Two and Three Dimensional Image Rotation Using the FFT

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Abstract

Techniques for rotating two and three dimensional images using FFTs are presented. The methods are applications of the multidimensional chirp algorithm. In the 2D case, one chirp transformation is sufficient, requiring four 2D FFTs. In the 3D case, two successive chirp transformations are required, needing six 3D FFTs.

I. Introduction

Rotating an image to a new pixel grid is a basic operation required in many applications. Translation without rotation can be accomplished using the Fourier transform shift property

$$f(\mathbf{x}+\mathbf{a}) = \mathcal{F}^{-1}\left[\mathcal{F}[f]\,\exp\{2\pi i\,\langle\mathbf{a},\mathbf{k}\rangle\}\right] \;, \qquad (1)$$

where $\mathcal{F}[f] = \int_{R^n} f(\mathbf{x})\,\exp\{-2\pi\,\langle\mathbf{k},\mathbf{x}\rangle\}\,d^n x$ is the continuous Fourier transform; R^n denotes n-dimensional real Euclidean space with inner product $\langle\mathbf{x},\mathbf{y}\rangle = \mathbf{x}^T\mathbf{y}$. The discrete application of this requires an FFT, scaling by the $\exp\{2\pi i\,\langle\mathbf{a},\mathbf{k}\rangle\}$ factor, and an inverse FFT. The effect is trigonometric interpolation of the original image onto the translated grid. The purpose of this Correspondence is to show how FFTs can be used to interpolate 2D and 3D images to a new pixel grid that is both translated and rotated from the original grid.

II. 2D Image Rotation

Given the image $f(p,q)$ on an $N\times N$ square grid, we want to compute f on a grid rotated by an arbitrary angle θ and shifted by an arbitrary translation $[a\ b]^T$. The first step is to compute the 2D FFT of f, so that

$$\begin{aligned}\hat{f}(l,m) &= \sum_{p,q} f(p,q)\,e^{-2\pi i(lp+mq)/N} \\ f(p,q) &= \frac{1}{N^2}\sum_{l,m}\hat{f}(l,m)\,e^{2\pi i(lp+mq)/N} \;. \end{aligned} \qquad (2)$$

We evaluate (2) on the desired output grid:

$$\begin{aligned}&f(p\cos\theta - q\sin\theta + a, p\sin\theta + q\cos\theta + b) \\ &= \quad \frac{1}{N^2}\sum_{l,m}\hat{f}(l,m)\,e^{2\pi i(la+mb)/N}\;e^{2\pi i(lp+mq)\cos\theta/N}\;e^{2\pi i(mp-lq)\sin\theta/N} \;. \end{aligned} \qquad (3)$$

To evaluate (3), we need to be able to evaluate the sum

$$\tilde{g}(p,q;\alpha,\beta) = \sum_{l,m} g(l,m)\, e^{2\pi i[(lp+mq)\alpha+(mp-lq)\beta]}\ , \tag{4}$$

for arbitrary α and β, where $g(l,m) = N^{-2}\exp\{2\pi i(la+mb)/N\}\hat{f}(l,m)$. The 1D analog of (4) is the computation of

$$\tilde{h}(p;\alpha) = \sum_{l} h(l)\, e^{2\pi i l p\alpha}\ .$$

This can be done efficiently using the chirp-z algorithm [1, 2]; that is, by expanding $2lp = l^2+p^2-(l-p)^2$, so that

$$\tilde{h}(p;\alpha) = e^{\pi i p^2\alpha}\sum_{l}\left\{h(l)\, e^{\pi i l^2\alpha}\right\}\, e^{-\pi i(l-p)^2\alpha}\ .$$

This expansion is a multiplication, a convolution, and a multiplication. The convolution can be done quickly using 3 FFTs (of $h(l)\exp\{\pi i l^2\}$, of $\exp\{-\pi i l^2\}$, and of the product of their transforms).

A similar method can be used to evaluate (4) efficiently. The integer expansion required is

$$\begin{aligned} 2(lp+mq) &= l^2+p^2-m^2-q^2-(l-p)^2+(m+q)^2 \\ 2(mp-lq) &= 2lm-2pq-2(l-p)(m+q)\ . \end{aligned}$$

With this, (4) can be written as

$$\tilde{g}(p,q;\alpha,\beta) = Z(q,p)^*\sum_{l,m}\{g(l,m)\,Z(l,m)\}\,Z(l-p,m+q)^*$$

where $Z(p,q) = \exp\{\pi i[(p^2-q^2)\alpha+2pq\beta]\}$ and $Z()^*$ denotes complex conjugation. This again is 2 multiplications and a convolution, which can be done with 3 2D FFTs. Counting the original FFT to get $\hat{f}(l,m)$, four 2D FFTs are required to carry out the rotation. (If the original data is available in Fourier space, as in magnetic resonance imaging [3], the initial FFT can be avoided—this amounts to reconstructing the raw MRI data directly to a rotated grid. If the same θ is to be applied to many images, then the FFT of Z need only be computed once.) We note that $Z(p+s,q) = \exp\{2\pi i s^2\alpha\}Z(p-s,q)^*Z(p,q)^2$, and $Z(p,q+s) = \exp\{-2\pi i s^2\alpha\}Z(p,q-s)^*Z(p,q)^2$, for arbitrary s. These relationships mean that the coefficient array $Z(p,q)$ can be computed recursively, in much the same way as the sine/cosine tables needed for the FFT can be generated. (Accuracy must be a consideration when using such a recursive table generator.)

III. 3D Image Rotation

The factoring trick that allows the 2D rotation to be done from Fourier space to image space with a single chirp transformation cannot be generalized to arbitrary higher dimensional rotations. Instead, two chirp transformations are necessary, one of which takes the original image to an intermediate rotated Fourier grid, and the second of which takes this transform to the final rotated image space grid.

As shown by Lawton [4], the chirp algorithm generalizes in n dimensions to

$$\hat{f}(\mathbf{Mq}) = Z(\mathbf{q})\sum_{\mathbf{p}}\{f(\mathbf{p})\,Z(\mathbf{p})\}\ Z(\mathbf{q}-\mathbf{p})^*\ , \tag{5}$$

where $\mathbf{M}$ is a real symmetric $n\times n$ matrix, $\mathbf{p}$ and $\mathbf{q}$ are integer n-vectors, and $Z(\mathbf{x}) = \exp\{-\pi i\,\langle\mathbf{x},\mathbf{Mx}\rangle\,/N\}$. (Our presentation is limited to N^n cubic lattices, unlike [4].) This expression is the discrete Fourier transform, since

$$\begin{aligned} Z(\mathbf{q})Z(\mathbf{q}-\mathbf{p})^* Z(\mathbf{p}) &= \exp\{-\pi i[\langle \mathbf{p}, \mathbf{Mq}\rangle + \langle \mathbf{q}, \mathbf{Mp}\rangle]/N\} \\ &= \exp\{-\pi i \left\langle \mathbf{p}, (\mathbf{M}+\mathbf{M}^T)\mathbf{q}\right\rangle /N\} . \end{aligned}$$

Since $\mathbf{M} = \mathbf{M}^T$, this is the DFT kernel for evaluation at $\mathbf{k} = \mathbf{Mq}$. The convolution can be evaluated efficiently using 3 n-dimensional FFTs (of $f(\mathbf{p})Z(\mathbf{p})$, of $Z(\mathbf{p})$, and of the product of their transforms). The coefficients $Z(\mathbf{p})$ can be generated recursively using the relation $Z(\mathbf{p}+\mathbf{s}) = Z(\mathbf{p}-\mathbf{s})^* Z(\mathbf{p})^2 \exp\{-2\pi i \langle \mathbf{s}, \mathbf{Ms}\rangle /N\}$, for any integer vector $\mathbf{s}$.

Lawton's chirp algorithm can be generalized to allow evaluation on some grids generated by unsymmetric matrices. We have

$$\hat{f}(\mathbf{A}^T\mathbf{Mq}) = Z(\mathbf{q}) \sum_{\mathbf{p}} \{f(\mathbf{p})\, Z(\mathbf{Ap})\}\ Z(\mathbf{q}-\mathbf{Ap})^* , \tag{6}$$

where $\mathbf{M}$ is still required to be symmetric and $\mathbf{A}$ is an arbitrary real matrix. For this generalized convolution to be done using the FFT, $\mathbf{A}$ must be an integer matrix. Not all unsymmetric real matrices $\mathbf{B} = \mathbf{A}^T\mathbf{M}$ can be so factored. The real line R is an infinite dimensional linear space over the field of rationals Q. $\mathbf{B}$ has n^2 elements, while $\mathbf{M}$ has only $n(n+1)/2$. If the elements of $\mathbf{B}$ are linearly independent over Q, they cannot be expressed as finite integer combinations of elements from any symmetric $\mathbf{M}$. Thus, (6) is more useful than (5) only in special circumstances. In the case of a 2D rotation, the output grid is generated by the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -\cos\theta & \sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

This factorization shows that the 2D rotation algorithm in section II is an example where (6) is useful, which is why the rotation can be accomplished with a single chirp transformation.

The general $n \times n$ orthogonal matrix $\mathbf{U}$ can be factored into two symmetric orthogonal matrices using the technique of Lemma 2 in [4]. In the case $n = 3$, the factor matrices can both be expressed in Householder form:

$$\mathbf{U} = \mathbf{VW} = \left[\mathbf{I} - 2\mathbf{v}\mathbf{v}^T\right] \left[\mathbf{I} - 2\mathbf{w}\mathbf{w}^T\right]$$

where $\|\mathbf{v}\| = \|\mathbf{w}\| = 1$. Let $\mathbf{U}$ represent rotation about the fixed axis $\mathbf{r}$ by angle θ. Then $\mathbf{v}$ and $\mathbf{w}$ are determined by requiring that $\langle \mathbf{v}, \mathbf{r}\rangle = \langle \mathbf{w}, \mathbf{r}\rangle = 0$ and $\langle \mathbf{v}, \mathbf{w}\rangle = \cos\frac{1}{2}\theta$. (We note that $\cos\frac{1}{2}\theta = \frac{1}{2}(1+u_{11}+u_{22}+u_{33})^{1/2}$; the rotation is in the direction from $\mathbf{w}$ towards $\mathbf{v}$; $\mathbf{r}$ is the eigenvector of $\mathbf{U}$ corresponding to eigenvalue 1.) The vectors $\mathbf{v}$ and $\mathbf{w}$ are not unique: replacing $\mathbf{v}$ by $\mathbf{U}^\alpha\mathbf{v}$ and $\mathbf{w}$ by $\mathbf{U}^\alpha\mathbf{w}$ does not change the result. Here, α is any real constant; $\mathbf{U}^\alpha$ represents rotation about $\mathbf{r}$ by angle $\alpha\theta$ (*i.e.*, $\mathbf{v}$ and $\mathbf{w}$ can be rotated together along the unit circle perpendicular to $\mathbf{r}$). For some orthogonal matrices $\mathbf{U}$, it is possible to choose $\mathbf{v}$ such that $\mathbf{v}\mathbf{v}^T$ is rational. In such cases, $\mathbf{U}$ can be factored into an integer matrix (a multiple of $\mathbf{V}$) and a symmetric matrix (a multiple of $\mathbf{W}$). For general orthogonal $\mathbf{U}$, this will not be possible, since the set of unit vectors $\mathbf{v}$ with $\mathbf{v}\mathbf{v}^T$ rational is only countable. The set of fixed axis vectors $\mathbf{r}$ is uncountable, so not every unit circle can contain such a $\mathbf{v}$. (When the 2D rotation is embedded in R^3 with $\mathbf{r} = \hat{\mathbf{z}}$, the appropriate unit circle is in the xy-plane, and the vector $\mathbf{v} = [-1\ 0\ 0]^T$ can be used.)

Thus, for general 3D rotations $\mathbf{U}$, it is necessary to perform 2 chirp transformations. The first transformation takes $f(\mathbf{p})$ to $\hat{f}(\mathbf{Wq})$; that is, it evaluates the Fourier transform of the 3D image on an intermediate rotated grid. (Any phase factors needed for translation should be multiplied into $\hat{f}$

after this first transformation.) The second chirp transformation takes $\hat{f}(\mathbf{Wp})$ to $f(\mathbf{VWq})$, back in image space on the final rotated grid.

A total of six 3D FFTs are needed to perform these chirp convolutions. The computational load in the first chirp transformation can be reduced somewhat by one further trick. It is always possible to choose $\mathbf{w}$ such that $w_3 = 0$. In this case, $\mathbf{W}$ is really a 2D rotation and the 2D algorithm of section II can be applied to each (p_1, p_2)-plane of f separately. A 1D FFT from p_3 to q_3 at the end will result in $\hat{f}(\mathbf{Wq})$.

IV. Discussion

Realigning image data to a common orientation is a necessary first step in longitudinal studies. This requires resampling the images to rotated and translated grids. In biomedical applications, high order interpolation is often needed in order to make subtle changes over time more visible. For this reason, sinc interpolation has been recommended for such applications [5]. The image space implementation of sinc interpolation can be very slow unless the interpolation kernel is severely truncated, which may result in ringing artifacts. Unser *et al.* [6], Fraser *et al.* [7], and Eddy *et al.* [8] each separately proposed a faster technique using 3 shearing transformations to perform a 2D rotation, and implementing the interpolation within each sheared row/column using (1). The methods put forward in the present Correspondence clarify the results of [4] and show how multi-dimensional Fourier transforms can be used to perform image rotations directly. These techniques can be combined with other Fourier space processing, if desired (*e.g.*, sharpening, differentiation, ...).

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