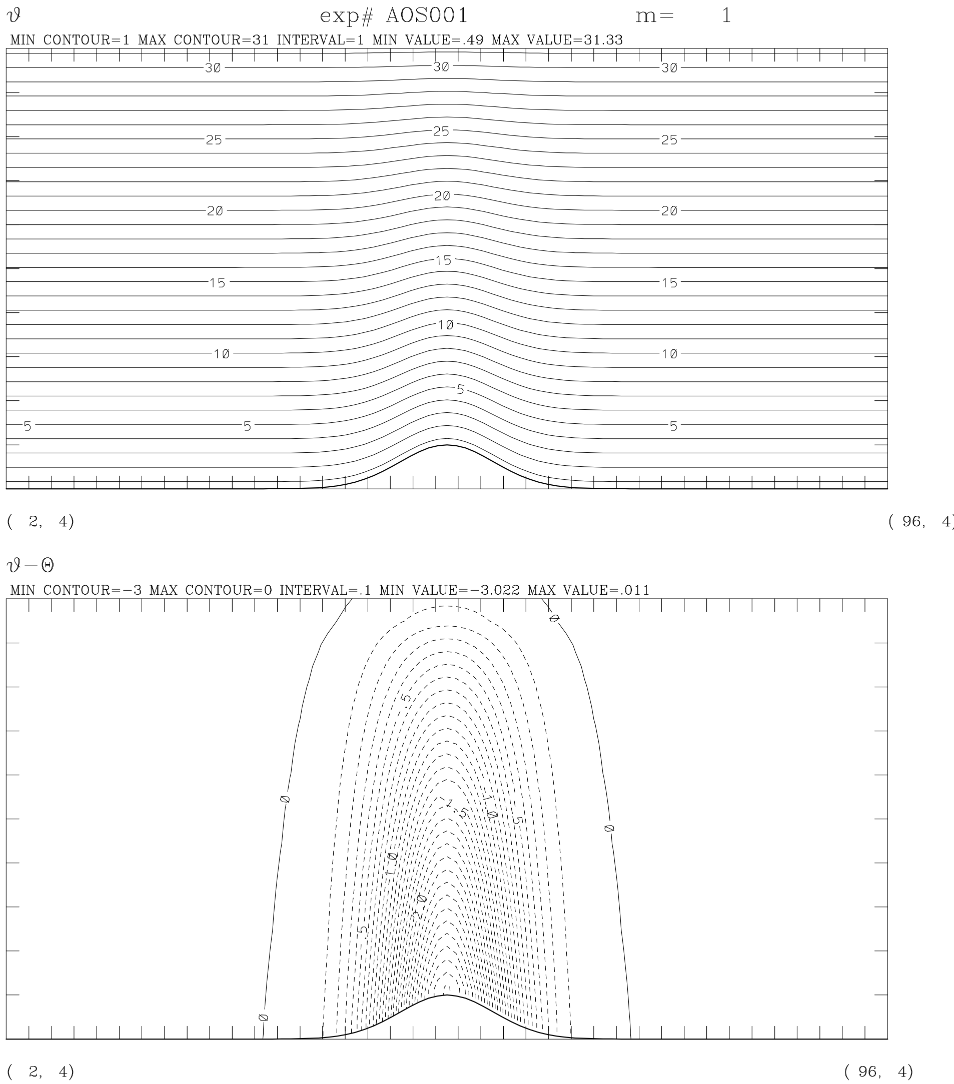


Figure 5. Distribution of potential temperature b_r+b_0 (top) and b_0 (bottom) illustrating how a cold perturbation can produce an isentropic lower boundary. This is *not* a solution to the quasigeostrophic system.

3.4.2 Two-dimensional solutions

Consider a uniform flow (of nondimensional magnitude 1) over a ridge of infinite north-south extent. The flow perturbations induced by the ridge are assumed to be independent of y. In this case, the governing equations become

$$\frac{\partial^2 \varphi_0}{\partial x^2} + \frac{\partial^2 \varphi_0}{\partial z^2} = 0 \qquad \text{Eq. 3.27}$$

with

$$b_0 = \frac{\partial\varphi_0}{\partial z} = -Frh(x) \text{ at z=0} \qquad \text{Eq. 3.28}$$

The corresponding solutions to Laplace's equation (3.27) will be determined by the boundary condition at the upper boundary

3.4.3 Semi-infinite atmosphere

The appropriate solution to (3.27) in an unbounded atmosphere is given by (Smith 1979a[1])

$$\varphi_0 = -y - \frac{\mu}{4\pi}\log[x^2 + (z+d)^2] \qquad \text{Eq.3.29}$$

which corresponds to a source of strength μ at (*x*,*z*)=(0,-*d*). Then

$$u_0 = -\frac{\partial\varphi_0}{\partial y} = 1 \qquad \text{Eq. 3.30}$$

corresponds to the incoming uniform flow,

$$v_0 = \frac{\partial\varphi_0}{\partial x} = -\frac{\mu}{2\pi}\frac{x}{x^2 + (z+d)^2} \qquad \text{Eq. 3.31}$$

is the meridional velocity induced by the ridge, and

$$b_0 = \frac{\partial\varphi_0}{\partial z} = -\frac{\mu}{2\pi}\frac{z+d}{x^2 + (z+d)^2} \qquad \text{Eq. 3.32}$$

is the induced buoyancy perturbation, representing the cold air over the ridge. Although the actual geopotential streamfunction (3.29) is unbounded, both the velocity and buoyancy perturbations decay away from the ridge. According to (3.28), the topographic profile is a bell-shaped curve given by

$$Frh(x) = \frac{\mu}{2\pi}\frac{d}{x^2 + d^2}\,. \qquad \text{Eq. 3.33}$$

The strength μ and the position *d* of the source under the "ground" (z=0) may be used to create a

1. Smith, R.B., 1979a: The influence of mountains on the atmosphere. *Advances in Geophysics*, Vol. 21, Academic Press, 87-230.

ridge of the desired shape. An example of this solution is shown in Fig. 6.

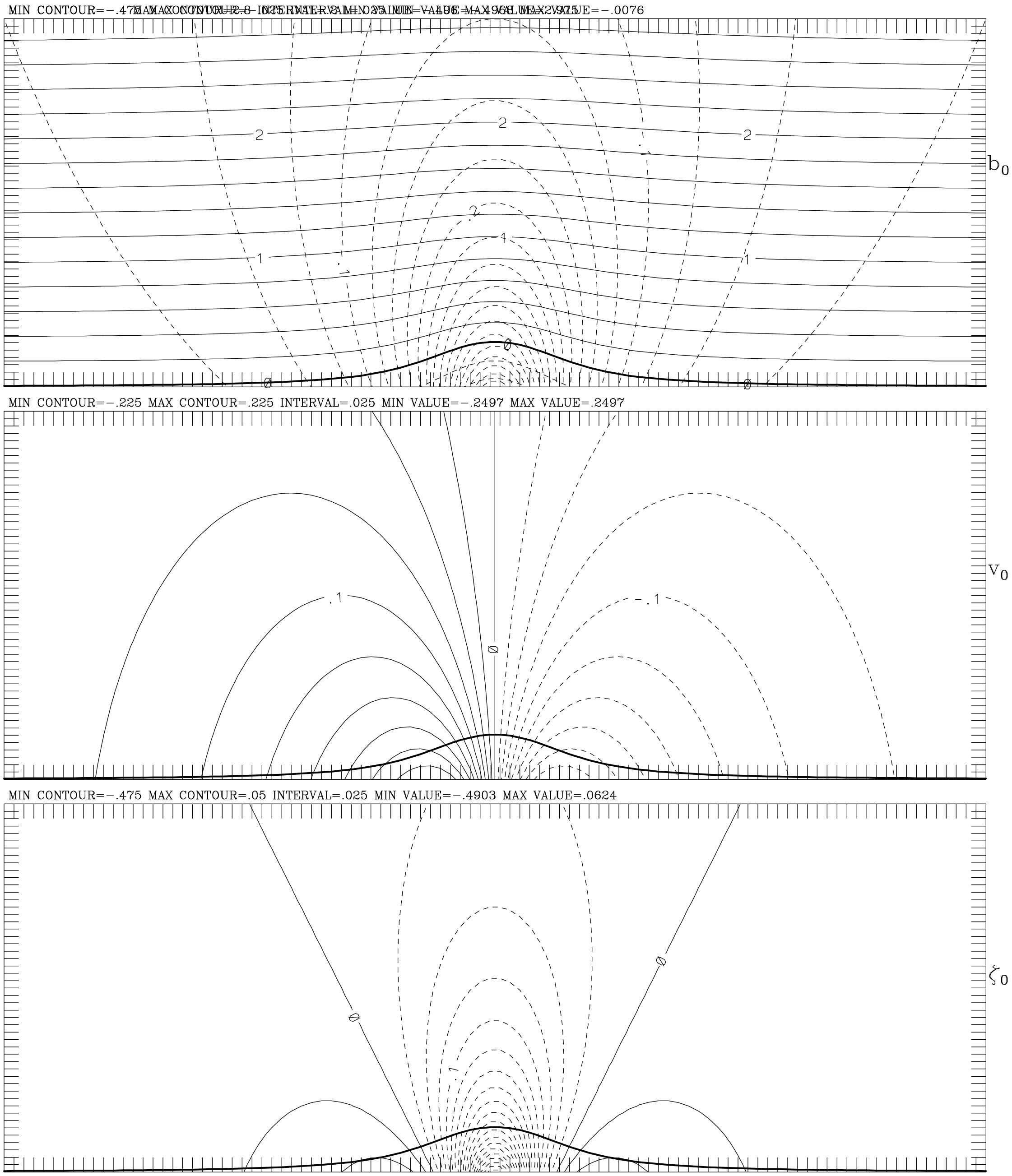


Figure 6. The quasigeostrophic solution for steady flow over a semi-infinite ridge in a semi-infinite atmosphere. The plots show (top) total buoyancy (solid) and perturba-

Now according to the thermodynamic equation

$$w_0 = \frac{dz}{dt} = -\frac{db_0}{dt}, \qquad \text{Eq. 3.34}$$

the displacement of θ-surfaces from their upstream value is

$$\eta(x, z) = -b_0 = -\frac{\partial\varphi_0}{\partial z} = \frac{\mu}{2\pi}\frac{z+d}{x^2+(z+d)^2} \qquad \text{Eq. 3.35.}$$

and the displacement area under each isentrope is

$$\int_{-\infty}^{\infty} \eta(x, z)dx = \frac{\mu}{\pi} \qquad \text{Eq. 3.36,}$$

which is a constant independent of height. According to (3.36), as the maximum displacement of a θ-surface decreases with height, its horizontal breadth increases.

One consequence of (3.36) is the creation of regions of cyclonic relative vorticity upstream and downstream of the ridge, as shown in Fig. 6, which are associated with vortex tube stretching as fluid parcels enter regions where the θ-surfaces become vertically separated. Directly over the ridge, however, the separation of θ-surfaces decreases from its upstream value, and vortex tube compression generates anticyclonic relative vorticity.

Implicit in this potential-vorticity conservation argument is that mass conservation requires parcels to decelerate where the θ-surfaces become vertically separated upstream and downstream of the ridge and accelerate where θ-surfaces approach one another over the ridge. These accelerations imply a disruption of pure geostrophic balance, and cyclonic and anticyclonic curvature are produced in these respective regions by the unbalanced pressure gradient forces. Indeed, the horizontal divergence associated with vortex tube stretching and compression *requires* the next higher order approximation to pure geostrophic flow that is the hallmark of quasigeostrophic theory.

3.4.4 3-D Solution for Semi-infinite atmosphere

The solution to (3.27) in a semi-infinite atmosphere also corresponds to a source of strength μ at $(x, y, z) = (0, 0, -d)$, given in three-dimensions by

$$\varphi_0 = -y + \frac{\mu}{4\pi r} \qquad \text{Eq. 3.37}$$

where

$$r = \sqrt{x^2+y^2+(z+d)^2} \qquad \text{Eq. 3.38.}$$

Then

$$u_0 = 1 + \frac{\mu y}{4\pi r^3} \qquad \text{Eq. 3.39,}$$

$$v_0 = -\frac{\mu x}{4\pi r^3} \qquad \text{Eq. 3.40,}$$

$$b_0 = -\frac{\mu(z+d)}{4\pi r^3} \qquad \text{Eq. 3.41,}$$

and

$$Frh(x, y) = \frac{\mu d}{4\pi(x^2 + y^2 + d^2)^{3/2}} \qquad \text{Eq. 3.42}$$

which corresponds to an isolated mountain with circular height contours. Appropriate values of μ and *d* for a given topographic profile may be determined from (3.42). The streamfunction at *z=Frh(x,y)* and *z*=3.0 is shown in Fig. 7. Note that the closer spacing of the streamlines north of

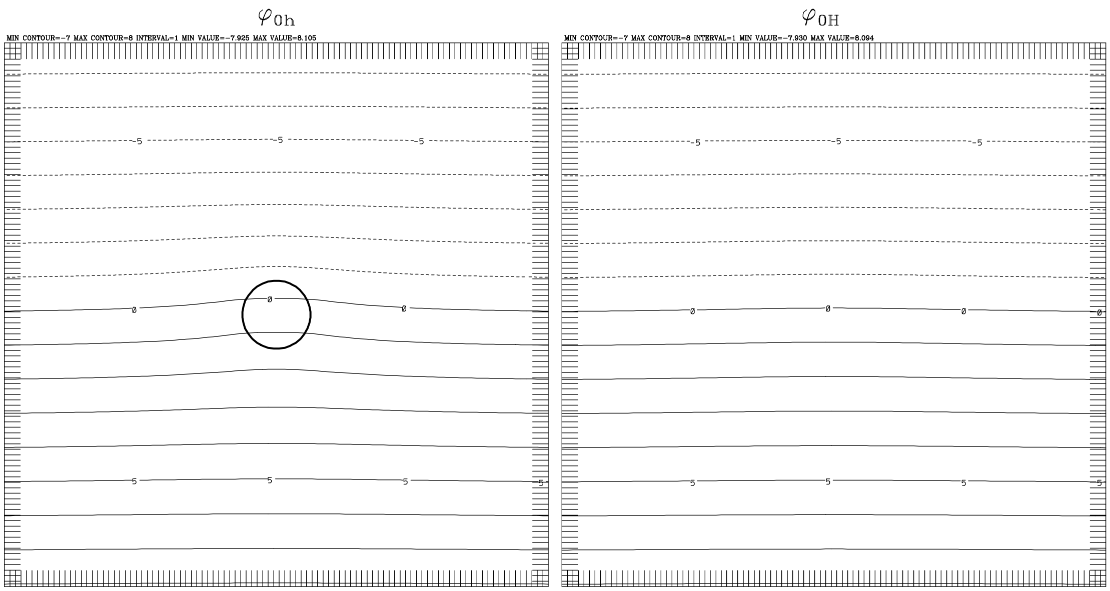


Figure 7. Geostrophic streamfunction (26) at z=Frh (left) and z=3.0 (right). The

the mountain in Fig. 7 indicate larger zonal velocities there, while the zonal flow is decreased south of the mountain. In fact, if the Froude number is large enough (>2.5 in the present case), the geostrophic flow may be decelerated to rest. However, this would violate the scaling assumptions used in deriving quasigeostrophic theory, in particular (3.).

The vertical component of relative vorticity is

$$\zeta_0 = -\frac{\mu}{4\pi r^3}\left(2 - \frac{3R^2}{r^2}\right) \qquad \text{Eq. 3.43,}$$

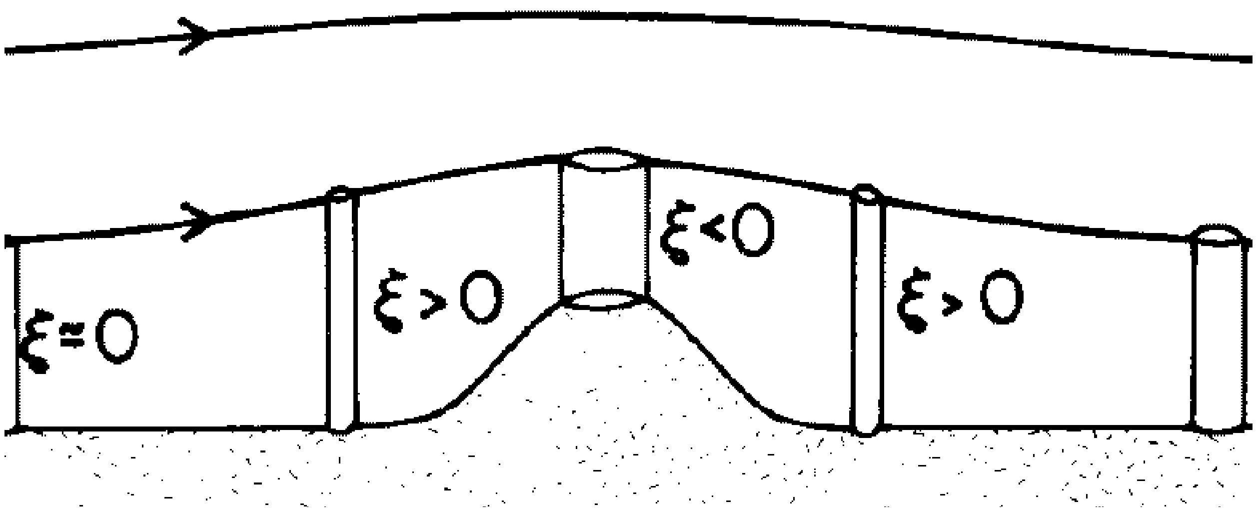
which is negative (anticyclonic) near the mountain (small R) and positive (cyclonic) but weak far from the mountain (large R). All of the dependent variables (3.37)-(3.43) possess axial symmetry for the circular mountain (3.42).The displacement of θ-surfaces satisfies

$$\int_{-\infty} \eta(x, y, z)dxdy = \frac{\mu}{2} \qquad \text{Eq. 3.44}$$

so that the volume under any θ-surface is constant. The upward displacement of θ-surfaces decreases with height while the horizontal extent of the displacement increases, which produces the cyclonic vorticity far from the mountain by the same mechanism as in the two-dimensional solution.

Some aspects of the solution (3.37) are shown in Fig. 5. Most features of this solution are very similar to those of the two-dimensional solution in a semi-infinite atmosphere shown in Fig. 6. The major distinguishing feature is that the anticyclonic vorticity is stronger and the cyclonic vorticity away from the mountain is weaker in the isolated mountain solution.

Schematics of the Stratified quasi-geostrophic flow over an isolated mountain.



The vorticity dynamics in a stratified quasi-geostrophic flow over an isolated mountain. The magnitude of the lifting of Q surfaces aloft is less than the mountain height, but the lifting is more widespread. As parcels near the ground approach the mountain, they are first streched producing cyclonic vorticity. Over the mountain, the parcels are shortened producing anticyclonic vorticity. The total amount of cyclonic and anticyclonic vorticity are equalat each level and, as a result there

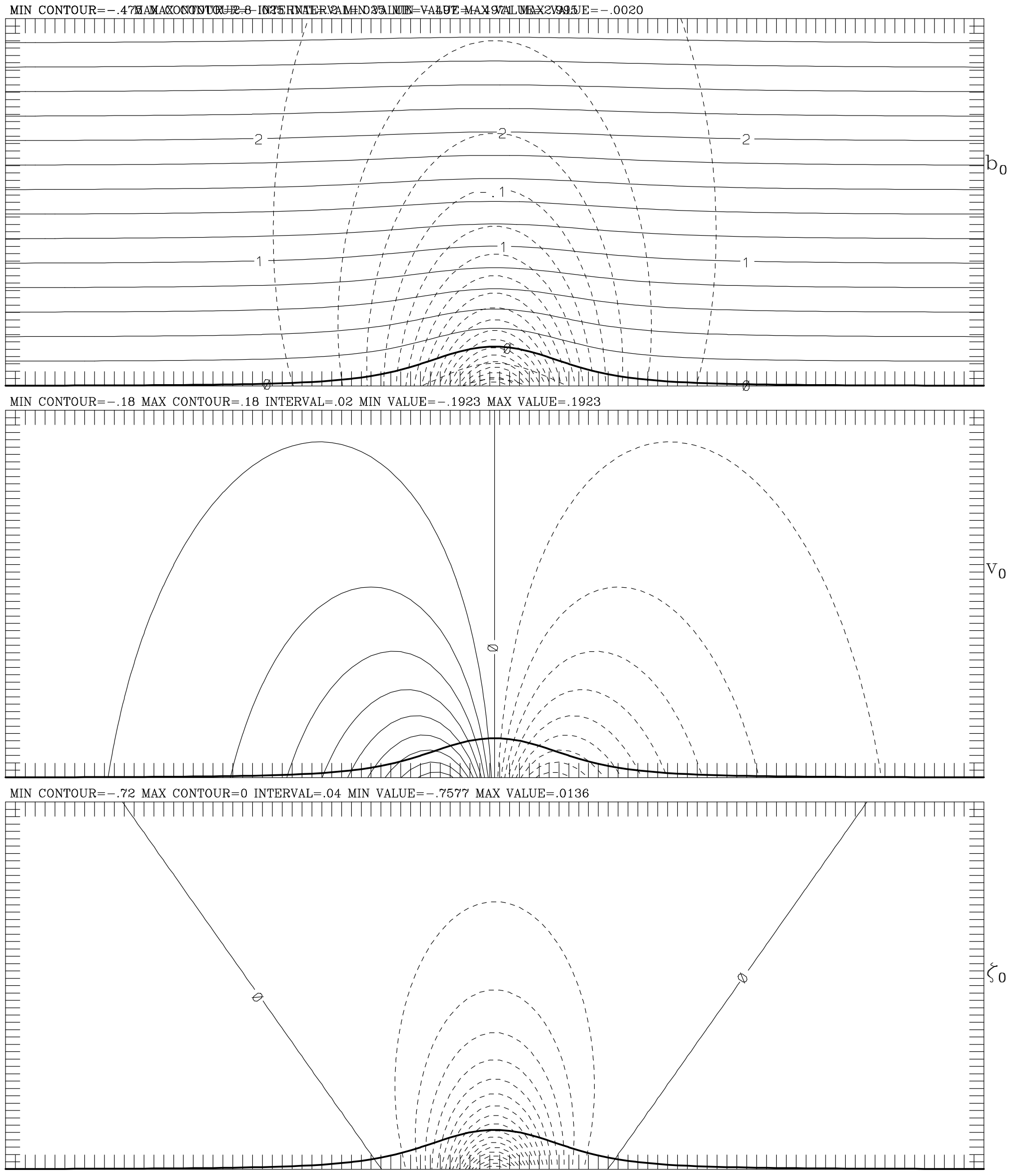
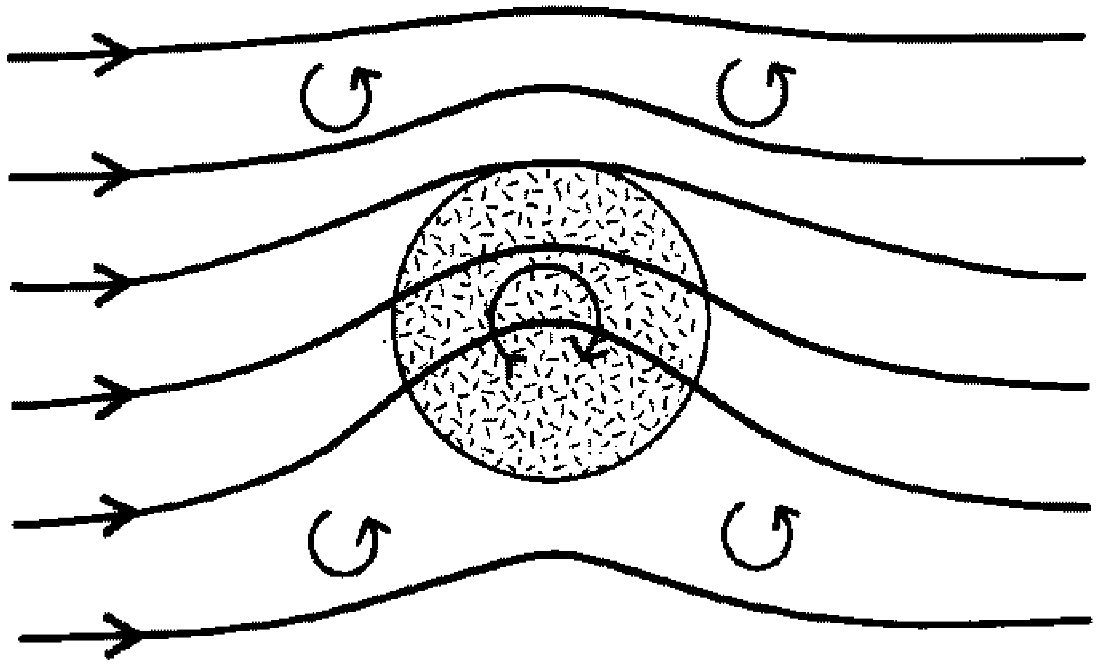


Figure 8. The quasigeostrophic solution at y=0 for steady flow over a circular mountain in a semi-infinite atmosphere. The plots show (top) total buoyancy

is not far-field circulation.



The incoming flow is distorted by the mountain anticyclone. The perturbation velocity and pressure field decay away from the mountain. (After Buzzi and Tibaldi 1977).

3.5 Planetary Scale

3.5.1 The Barotropic Response:

Let us assume that we are interested in the vertical integrated response. For simplicity let us assume constant density and uniform Q. However, a similar results can be obtained by the vertically averaged eq 3.19, weighted by the density. The averaged Q is then:

$$\langle Q_0\rangle=\nabla^2\langle\phi o\rangle +(f_0 2/HN(z)2\phi oz)_{top} -(f_0 2/HN(z)2\phi oz)_{bottom} \qquad \text{Eq. 3.5.1}$$

where <F> indicates the vertical integral of F divided by H the height of the fluid layer.

recalling that $f_0\phi oz=b=g\ \theta/\Theta_0$

$$\langle Q_0\rangle=\nabla^2\langle\phi o\rangle +(f_0/HN(z)2g\ \theta/\Theta_0)_{top}-(f_0/HN(z)2g\ \theta/\Theta_0)_{bottom} \qquad \text{Eq. 3.5.2}$$

The average Q is affected by the averaged relative vorticity and by the temperature anomalies at the boundaries. Since $\langle Q_0\rangle$=cte, the deviations of a state at rest will imply that:

$$\nabla^2\langle\phi o\rangle =-(f_0/HN(z)2g\ \theta/\Theta_0)_{top}+(f_0/HN(z)2g\ \theta/\Theta_0)_{bottom} \qquad \text{Eq. 3.5.3}$$

It is easily recognized that the RHS represents the thickness between the material surfaces at both boundaries. In fact the time variation of the vertically averaged vorticity is given by:

$$d\nabla^2\langle\phi o\rangle/dt=f_0(w_{top}-w_{bottom})/H, \qquad \text{Eq. 3.5.4}$$

remember that $w=-d(f_0/N(z)2g\ \theta\ /\Theta_0)\ /dt$. The relation between temperature anomalies at the boundaries and the vertically averaged vorticity can be simply seen if we assume that the upper boundary is flat w=0 no temperature anomalies at that boundary, Then,

$$\nabla^2\langle\phi o\rangle =+(f_0/HN(z)2g\ \theta\ /\Theta_0)_{bottom}$$

a warm anomaly at the surface (lower boundary) corresponds to positive vorticity (cyclonic) and a cold anomaly corresponds to anticyclonic vorticity. Furthermore, since $\phi o=\pi_0/f_0$, a warm anomaly (cyclonic vorticity) implies $\phi o<0$ and π_0 and pressure, to be minimum (Low) anticyclonic vorticity corresponds to a maximum in pressure (High). These conclusions can be extended to the topographic response.

Since the lower boundary is assumed isentropic

$$(f_0/HN(z)2g\ \theta\ /\Theta_0)_{bottom} =-(f_0/H)*h(x,y)$$

[h>0, cold anomaly and h<0 warm anomaly]

$$\nabla^2\langle\phi o\rangle =-(f_0/H)*h(x,y)\ ,\qquad h>0 \text{ produce anticyclonic vorticity}$$

$\nabla^2\langle\phi o\rangle<0$. as discussed in 3.4.

3.5.2 Orographic response in a beta plane.

If beta is different than zero, from eq 3.19

$$\langle Q_0\rangle=\nabla^2\langle\phi o\rangle +\beta(y-y_0)+(f_02/HN(z)2\phi oz)_{top} -(f_02/HN(z)2\phi oz)_{bottom} \qquad \text{Eq. 3.5.5}$$

since in the steady state:

$$J(\phi o,Q_0)=0 \qquad \text{Eq. 3.5.6}$$

assuming a flow bounded by a rigid lid at z=H and a topography disturbance h(x,y) at z=0.

The boundary conditions at z=0 and z=H are

$$w(x,y,0)= J(\phi o,h(x,y))=-J(\phi o,(f_0/N(z)2\phi oz))_{bottom} \text{ and } w(x,y,H)=0 \qquad \text{Eq. 3.5.7}$$

Eq 3.14 with b.c (3.17) has the solution:

$$\langle Q_0\rangle=F(\phi o) \text{ and } h(x,y)+f_0/N(z)2\phi oz=G(\phi o) \qquad \text{Eq. 3.5.8}$$

if G=0 is the isentropic b.c, the temperature surfaces at the ground are parallel to the surface topography. The first relation in (3.5.8) is the conservation of Q along streamlines. The functional form of F can be determined by the knowledge of the relation of Q and ϕo at an inflow boundary.

3.5.3 Channel flow over a ridge.

Let us determine the solution for this case:

at the left boundary say $x=-\infty$:

$$\phi o(-\infty)=\Psi(y) =-U_0(y-y_0) \text{ and}$$

$$Q_0(-\infty)=\beta(y-y_0)=-\beta/U_0\Psi(y)$$

Then $Q_0(x,y)=-\beta/U_0\ \phi o(x,y)$ where $\phi o(x,y)=\Psi(y)+\phi(x,y)$ Eq. 3.5.9

Substituting eq (3.5.9) in (3.5.8) we get:

$\nabla^2\phi+\beta/U_0\ \phi=-f_0/h(x,y)/H$ as in the case with $\beta=0$ the vorticity is given by the planetary vorticity times the mountain height scaled by the total height. The difference is the extra term do to the meridional advection of planetary vorticity $\beta/U_0\ \phi$. In this case waves can exist even in the region of $h(x,y)=0$.

$$\text{If } U0>0 \qquad \nabla^2\phi= -\beta/U_0\ \phi$$

Rossby waves are possible. Solving the Poisson equation the wavenumber for the steady response is given by:

$$k^2+l^2=\beta/U_0 \qquad \text{Eq. 3.5.10}$$

where k and l are the zonal and meridional wavenumbers respectively. The wavelength of the response downstream of the ridge can be estimated assuming the gravest mode in the meridional direction $l=\pi/L$ where L is the width of the channel~6600km, $l=0.476\times10^{-6}$,

$\beta/U_0=1\times10^{-12}m^{-2}$ gives a value for the horizontal wavelenght λ ~ 7000km.

Energy arguments can be used to explain the fact of the response is only in the downstream direction. The dispersion relation for Rossby waves is:

$$\omega = U_0k-\beta k/(k^2+l^2) \qquad \text{Eq. 3.5.11}$$

The observed frequency is equal to the intrinsic frequency $-\beta k/(k^2+l^2)$ plus the doppler shifted frequency due to U_0. For the steady response $\omega=0$ and the wavenumbers satisfy (3.5.10). The phase velocity $Cp=\omega/k=U_0-\beta/(k^2+l^2)$, the waves tend to propagate westward but in the present of an eastward mean flow can became stationary. The group velocity

$$Cg=(U_0+\beta(k^2-l^2)/(k^2+l^2)^2,\ 2\beta(kl)/(k^2+l^2)^2) \qquad \text{Eq. 3.5.12}$$

The energy in the downstream direction will be advected with the group velocity in the x direction Cgx >0 for $k^2>l^2$ as it is in the case of the figure. It is easy to show that for the steady response the wavenumbers satisfy (3.5.10) the group velocity is in the downstream direction. It is also noted from (3.5.10) that for U_0 <0 the flow is westward and the solutions do not generate waves (the solution decays exponentially from the forcing area), because Q cannot be conserved.

Numerical simulations with a barotropic flow (westerly/easterly) were performed by runing the ZETA model with constant basic velocity on a channel in spherical coordinates. The results are displayed in figures 9 and 10. In the first case, Uo=40m/s will generate a quasi-stationary wave with a wavenumber that approximately satisfies 3.5.11, sqrt $[(k^2+l^2)\}$=sqrt$\{\ \beta/U_0\ \}$=$0.5\text{x}10^{-6}$ m^{-1} or the wavelength l=2p/|k|=12.48 x 10^{+6} m or 12480 Km.

U_0 > 0
Westerly
flow

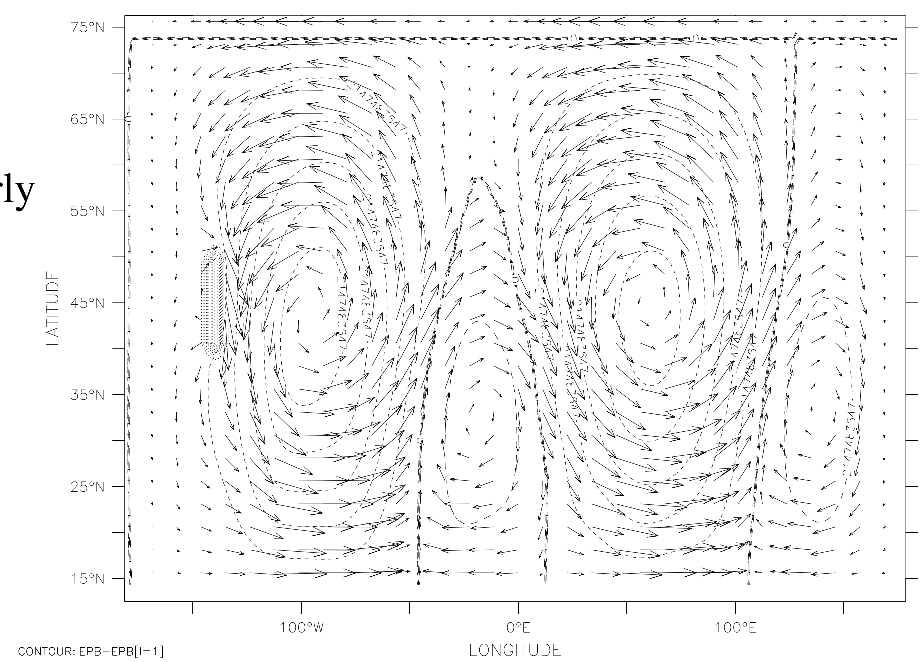


Figure 9 shows the initial value solution of a constant flow U=U_0 =40m/s at the left boundary on a spherical channel. The figure shows the barotropic streamfunction (anomaly) velocity vector and topography at 15 days. A wave pattern is visible on the lee side of the mountain ridge (right side). Note that the wavelength is much larger than the width of the mountain ridge.

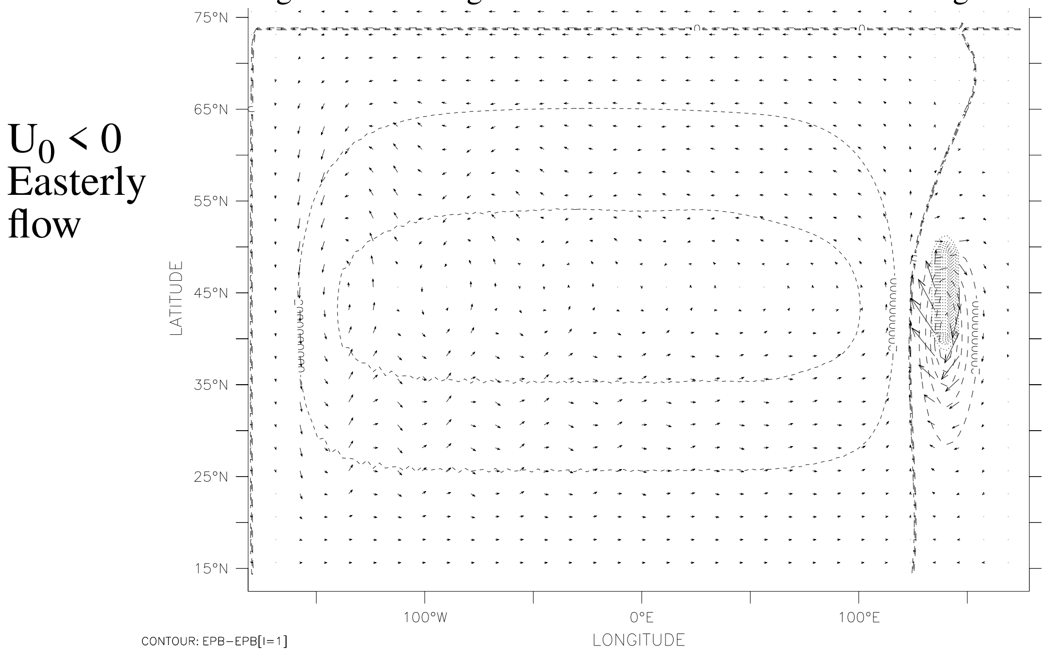
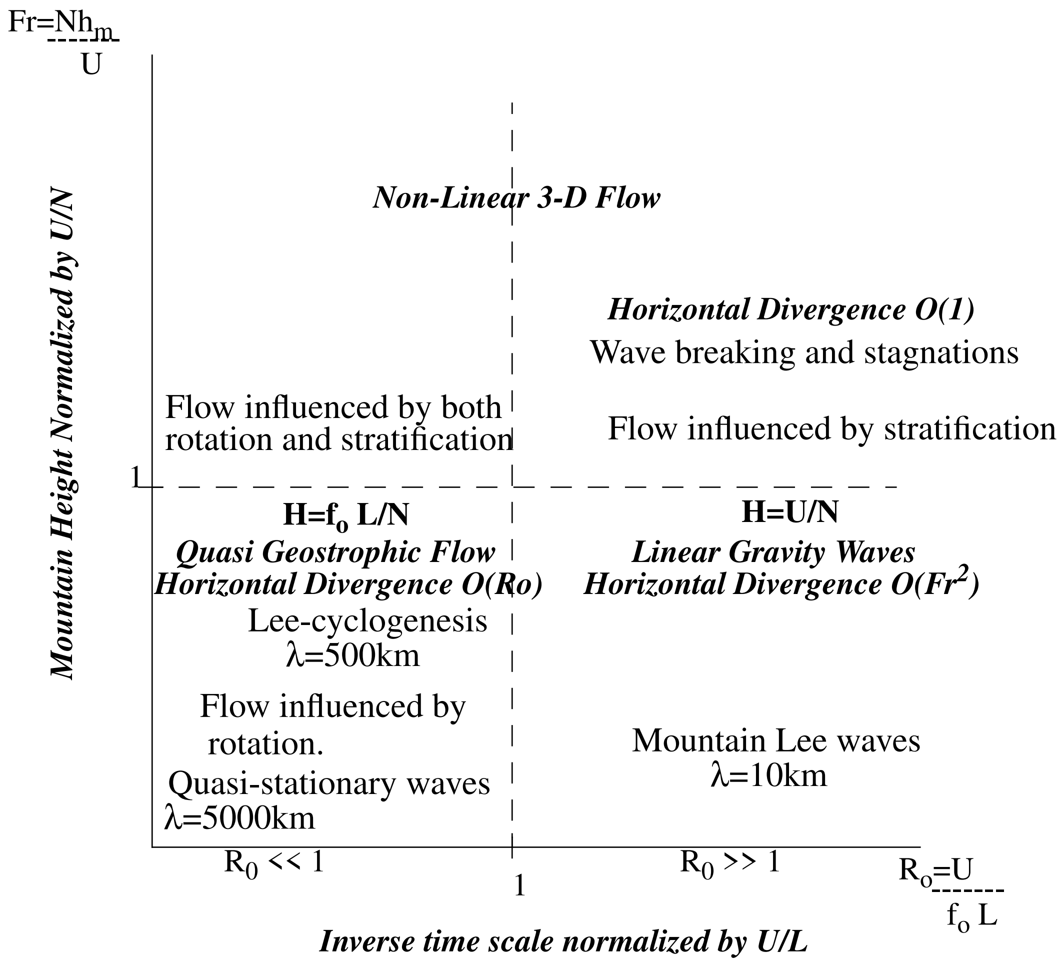


Figure 10. Shows the solution for an easterly wind Uo=-40m/s. No planetary waves are generated in this case. Only the anti-cyclone over the mountain is generated.

3.6. **Summary of the possible flows in a stratified rotating atmosphere.**

In light of the previous discussion, let us summarize the mountain response for the different range of parameters, Ro and Fr numbers.



II) UNSTABLE WAVES

4. Barotropic Instability.

This instability is a consequence of the variations in vorticity of the basic state, and is equally present in non-geostrophic as well as quasi-geostrophic flows.

What is an **unstable** wave?

If we have the dispersion relation, $\omega = f(k, \gamma, U_z, N^2, f)$ such that ω is **complex**. If Φ (k,y,z) is and eigenfunction that satisfy the boundary conditions:

$$\phi_k = \phi(k, y, z)e^{i\omega t}$$

and $\omega = \omega_r \pm i\omega_i$, then

$$\phi = e^{\pm\omega_i t}\phi(k, y, z)e^{i\omega_r t}$$

The eigenmode will be an exponentially growing solution. We will called them unstable eigenmodes.

4.1 Edge waves in a shear flow:

Assume that **(β = 0) (f = f_0 = const)**

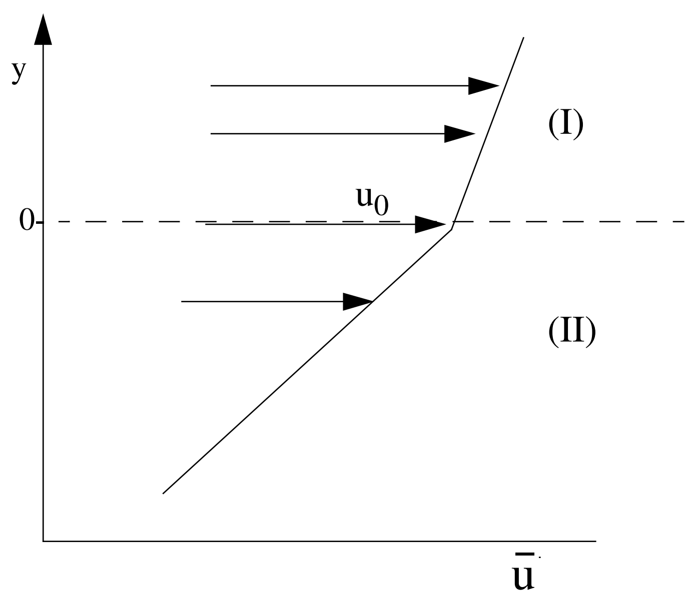


Fig 4.1.1 Zonal basic state

The absolute vorticity is conserved and since **f =f$_0$** the relative vorticity is also conserve.:

$$\frac{\partial\zeta}{\partial t} + \vec{v}\nabla\zeta = 0$$

where $\zeta = \nabla^2\psi$ and $\psi \equiv \text{total streamfunction}$ is given by $\psi = \bar{\Phi} + \phi$

The basic zonal velocity is given in layer I by: $\bar{u} = u_0 + \bar{u}_{yI}y$ and in layer II by $\bar{u} = u_0 + \bar{u}_{yII}y$. Since the basic state vorticity is constant in both layers. The perturbation vorticity equation from eq4.1.1 is given by:

$$\frac{d}{dt}\zeta' + v'\cancelto{0}{\bar{\zeta}_y} = 0 \qquad (4.1.2)$$

$\bar{\zeta} = \psi_{yy} = \frac{\partial}{\partial y}(-u) = -u_y$ As a conclusion is the perturbation vorticity is zero initially

(t=0) will be zero for all t. $\zeta' = \nabla^2\phi = 0$

The solutions in each layer will be:

$$\nabla^2\phi_I = 0$$

$$\nabla^2\phi_{II} = 0$$

$$\phi_I = \tilde{\phi}_I(y)e^{i(kx-\omega t)}$$

$$\phi_{II} = \tilde{\phi}_{II}(y)e^{i(kx-\omega t)}$$

$$\nabla^2\phi_{II} = -k^2\phi_{II} + \phi_{II_{yy}} = 0$$

$$\tilde{\phi}_{II}(y) = e^{\pm ky}$$

So,

$$\phi_I = \phi_I^{+/-}e^{\pm ky}e^{i(kx-\omega t)}$$

$$\phi_{II} = \phi_{II}^{+/-} e^{\pm ky} e^{i(kx - \omega t)}$$

Now, as y $\longrightarrow$ +∞ or -∞, ϕ should $\longrightarrow$ 0, so

$$\phi_I = \phi_I^{-} e^{-ky} e^{i(kx - \omega t)} \tag{4.1.3}$$

$$\phi_{II} = \phi_I^{+} e^{ky} e^{i(kx - \omega t)} \tag{4.1.4}$$

All the activity occurs at the interface of these two layers. The pressure and the normal velocities should be continuous across the interface η.

$$\phi_I = \phi_{II} \tag{4.1.5}$$

For v to be continuous at y=0 and

$$\frac{\partial u_I}{\partial t} + \bar{u}_0 \frac{\partial u_I}{\partial x} + v_I \bar{u}_{yI} = \frac{\partial u_{II}}{\partial t} + \bar{u}_0 \frac{\partial u_{II}}{\partial x} + v_{II} \bar{u}_{yII}$$

Replacing eqs 4.1.3 and 4.1.4 and using 4.1.5 we obtain the dispersion relation for the edge wave.

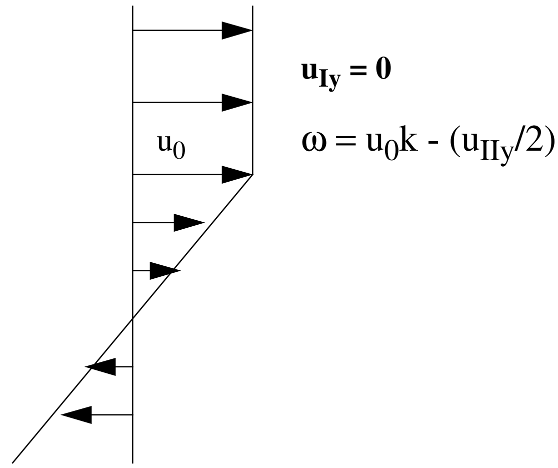
$$c = \frac{\omega}{k} = \bar{u}_0 + \frac{(\bar{u}_{yI} - \bar{u}_{yII})}{2k} \tag{4.1.6}$$

It shows that the edge wave propagates with the speed of the interface velocity U_0 modified by the difference of the vorticity of both layers. If the vorticity in layer I and II are the same $c=U_0$. The interpretation is similar to Rossby waves generated by a the variation of planetary vorticity, here is due to the variation of basic state vorticity.

4.2 The Unstable Mode (Interaction of two edge waves)

Let us take a simple flow such U_I=constant:

So for



The case in which we include a third layer with constant flow like the following:

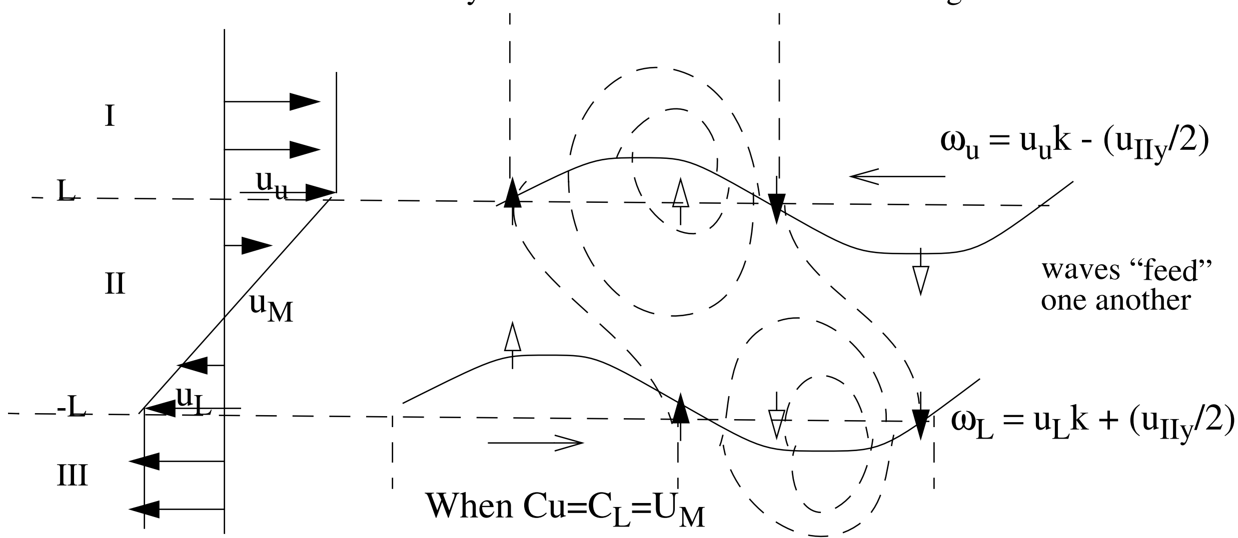


Fig. 4.1.3 A shear flow sandwiched between two layers of uniform flow. The two interfaces and the perturbation stream function are also shown. Arrows of normal velocity at each interface (full) and the induced velocity (open) are also shown.

The edge waves are phase lock and instability can occur.

$$\omega^2 = -\left(\frac{d}{dy}\bar{u}\right)^2\left(\frac{1}{4}\exp(-4kL) - \left(\frac{1}{2} - kL\right)^2\right) \qquad (4.2.1)$$

The maximum growth rate is given by:

$$\mathrm{Imag}(\omega)_{max}=0.2012\mathbf{U_y} \quad \text{for} \quad \mathbf{kL}=0.3984$$

This barotropic instability is exactly the same as in the nonrotating case. This unstable flow was studied by Rayleigh (1880). Since in the nonrotating case the basic state can be consider equally a function of the vertical coordinate z instead of a function of y. The stabilized effect of buoyancy due to density variations in the mean state modifies this instability. The instability due to the vertical shear and stratification is the well know “Kelvin-Helmholtz instability” (H. Helmholtz(1888). The dispersion relation (4.2.1) only has imaginary roots when the wavenumber K is less than a cutoff wavenumber K_0, $KL < K_0L=0.6392$.

It can be deduced from Fig 4.1.3 that **u’v’ <0** in the middle layer. For unstable waves the eddy kinetic energy should grow at expense of the basic state.

$$\frac{\partial}{\partial t}Ke = \frac{\partial}{\partial t}\left(\overline{u'^2} + \overline{v'^2}\right) = -\overline{u'v'}\frac{dU}{dy}$$

Since $U_y>0$, **u’v’ <0** for K_e to grow.

4.3 Necessary condition for instability.

It should be note from the previous analysis, that in order for the flow to became unstable, the existence of two waves propagating in opposite directions is a “necessary condition” for phase locking.

The phase velocity for each edge wave was given from 4.1.6 as:

$$c = \frac{\omega}{k} = \bar{u}_0 + \frac{(\bar{u}_{yI} - \bar{u}_{yII})}{2k} \qquad 4.3.1$$

The relative phase speed depends on the vorticity differences at each interface, for a continuous flow it will be proportional to dζ/dy=-U_{yy}. Both edge waves should have the same phase speed for phase lock. The effect of the vorticity gradient will be reduce the phase speed for the upper edge wave and increase for the lower wave. It Is a necessary condition for instability that dζ/dy has different signs in both interfaces, such to make one edge wave move faster than the interface velocity and the other to be slower than its . Rayleigh (1880) derived the necessary condition for a general barotropic flow.

Assume a periodic channel in x bounded by two rigid walls at y=+/-L.The x-momentum equation is:

$$U_t + (UU)_x + (vU)_y = -\frac{p_x}{\rho}$$

Where U=U(y)+u and v=v,.Take a <u>zonal</u> average

$$\overline{U}_t + \cancelto{0}{(UU)_x} + (vU)_y = \cancelto{0}{-\frac{\overline{p}_x}{\rho}}$$ any derivative w.r. t. **x** is zero.

$$\overline{U}_t + \frac{\partial}{\partial y}(\overline{vu}) = 0$$ can take $\frac{\partial}{\partial y}$ outside v since $u_x + v_y = 0$.

$$\int_{-L}^{L} \overline{U}_t dy = -\int_{-L}^{L} \frac{\partial}{\partial y}(\overline{vu})dy \equiv 0 \qquad 4.3.2$$

If we integrated over the meridional extent of the channel

due to the boundary conditions v(+/-L)=0. The total mass transport is conserved.

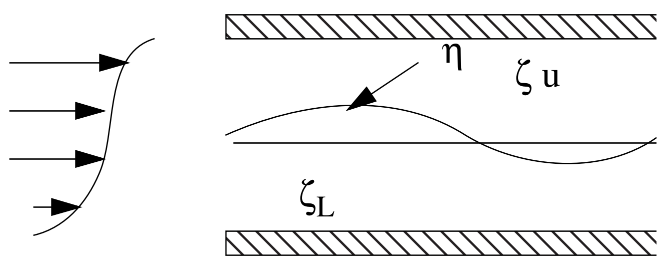
The vorticity is given by: $\bar{\zeta} = -\overline{U}_y$ and the mean flow is given by:

$$u = \bar{u}(-L) - \int_{-L}^{y} \zeta dy$$

the zonal flow at the boundary y=L channel is given then by:

$$\overline{U}(L, t) = \overline{U}(-L) - \left[\int_{0}^{\lambda}\int_{-L}^{L} \zeta dy dx\right] \cdot \frac{1}{\lambda} \qquad 4.3.3$$

Imagine that an interface η divides the channel in two layers the upper and lower as in the figure below.



The flow at y=L given by 4.3.3 is constant in time due to condition 4.3.2 and is

$$\overline{U}(L, t) = \overline{U}(-L) - \frac{1}{\lambda}\left(\int_{0}^{\lambda}\int_{\eta}^{L} \zeta_u dy dx + \int_{0}^{\lambda}\int_{-L}^{\eta} \zeta_L dy dx\right) \qquad 4.3.4$$

the vorticity at each side of the interface can be expressed to the first order Taylor expansion as:

$$\zeta'_u = -\frac{d\zeta}{dy}\eta \qquad\qquad \zeta'_L = \frac{d\zeta}{dy}\eta \qquad\qquad 4.3.5$$

The first integral can be split it in two: from L to zero and from zero to η , similarly the second one from η to zero and from zero to -L . Integrating the contribution from η, and using the expansions for the vorticity (4.3.5). The integral (4.3.4) can be expressed as:

$$\overline{U}(L,t) = \overline{U}(-L) - \frac{1}{\lambda}\left(\int_0^{\lambda}\int_0^{L}\zeta_u dydx + \int_0^{\lambda}\int_{-L}^{0}\zeta_L dydx + \int_0^{\lambda}\frac{d\zeta}{dy}\eta^2 dx\right) \qquad 4.3.6$$

It is easy to see that can't be any contributions from the last integral since the two first integrals cover the entire channel. For unstable waves it should be expected that η will grow in time and since η^2 is definitely positive, the necessary condition for instability is:

$$\int_0^{\lambda}\frac{d\zeta}{dy}\eta^2 dx = 0$$

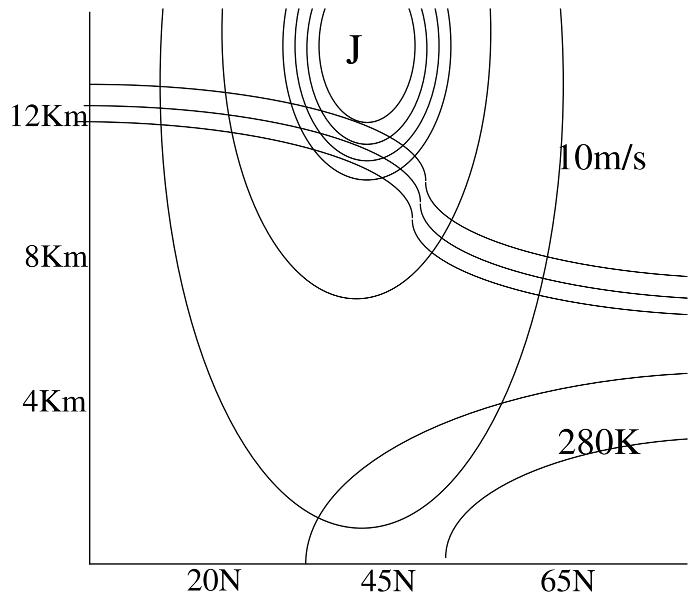
The vorticity gradient should change sign in the interior of the domain

It is clear also that if the vorticity gradient of the basic state does not change sign in the interior of the channel the condition is sufficient for stability.

4.4 Necessary Condition for Instability in a baroclinic flow.

The previous condition derived by Rayleigh was generalized by Charney and Stern[1] (1962) for a quasi-geostrophic baroclinic zonal jet, characteristic of the mid-latitude winter circulation.
Let us consider the stability of a flow as shown in the figure.

$U(y,z)=-\Psi_y(y,z)$



Where the potential temperature is given by the thermal wind relation:

$$\Psi_z(y,z)=(g/f)\Theta(y,z)/\Theta_0$$

The quasi-geostrophic potential vorticity $Q(y,z)$ is given by:

$$\overline{Q}(y,z) = \overline{\psi}_{yy} + \beta(y-y_0) + \frac{f_0^2}{\rho_r}\left(\frac{\rho_r}{N^2(z)}\overline{\psi}_z\right)_z \qquad 4.4.1$$

If the system is perturbed with a periodic disturbances $q(x,y,z,t)$, and since in the quasi-geostrophic system Q is conserved.

$$\frac{dQ}{dt} = 0 \qquad Q = \overline{Q}(y,z) + q(x,y,z,t) \qquad 4.4.2$$

expanding d/dt to the first order: $\frac{\partial q}{\partial t} + \overline{U}(y,z)q_x + v\overline{Q}_y = 0$ 4.4.3

1. On the stability of internal baroclinic jets in a rotating atmosphere. J. Atmos. Sci 19, 159-172.

$$\text{Let} \quad q \propto q(y,z)e^{i(kx-\omega t)}$$

$$v = \phi_x$$

$$q = \nabla^2\phi + \frac{f_0^2}{\rho_r}\left(\frac{\rho_r}{N^2}\phi_z\right)_z \qquad 4.4.4$$

$$ik\left(\overline{U}(y,z) - \frac{\omega}{k}\right)q + ik\phi\overline{Q_y}(y,z) \equiv 0 \qquad c = \frac{\omega}{k} \qquad 4.4.5$$

we get:

$$q = -\frac{\phi\overline{Q_y}(y,z)}{(\overline{U}(y,z)-c)} \qquad 4.4.6$$

Now multiply q by $\rho_r\phi^*$; ϕ^* = complex conjugate of ϕ

$$\rho_r\phi^*\nabla^2\phi + f_0^2\phi^*\left(\frac{\rho_r}{N^2}\phi_z\right)_z = -\frac{|\phi|^2\rho_r\overline{Q_y}(y,z)}{(\overline{U}(y,z)-c)} \qquad 4.4.7$$

Taking the volume integral of 4.4.7

$$\int_0^H\int_{-L}^L\int_0^\lambda\left[\rho_r\phi^*\nabla^2\phi + f_0^2\phi^*\left(\frac{\rho_r}{N^2}\phi_z\right)_z\right]dxdydz = \iiint\left(-\frac{|\phi|^2\rho_r\overline{Q_y}(y,z)}{(\overline{U}(y,z)-c)}\right)dxdydz \quad 4.4.8$$

assuming that the flow is bounded by rigid walls in the north and south v(+/-L)=0 then the first left terms gives:

$$-\int_0^H\int_{-L}^L\rho_r(|\phi_x|^2 + |\phi_y|^2)dydz \Leftarrow \text{The Eddy Kinetic Energy}$$

Integrating by parts the second left term of eq 4.4.8 leads to:

$$\iiint f_0^2\phi^*\left(\frac{\rho_r}{N^2}\phi_z\right)_z dv = -\int_0^H\int_{-L}^L\frac{f_0}{N^2}\rho_r|\phi_z|^2dydz + \iiint f_0^2\frac{\partial}{\partial z}\left(\frac{\rho_r\phi^*\phi_z}{N^2}\right)(dydz)$$

The first term on the right side is the Eddy Potential Energy and the second one gives the contributions of the upper and lower boundary.

$$\int_0^H \int_{Y_S}^{Y_N} f_0^2 \frac{\partial}{\partial z}\overline{\left(\rho_r \frac{\phi^*}{N^2}\phi_z\right)} dy dz = \int_{Y_S}^{Y_N} f_0^2 \left(\rho_r \frac{\phi^*}{N^2}\phi_z\right) dy \Big|_0^H \qquad 4.4.9$$

Since the boundaries at z=0 and z=H is given by w=0

$$w = -\frac{1}{N^2}\frac{db}{dt} = -\left[\frac{f_0}{N^2}\left(\frac{\partial}{\partial t} + U(z{=}0)\frac{\partial}{\partial x}\right)\phi_z + v f_0 \phi_{zy}\right] = 0 \qquad 4.4.10$$

It easy to show that at the boundaries

$$\phi_z = \frac{\overline{U_z}\phi}{(\overline{U} - c)} \qquad 4.4.11$$

substituting 4.4.11 in 4.4.9 we get:

$$\int_{-L}^{L} f_0^2 \rho_r \phi^* \phi_z dy \Big|_0^H = \int_{-L}^{L} \frac{f_0^2 \rho_r}{N^2} \frac{\overline{U}_z |\phi|^2}{\overline{U} - c} dy \Big|_0^H$$

Since $c = c_r + ic_i$, appears only in the above boundary contribution and on right term of eq. 4.4.8. The real and imaginary parts could be separated from eq. 4.4.8 as follow:
We can break this up into real and imaginary parts...

$$= \iint \frac{\rho_r \overline{Q}_y (\overline{U} - c_r + ic_i)|\phi|^2}{(\overline{U} - c_r)^2 + c_i^2} dy dz$$

the real part

$$-\iint (\overline{K}_e + \overline{P}_e) dy dz + \int_{Y_S}^{Y_N} \frac{\overline{U}_z |\phi|^2 (\overline{U} - c_r)}{(\overline{U} - c_r)^2 + c_i^2} dy \Big|_0^H = -\int \frac{\bar{\rho}_r |\phi|^2 \overline{Q}_y (\overline{U} - c_r)}{(\overline{U} - c_r)^2 + c_i^2} dy dz \qquad 4.4.12$$

and the imaginary part gives the necessary condition for instability for Ci non zero:

$$-c_i \int_{Y_S}^{Y_N} \frac{f_0 \rho_r}{N^2} \frac{\overline{U}_z |\phi|^2}{(\overline{U} - c_r)^2 + c_i^2} dy \Big|_0^H = c_i \int \frac{\rho_r |\phi|^2 \overline{Q}_y}{(\overline{U} - c_r)^2 + c_i^2} dy dz \qquad 4.4.13$$

Then we have a **<u>necessary</u>** condition for instability. (But, c_i could still be zero anyway)

If the above equation is not equal to zero, then it is a sufficient proof of stability ($c_i = 0$).

The real part (eq. 4.4.12) gives a sufficient condition for instability.

{I} Internal Jet ($\overline{Q}_y \neq 0$) but $\overline{U}_z \equiv 0$ at the boundaries

Inspecting eq. 4.4.13 for this case. Since $\overline{U}_z \equiv 0$ at the boundaries, then

$Q_y = 0$ should be satisfy for an instability to occur. Q_y <u>must</u> <u>change</u> <u>sign</u> in the interior.

(i) In the <u>barotropic</u> case, where

$$\overline{Q}_y = -\overline{U}_{yy} + \underbrace{\beta}_{\text{positive}} \qquad 4.4.14$$

For $\overline{Q}_y = 0$, our

$\overline{U}_{yy}$ term must be positive.

(Only happens in the tropics)

{II} Baroclinic flow $z \rightarrow \infty$ (Only one boundary effect)

Let us make the assumption that

$$U = U_0 + \Lambda z \qquad \text{and} \quad \rho_r = \text{Constant}$$

Now,

$$\overline{Q}_y = -\cancelto{0}{\overline{U}_{yy}} + \beta - \cancelto{0}{\left(\frac{f_0^2}{N_0^2}\overline{U}_z\right)_z}$$

for this case it reduces to:

$$\bar{Q}_y = \beta$$

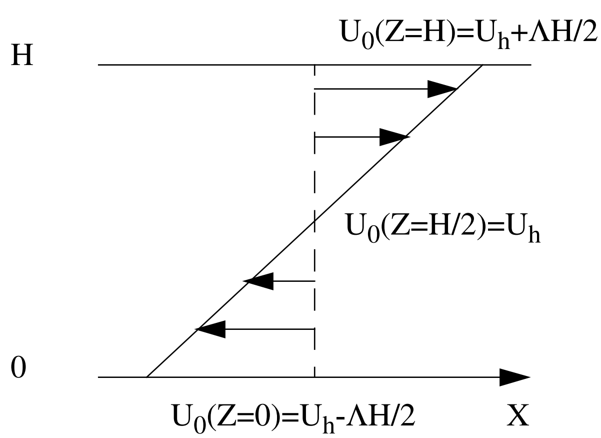
Then balance in the equation 4.4.13 is between the volume integral involving Qy $\bar{Q}_y = \beta$ and the integral for the contribution of the lower boundary $\bar{U}_z$ (z=0); i.e., (boundary effect at z = 0). As we will discuss model later in this section. Charney's unstable modes are due to the interaction between a westward propagating wave $\bar{Q}_y = \beta$ and an surface edge wave due to $\bar{U}_z$ at z-0.

A similar instability arise in a more simplified flow. The Eady model, is perhaps the simplest baroclinic unstable flow. Consider a f-plane: $\bar{Q}_y = \beta = 0$, but the model has a rigid lid.

Equation 4.4.13 for this case, even though $\bar{Q}_y = 0$, there is a balance between our boundary effects at the lower boundary and at the rigid lid (e.g., tropopause) that will also make (4.4.13) equal zero, and give us an unstable solution.

4.5 The Eady model[1]:

For both models, we have $\bar{U} = U_0 + \Lambda z$, so our potential temperature is: $\theta = \theta_0 + \Gamma z - \frac{\theta_0 \Lambda y f_0}{g}$



1. Eady, E/. J. 1949: Long wave and cyclone waves, Tellus 1, 3, 33-52

Since the basic state Qy=0 the perturbation q' is also conserved.

$$\frac{dq'}{dt} + v\cancelto{0}{\bar{Q}_y} = 0$$

If q'=0 at t=0 is zero for all times. The interior equation is then:

$$\phi_{xx} + \phi_{yy} + \frac{f_0^2}{N^2}\phi_{zz} = 0 \qquad 4.5.2$$

and with the boundary conditions:

$$\frac{d\phi_z}{dt} + \Lambda\phi_x = 0 \quad \text{at } z = 0, H \qquad 4.5.3$$

the solution to 4.5.2 is

$$\phi = (\tilde{\phi}_p e^{\mu z} + \tilde{\phi}_m e^{-\mu z})e^{i(kx + ly - \omega t)} \qquad 4.5.4$$

where $\mu = (N/f)(k^2+l^2)^{0.5}$ from eq. 4.5.2 and the boundary conditions at z= and H are:

$$\text{at } z=H \quad \left[U_h + \frac{\Lambda H}{2} - c\right]\phi_z - \phi\Lambda = 0 \qquad 4.5.5$$

$$\text{and at } z=0 \quad \left[U_h - \frac{\Lambda H}{2} - c\right]\phi_z - \phi\Lambda = 0$$

Substituting the solution 4.5.4 in both boundary conditions 4.5.5 we have an homogeneous system for both amplitudes;

$$\left[\left(U_h + \frac{\Lambda H}{2}\right) - c\right](\mu A e^{(\mu H)/2} - \mu B e^{-(\mu H)/2}) - (A e^{(\mu H)/2} + B e^{-(\mu H)/2})\Lambda = 0$$

4.5.6

$$\left[\left(U_h - \frac{\Lambda H}{2}\right) - c\right](\mu A e^{-(\mu H)/2} - \mu B e^{(\mu H)/2}) - (A e^{-(\mu H)/2} + B e^{(\mu H)/2})\Lambda = 0$$

For non trivial solutions the determinant should vanish, given a condition for C.

$$c = U_h \pm \frac{\Lambda}{\mu}[((\mu H)/2 - \tanh(\mu H)/2)((\mu H)/2 - \coth(\mu H)/2)]^{1/2} \qquad 4.5.7$$

There are two real roots for C is $\mu H/2 > 1.1997$ and two imaginary roots for $\mu H/2 > 1.1997$.

The maximum wave number is for $\mu H/2 = 0.8$.

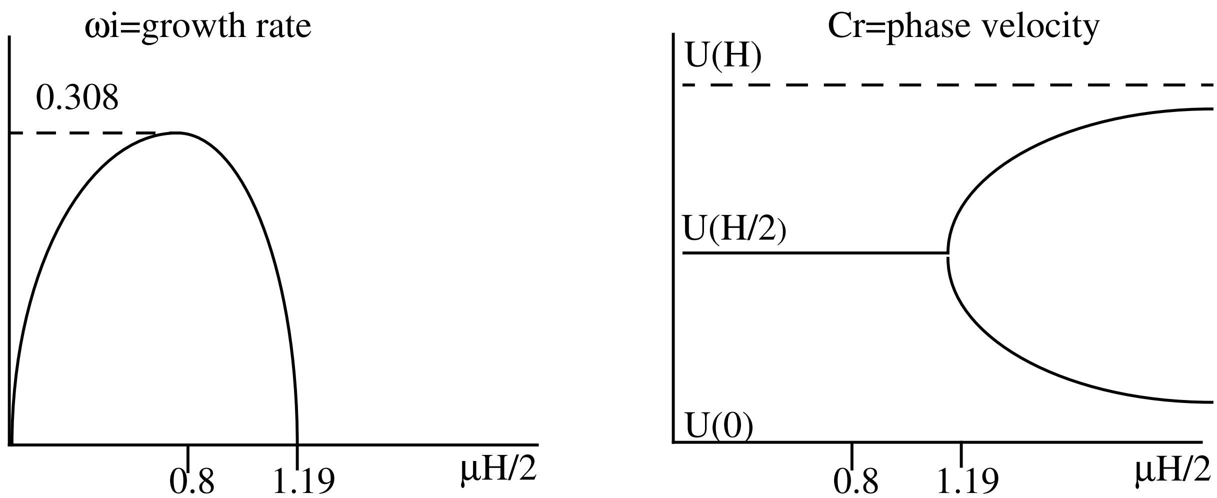
$$\lambda_{max} = \frac{2\pi}{\mu H} \cong 4000 \text{ km} \qquad \left(\text{for } \frac{f_0^2}{N^2} = 10^{-4}\right) \qquad 4.5.8$$

The growth rate for the most unstable wave is

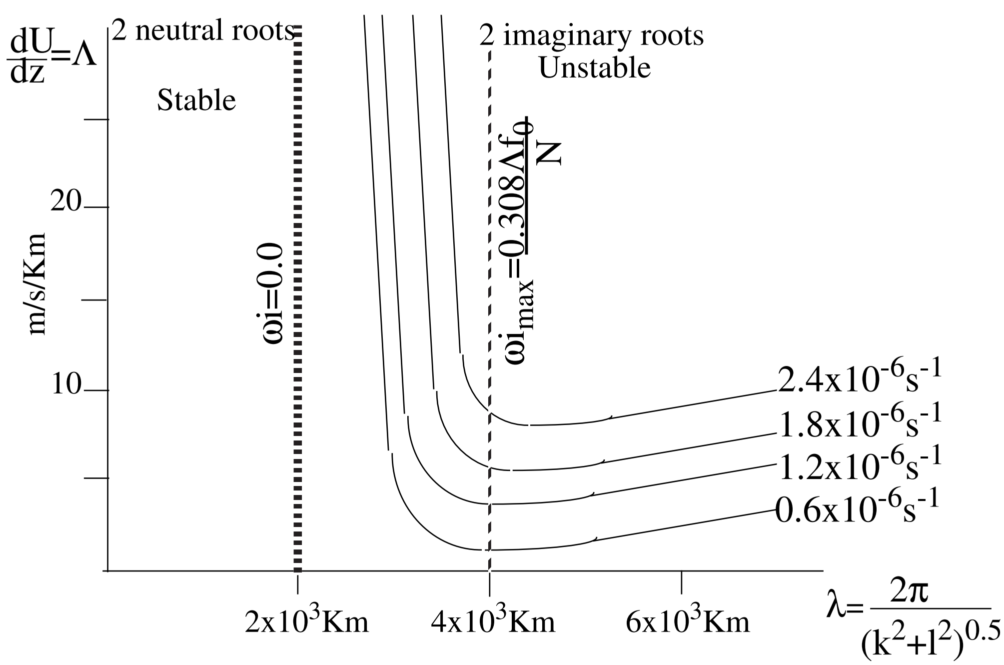
$$\omega_i = 0.308\frac{f_0}{N}\Lambda \quad \text{(the imaginary part of } \omega\text{)}$$ with a doubling time

$$\ln\frac{\phi}{\phi_0} = \omega_i t \qquad \ln 2 = \omega_i t_D$$

The growth rate and phase velocity vs. the horizontal wavenumber or μH are:



The regions of instability as a function of vertical shear and wavenumber is shown in the next figure.



The structure of the wave can be seen in the following cross section.

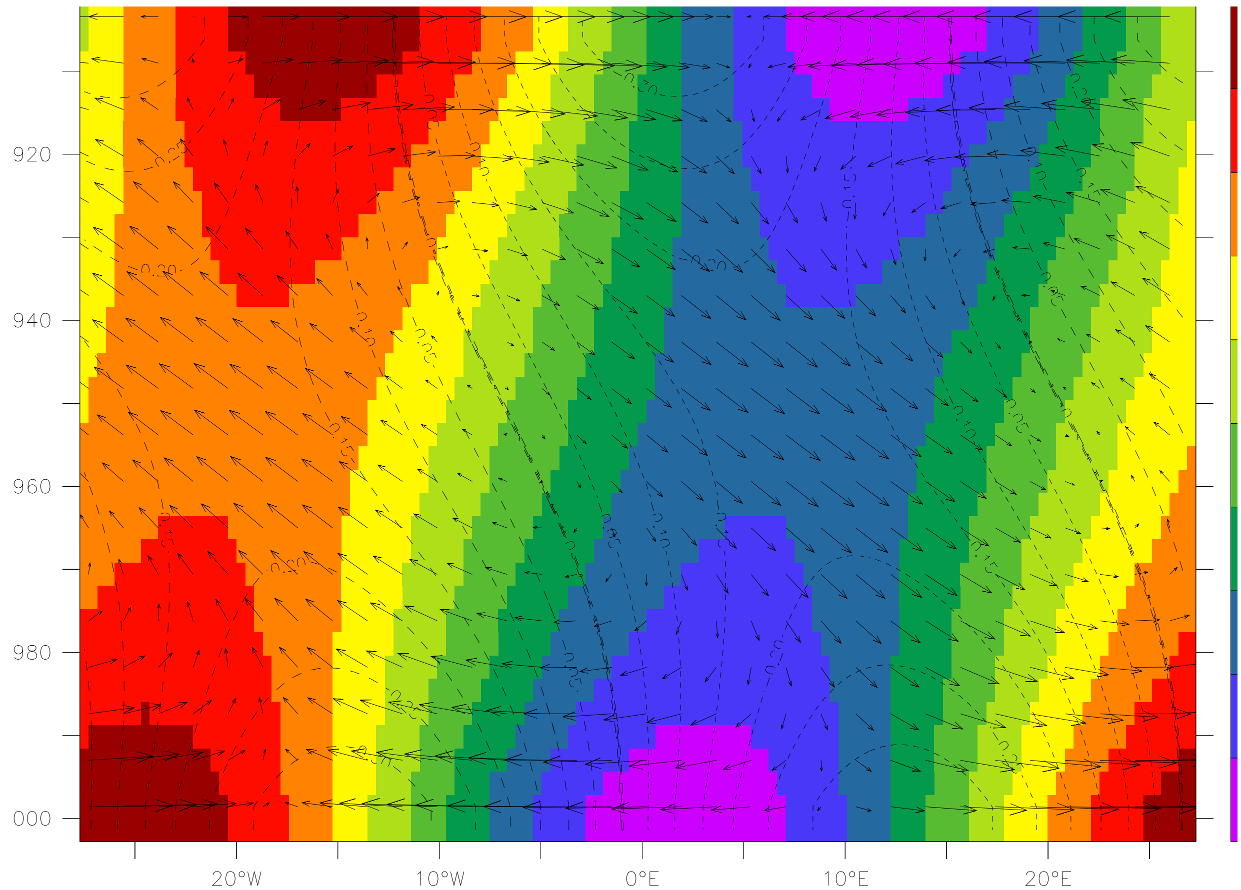


Fig 4.5.1 The potential temperature anomaly (shading), the meridional velocity v (contours) and the zonal and vertical component of the perturbation velocity as function of z and longitude in the middle of the channel.

Notice the westward tilt of the meridional velocity (similar to the stream function) about π/2 (13 degrees. is one quarter of the total length 55 degrees). This tilt is require for baroclinic instability since provides a poleward heat flux. Warm air moves to the north (V>0) cold air to the south (v<0). The air is ascending in the warm side and sinking in the cold side. This structure could be better seen is eq 4.5.4 is cast in the following form

$$: \ \phi = |\hat{\phi}(z)| e^{\omega_i t} \cos(ly) \cos(kx + \delta(z)) \qquad 4.5.9$$

$$|\hat{\phi}(z)| = \left[\cosh^2 \mu z + \frac{c_i}{\mu |c|^2} \sinh^2 \mu z\right]^{1/2} \text{ and } \delta(z) = \tanh^{-1}\left[\frac{c_i}{\mu |c|^2} \frac{\sinh \mu z}{\cosh \mu z}\right]$$

for the amplitude and the phase. The phase $\delta(H)\sim\pi/2$ for the most unstable wave as shown in fig 4.5.1. The potential temperature however has two components $\Phi_z\cos(kx+\delta(z))$ and $\Phi\delta_z\sin(kx+\delta(z))$ and is tilted to the east. The surface fields are shown in fig 4.5.2

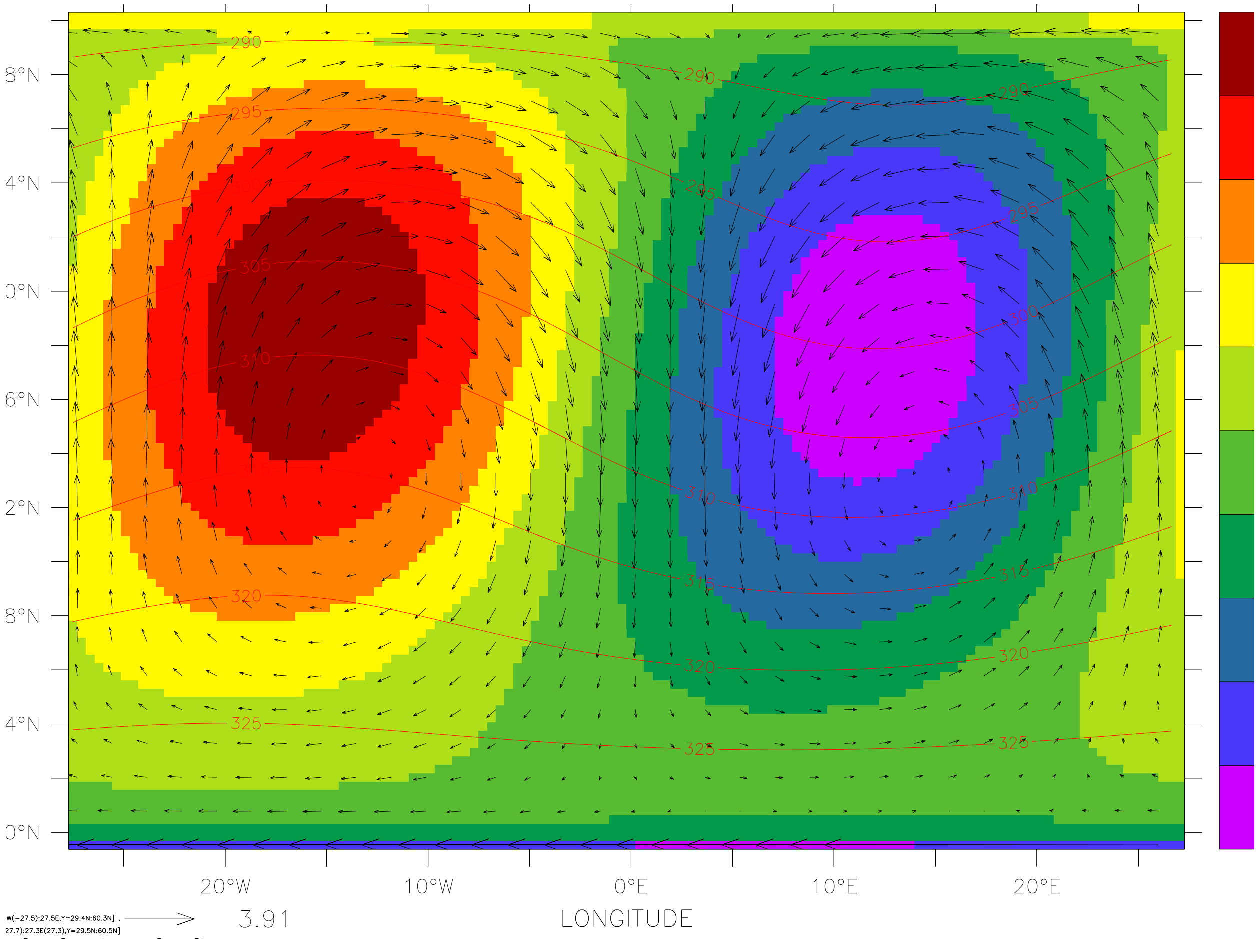


Fig 4.5.2. The potential temperature anomaly the surface wind and contour of total potential temperature is shown for the solution at 3.5 days.

As previously discussed a neutral edge wave would have relative vorticity collocated with temperature field. In the unstable solutions due to the interaction with the upper edge wave the centers of vorticity are slightly out of phase such that there is a net meridional advection over the maximum and minimum temperature disturbances. This circulation enhances the potential temperature anomaly that in turn strengthening the cyclonic (warm) and anticyclonic (cold) vorticity field.

4.5.2 Meridional momentum, heat and PV fluxes

The zonal momentum equation reduces to:

$$\overline{U}_t \approx -\frac{\partial}{\partial y}(\overline{u'v'}) + f\overline{v}_a \qquad \text{the last term is the ageostrophic circulation. 4.5.10}$$

the heat equation $\bar{\theta}_t \approx -\frac{\partial}{\partial y}(\overline{v'\theta'}) - \frac{\partial}{\partial z}(\overline{w'\theta'}) - \overline{w}\bar{\theta}_z$ 4.5.11

and the zonal mean potential vorticity:

$\frac{\partial \overline{Q}}{\partial t} = -\frac{\partial}{\partial y}(\overline{v'q'})$ is possible to show that the meridional flux of quasi-geostrophic vorticity is related to the meridional momentum and heat flux divergences. For the replacing the definitions of v' and Θ and q':

$$\overline{v'q'} = \overline{\phi'_x\left(\nabla^2\phi' + \frac{f_0^2}{N^2}\phi'_{zz}\right)} = \overline{\phi_x\phi_{xx}} + \overline{\phi_x\phi_{yy}} + \overline{\phi_x\phi_{zz}\frac{f_0^2}{N^2}} \qquad 4.5.12$$

integrating by parts and using ()x=0 and remembering that $f_0\phi_z = b' = \frac{g\theta}{\theta_0}$ is easy to show:

$$\overline{v'q'} = -(\overline{u'v'})_y + \frac{f_0}{N^2}(\overline{v'b'})_z \qquad 4.5.13$$

In the case of (uniform PV) q'=0 and then:

$$\frac{\partial}{\partial y}(\overline{u'v'}) \equiv \frac{f_0}{N^2}\frac{\partial}{\partial z}(v'b') \qquad 4.5.14$$

For the particular case of the Eady solution 4.5.8 is possible to show that the momentum fluxes are: $\overline{\phi_x\phi_y} \equiv 0$ consequently $\frac{\partial}{\partial z}(\overline{v'b'}) \equiv 0$ or $v'b' = \text{constant}$ the heat fluxes are constant in the vertical column. As a consequence from eq 4.4.5 the z-derivative of the phase times the amplitude square should be constant with z.

$$\overline{v'b'} = \text{constant}=a\delta(z)_z|\phi(z)|^2$$

For poleward heat fluxes v'b'>0 the phase δ(z) should increase with height, the stream function phase for unstable waves then should tilt to the west with height. A neutral solution has no tilt $\delta(z)_z=0$ and $\delta(z)_z<0$ corresponds to a decaying solution.

4.6 The Charney Model[1]

A more realistic atmosphere it is consider in the Charney model, with the following changes: a beta plane instead of the f-plane consider by Eady, a semi-infinite atmosphere and the density varying with height.

$$\rho=\rho_0\exp(-z/Hs),\ f=f_0+b(y-y_0) \text{ and } U=U_0+\Lambda z$$

Potential Vorticity is not longer uniform

$$\overline{Q}_y = \beta - \frac{f_0}{\rho_r N_0^2}(\rho_r U_z)_z \qquad 4.6.1$$

and for a flow with constant vertical shear 4.6.1 became:

$$\overline{Q}_y = \beta + \frac{f_0}{HN_0^{\ 2}}\overline{U}_z \qquad 4.6.2$$

So for this case the conservation of Q gives:

1. Charney, J. C 1947: the dynamics of long waves in baroclinic westerly current, J. Meteor. 4, 5, 135-162.

$$\left(\frac{\partial}{\partial t}+U(z)\frac{\partial}{\partial x}\right)q'+v\overline{Q}_y = 0 \qquad 4.6.3$$

and assuming as before a wave solution for the stream function we can rewrite 4.6.3 as:

$$(U(z)-c)\left[\nabla^2\phi+\frac{1}{\rho_r}\frac{f_0^2}{N^2}(\rho_r\phi)_z\right]+\phi'\overline{Q}_y = 0 \qquad 4.6.4$$

with the same boundary condition at z=0 as before:

$$(U(0)-c)\phi_z-\Lambda\phi = 0 \quad \text{at } z=0 \qquad 4.6.5$$

the boundary for z-> infinite will be taken to be the vanishing kinetic energy density of the perturbation. $\lim_{z\infty}\rho\phi^2 = 0$

the substitutions

$$\begin{aligned}\xi &= \frac{2A}{\Lambda}(U-c) = 2Az+\frac{2A}{\Lambda}(U_0-c)\\ \chi &= \phi(z)\exp\left(A-\frac{1}{2H}\right)z\end{aligned} \qquad 4.6.6$$

and

$$A = \left(\frac{Nk}{f_0}\right)^2+\frac{1}{4H^2}$$

reduce equations (4.6.4) and (4.6.5) to:

$$\xi\frac{d^2\chi}{d\xi} - \xi\frac{d\chi}{d\xi} + r\chi = 0 \qquad 4.6.7$$

$$\frac{1}{\chi_0}\left(\frac{d\chi}{d\xi}\right)_0 = \alpha + \frac{1}{\xi_0} \text{ at } \xi_0 = \frac{2A}{\Lambda}(U_0 - c) \qquad 4.6.8$$

The constants r, λ, α and p are defined as follow:

r=0.5(λ+1)/sqrt(p^2+0.25), $\lambda=\beta N^2H/(\Lambda f^2)$, a=0.5(1-(sqrt(1+4$p^2$))$^{-1}$ and p=$A^{0.5}$H.

The problem became one of determining the eigenvalues ξ_0 in terms of the non-dimensional wave number p and the non dimensional parameter r which depends on the ratio β/Λ.

Equation (4.6.7) is a special case of the confluent hypergeometric equation

$$\xi\frac{d^2\chi}{d\xi} + (b-\xi)\frac{d\chi}{d\xi} - a\chi = 0 \qquad 4.6.9$$

with two independent solutions M(a,b,ξ) and ξ^{1-b}M(a-b+1,2-b,ξ). The second one diverges at infinite and the first one is regular.

By a method of trial and error Kuo (1952)[1] calculated c for the range 0<r<1. Green (1960)[2] calculated numerically with an atmosphere bounded rigidly at z=H (similar to Eady's model). The growth rates for the three models can be seen in the figure below.

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1. Three-dimensional disturbances in a baroclinic zonal current. J. Meteor. 9, 260-278
2. A problem in baroclinic stability, Quart. J. R. Meteor. Soc. 86, 237-251.

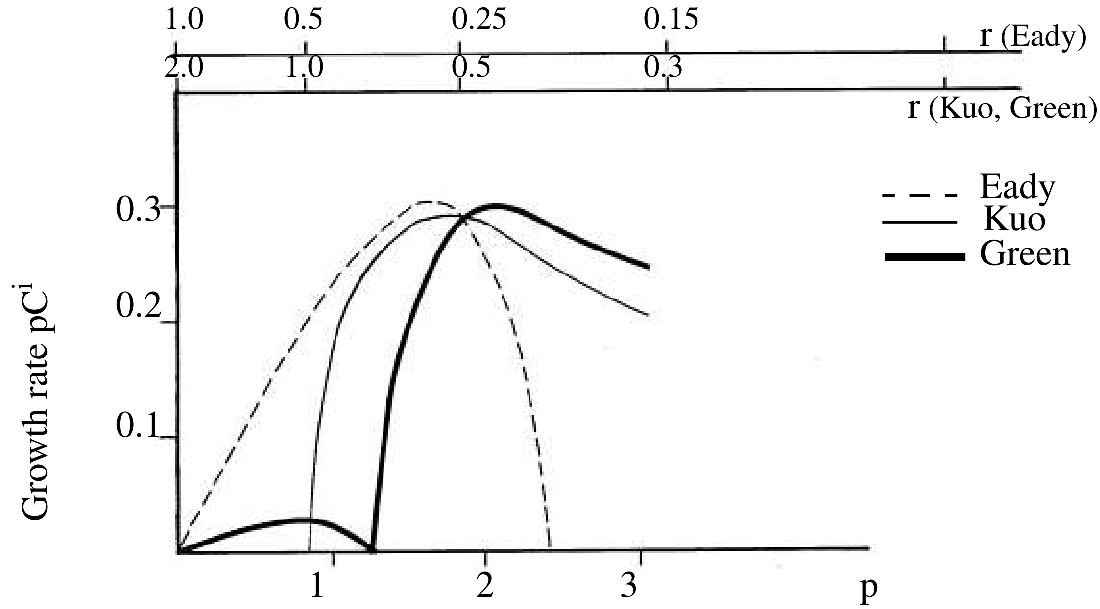


Fig 4.6.1 comparison of the different growth rates

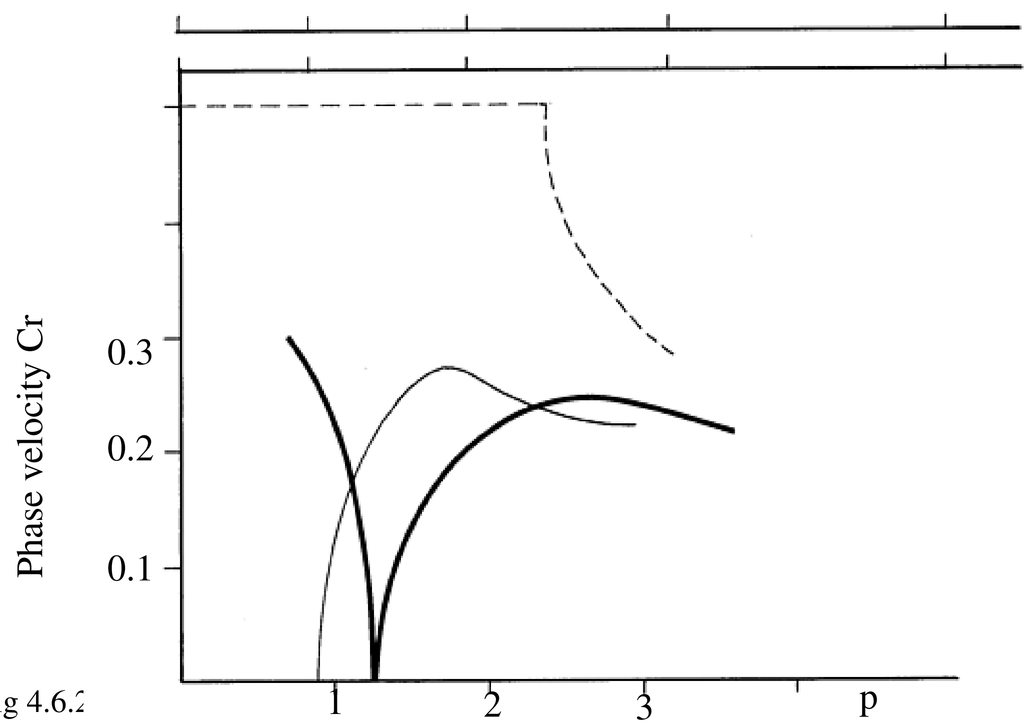
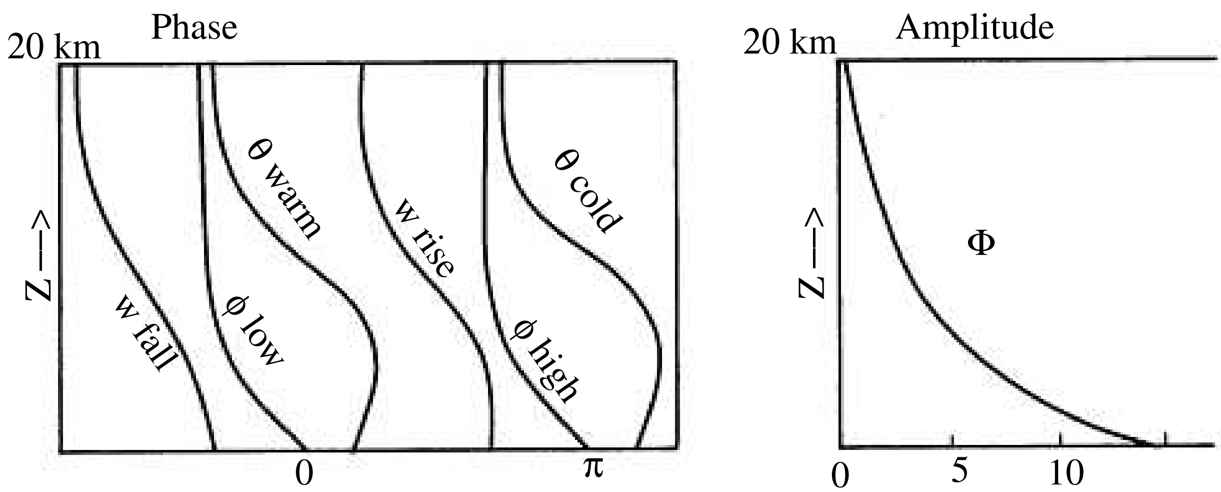


Fig 4.6.2

The structure of the Charney mode, Kuo's calculation for the most unstable wave is shown below.



Charney 's model (H. L Kuo)

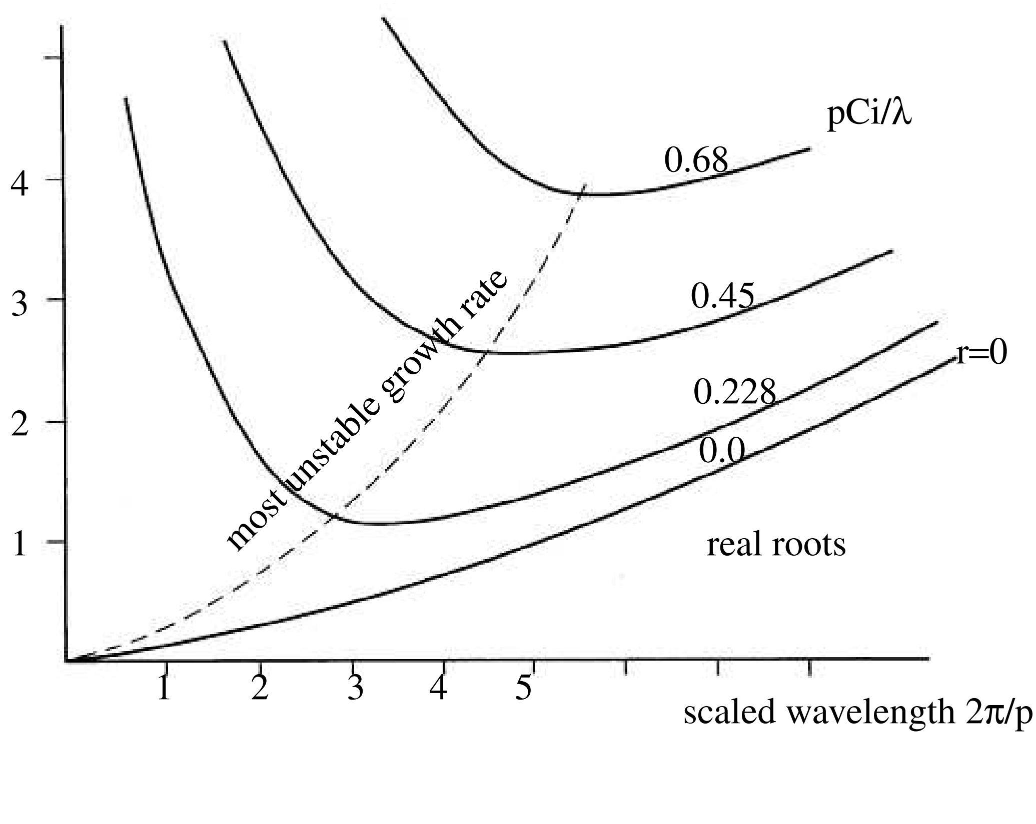


Fig 4.6.5 Contours of growth rates as a function of non-dimensional wavelength and shear for

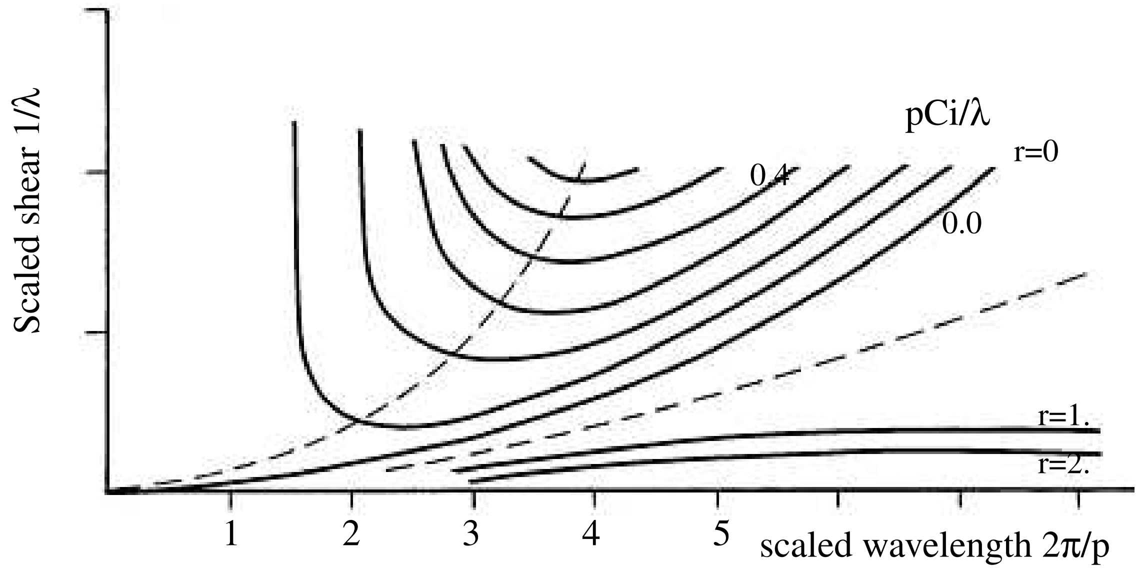


Fig 4.6.5 Contours of growth rates for the Eady's model for the same scaling variables.