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Decoherence Bounds on Quantum Computation with Trapped Ions

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Using simple physical arguments we investigate the capabilities of a quantum computer based on cold trapped ions. From the limitations imposed on such a device by spontaneous decay, laser phase coherence, ion heating and other sources of error, we derive a bound between the number of laser interactions and the number of ions that may be used. The largest number which may be factored using a variety of species of ion is determined.

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In a quantum computer binary numbers can be represented by quantum states of two-level systems ("qubits"), bringing a new feature to computation: the ability to compute with coherent superpositions of numbers [1]. Because a single quantum operation can affect a superposition of many numbers in parallel, a quantum computer can efficiently solve certain classes of problems that are currently intractable on classical computers, such as the determination of the prime factors of large numbers [2]. These problems are of such importance that there is now considerable interest in the practical implementation of a quantum computer [3,4]. There are three principal challenges which must be met in the design of such a device: the qubits must be sufficiently isolated from the environment so that the coherence of the quantum states can be maintained throughout the computation; there must be a method of manipulating the states of the qubits in order to effect the logical "gate" operations; and there must be a method for reading out the answer with high efficiency.

Cirac and Zoller [5] have made the most promising proposal for the implementation of a quantum computer so far. A number of identical ions are stored and laser cooled in a linear radio-frequency quadrupole trap to form a quantum register. The radio-frequency trap potential gives strong confinement of the ions in the Y and Z directions transverse to the trap axis, while an electrostatic potential forces the ions to oscillate in an effective harmonic potential in the axial direction (X). After laser cooling the ions become localized along the trap axis (the Lamb-Dicke regime) with a spacing determined by their Coulomb repulsion and the confining axial potential. The normal mode of the ions' collective oscillations which has the lowest frequency is the axial center of mass (CM) mode, in which all the trapped ions oscillate together. A qubit is the electronic ground state $|g\rangle(|0\rangle)$ and a long-lived excited state $|e\rangle(|1\rangle)$ of the trapped ions. The electronic configuration of individual ions, and the quantum state of their collective CM vibrations can be manipulated by coherent interactions of the ion with a laser beam, in a standing wave configuration, which can be pointed at any of the ions. The CM mode of axial vibrations may then be used as a "bus" to implement the quantum logical gates. Once the quantum computation has been completed, the readout is performed through the mechanism of quantum jumps. Several features of this scheme have been demonstrated experimentally, mostly using a *single* trapped ion [4,6].

The unavoidable interaction of a quantum computer with its environment places considerable limitations on the capabilities of such devices [7]. In this letter we make a quantitative assessment of these limitations for a computer based on the Cirac-Zoller cold-trapped-ion design, in order to determine the best physical implementation and the optimization parameters for quantum algorithms [8]. There are two fundamentally different types of decoherence during a computation: the intrinsic limitation imposed by spontaneous decay of the metastable states $|e\rangle$ of the ions; and practical limitations such as the random phase fluctuations of the laser driving the computational transitions or the heating of the ions vibrational motion. One could, in principle, expect that as experimental techniques are refined, the effects of these practical limitations may be reduced until the intrinsic limit of computational capability due to spontaneous emission is attained.

The influence of spontaneous emission on a quantum computation with trapped ions depends on: the natural lifetime of the $|e\rangle$ qubit; the number of ions, L, being used; and the quantum states of those ions. The number of ions which are not in their ground states varies as the calculation progresses, with ancillary ions being introduced and removed from the computation. The progression of the ions' states can be characterized well by an effective number of ions, L_e, which have a non-zero population in the excited state $|e\rangle$. In the case of Shor's factoring algorithm [2] (using long multiplication), a reasonable estimate is $L_e \approx 2L/3$. Therefore, to estimate the effect of decoherence during the implementation of Shor's algorithm, we will consider the following process: a series of laser pulses of appropriate strength and duration ($\pi/2$ pulses) is applied to $2L/3$ ions, causing each of them to be excited into an equal superposition state $(|g\rangle + |e\rangle)/2$. After an interval T, a second series of laser pulses ($-\pi/2$ pulses) is applied, which, had there been no spontaneous emission, would cause each ion to be returned to its ground state. This is the "correct" result of our pseudo-computation. If there were spontaneous emission from one or more of the ions, then the ions would finish in some other, "incorrect" state. This process involves the sort of superposition states that will occur during a typical quantum computation, and so the analysis of decoherence effects in this procedure will give some insight into how such effects influence a real computation. A simple calculation shows that the probability of obtaining a correct result is

$$P(T) \approx 1 - LT/6\tau_o \tag{1}$$

where is the natural lifetime of the excited state $|e\rangle$. Thus the effective coherence time of the computer is $6\tau_o/L$.

The total time taken to complete a calculation will be approximately equal to the number of laser pulses required multiplied by the duration of each pulse. The time taken to switch the laser beam from ion to ion is assumed to be negligible. There are two types of laser pulse that are required in order to realize Cirac and Zoller's scheme. The first requires pulses that are tuned precisely to the resonance frequency of the $|e\rangle$ to $|g\rangle$ transition, configured so that the ion lies at the node of the laser standing wave ("V-pulses"); the second requires pulses tuned to the CM phonon sideband of the transition, arranged so that the ion lies at the antinode of the standing wave ("U-pulses") [9]. The interaction of U-pulses with the ions is considerably weaker than the V-pulses, and so, assuming constant laser intensity, the U-pulse duration must be longer. Hence, in calculating the total time required to perform a quantum computation, we will neglect the time required for the V-pulses. Because the entire calculation must be performed in a time less than the coherence time of the computer, we obtain the following inequality:

$$N_U t_U < 6\tau_o/L\,, \tag{2}$$

where is the total number of U-pulses, each of which has duration t_u. The Hamiltonian for the interaction of these pulses with the ions is given by the following expression (ref. [5], eq. (1)):

$$\hat{H} = \frac{\hbar\eta}{2\sqrt{L}}\Omega[|e\rangle\langle g|\hat{a}e^{-i\phi} + |g\rangle\langle e|\hat{a}^\dagger e^{i\phi}] \tag{3}$$

In this formula, is the Rabi frequency for the laser-ion interaction, L is the number of ions in the trap, $\hat{a}$ ($\hat{a}^\dagger$) is the annihilation (creation) operator for phonons of the CM mode and $\eta = \sqrt{\hbar\omega^2\cos^2\theta/2Mc^2\nu_x}$ is the Lamb-Dicke parameter (here ω is the laser angular frequency, θ the angle between the laser and the trap axis, ν_x is the angular frequency of the ions' axial CM mode and M the mass of each ion). A careful calculation, based on a perturbative analysis of the excitation of phonon modes other than the CM mode, shows that this Hamiltonian is valid if $(\Omega\eta/2\nu_x\sqrt{L})^2 \ll 1$. The longest duration laser pulse that will be required to implement a quantum computing algorithm using a Cirac-Zoller quantum computer is a U-pulse of duration $t_U = 2\pi\sqrt{L}/\Omega\eta$. We will assume that all of the U-pulses required for the calculation are of this duration. Therefore the lower bound on the duration of laser pulses is $t_U = y\pi/\nu_x$, where y is a dimensionless "safety factor". This result can also be obtained from the naive uncertainty principle argument that there must not be appreciable power at the frequencies of the adjacent lattice vibrations.

In order to attain the highest possible computational capability, one will need to minimize the duration of each laser pulse. Hence, it will be advantageous to employ an ion trap with the largest possible value of the trap frequency ν_x. However, the axial frequency cannot be made arbitrarily large because, in order to avoid crosstalk between adjacent ions, the minimum inter-ion spacing must be much larger than the size of the focal spot of the laser beam. The minimum separation distance between two ions occurs at the center of the string of ions, which can be calculated by solving for the equilibrium positions of the ions numerically, resulting in the following expression :

$$x_{\min} \cong \left(\frac{Z^2e^2}{4\pi\epsilon_o\nu_x^2 M}\right)^{/13} \frac{2.0}{L^{0.56}}\,, \tag{4}$$

where Z is the degree of ionization of the ions, e is the electron charge and ϵ_o is the permittivity of a vacuum. The spatial distribution of light in focal regions is well known [10]. The approximate diameter of the focal spot is $x_{\text{spot}} \approx \lambda F$, where λ is the laser wavelength and F the f-number of the focusing system (i.e. the ratio of the focal length to the diameter of the exit pupil). Hence the requirement that the ion separation must be large enough to avoid cross-talk between ions, i.e. that $x_{\min} \gg x_{\text{spot}}$, leads to the following expression for the duration of the U-pulses:

$$t_U \equiv \frac{\pi y}{\nu_x} = 2.9[s\,m^{-3/2}]\sqrt{\frac{Ay^5\lambda^3F^3L^{1.68}}{Z^2}}\,. \tag{5}$$

where A is the atomic mass number of the ions. From eqs. 2 and 5 we obtain the following constraint on the number of ions L and the total number of U-pulses :

$$N_u L^{1.84} < 2.0[s^{-1}\,m^{3/2}]\frac{Z\tau_o}{y^{5/2}A^{1/2}F^{3/2}\lambda^{3/2}}\,. \tag{6}$$

We will now apply this bound to Shor's factor finding algorithm [2]. Let be the number of bits of the integer we wish to factor. A careful analysis of the implementation of the algorithm (using long multiplication) reveals that the required number of ions and U-pulses are given by:

$$L = 5l + 2\,, \tag{7}$$

$$N_U = 544l^3 + 78l^2 + 10l\,. \tag{8}$$

Note that there are asymptotically much more efficient implementations, but they do not become competitive for the small number of binary digits under consideration here. If the measured Fourier transform [11] and interleaving measurements were to be used in the computation, the number of ions required can be reduced to $3l+4$. However the intermediate measurements may increase the decoherence of the other ions due to scattered photons or unintended heating of the ions. It is for this reason that we have avoided use of this technique in the assumptions underlying the algorithm.

Equations 7 and 8 define a curve in (L, N_U) space, which taken in conjunction with the inequality (6) allow us to determine the largest number of ions that can be used to implement Shor's algorithm in an ion trap computer with bounded loss of coherence. The linear relationship between L and l, eq.(7), can then be used to determine the largest number that can be factored.

As specific examples, we will consider the intrinsic computational capacity of Cirac-Zoller quantum computers based on the following three ions:
(i) Hg II: Z=1, A = 198; $|e\rangle$ is a sublevel of the $5d^9 6s^2\ ^2D_{5/2}$ state, $|g\rangle$ is the $5d^{10}6s^2\ ^2S_{1/2}$, the two states being connected by an electric quadrupole transition: λ = 281.5 nm; $\tau_o \approx 0.1s$.
(ii) Ca II: Z=1, A = 40; $|e\rangle$ is a sublevel of the 3d $^2D_{5/2}$ state, $|g\rangle$ is the 4s $^2S_{1/2}$, the two states being connected by an electric quadrupole transition: λ = 729 nm; $\tau \approx 1.14s$.
(iii) Ba II: Z=1, A = 137; $|e\rangle$ is a sublevel of the 5d $^2D_{5/2}$ state, $|g\rangle$ is the 6s $^2S_{1/2}$, the two states being connected by an electric quadrupole transition: λ = 1.76 μm; $\tau_o \approx 47s$.
We shall assume that we have a very high numerical aperture focusing system, so that (although in practice such a high focal ratio would be difficult to achieve), and we will err on the side of optimism by putting the safety factor $y = 1$. In figure 1 we have plotted the curves which limit the allowed values of L and , as given by eq.(6). We have also plotted, with a solid line, the "curve of factorization" defined by eqs. (7) and (8). The interception of the limiting curves for the different ions with the curve of factorization gives us the largest allowed value for the number of ions. Examining these curves, we find that the size of the largest integer that can be factored by a Cirac-Zoller quantum computer based on Hg II, Ca II or Ba II ions is 6 bits, 9 bits and 13 bits respectively. If these calculations were repeated with the less optimistic value for the safety factor, $y = 3$, one obtains 3 bits, 5 bits and 7 bits for the three species of ions, respectively. We note that the spontaneous emission lifetime is proportional to an odd power of λ (λ^5 in the case of electric quadrupole transitions) and so, for greater capability, eq.(6) suggests either going to longer wavelength (as seen with the three ion species above) or more highly forbidden transitions.

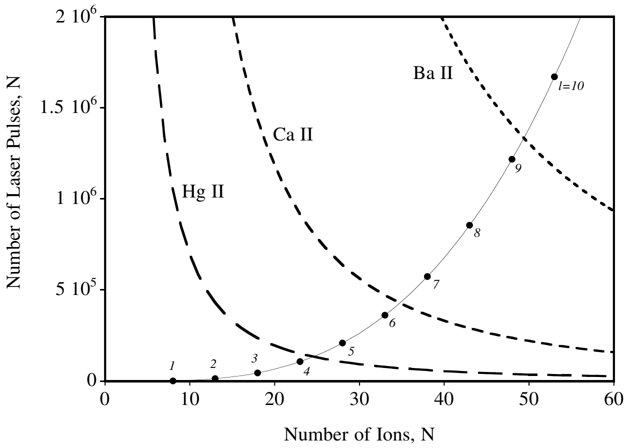

FIG. 1. The bounds on the numbers of ions, L, and the number of U-pulses, N_U , that may be used in a quantum computation without loss of coherence. The allowed values of N_U and L lie to the left of the curves. Curves for three ions are plotted. The unbroken line is the "factorization curve", specified by eqs.(7) and (8), which represents those values of L and which are required for execution of Shor's algorithm; the heavy black dots on this line represent the values of L and required to factor a number of l bits (l = 1, 2, ...15).

As a more dramatic illustration of the theoretical possibilities of the Cirac-Zoller scheme, one may consider a computer based on the $4f^{14}6s\ ^2S_{1/2} \leftrightarrow 4f^{13}6s^2\ ^2F_{1/2}$ electric octupole transition of Yb II. This very long lived transition, which has received considerable attention because of its potential applications as an optical frequency standard, has a wavelength of 467 nm and a calculated lifetime of 1533 days [12]. Performing a similar calculation to that given above suggests that, using this ion, it might be possible to factor a 438-bit number. Because such a calculation would require around 2200 trapped ions and 4.5×10^{10} U-pulses, taking about 100 hours, it would be difficult to over-emphasize the problems attendant on such an experiment.

One may calculate the limits on factoring due to other causes of decoherence by a similar procedure to that used above. In this case, we will assume that the loss of quantum coherence due to sundry effects such as random fluctuations of the laser phase or the heating of the ions' vibrational motion can be characterized by a single coherence time τ_e. The effects of other causes of error, such as imprecise measurement of the areas of π-pulses, which do not result in decoherence but nevertheless lead to incorrect results in a computation, can also be characterized by the time τ_e. Thus, in place of eq.(2) we now have the inequality . Using eq.(5) we obtain the following constraint on the values of the number of ions L and the number of laser pulses which can be used in a factoring experiment without significant loss of quantum coherence:

$$N_u L^{0.84} < 0.34[s^{-1}\, m^{3/2}] \frac{Z\tau_e}{y^{5/2} A^{1/2} F^{3/2} \lambda^{3/2}}\,, \qquad (9)$$

Using the "factorization curve" specified by eqs. (7) and (8), one can obtain as before a value for the number of bits l in the largest number which may be factored. In this case the value of l will depend on the value of the coherence time τ_e. In figure 2 we have plotted the values of l as a function of the experimental coherence time for the three species of ions discussed above. As τ_e increases, the largest number that can be factored also increases, until the limit due to spontaneous emission discussed above is attained. The slowest heating rate for a single trapped ion so far reported is 6 phonons per second (i.e. $\tau_e = 0.17s$) [13], and the laser phase coherence times longer than $10^{-3}s$ have been achieved by several groups [14]. Comparing these numbers with fig.2, we see that, in principle, current technology is capable of producing a quantum computer that could factor at least small numbers (several bits). Note that, in contrast to the spontaneous emission bounds from eq. (6) (where τ_o is, for quadrupole transitions, proportional to λ^5), eq.(9) argues for using shorter wavelength transitions. So we see from figs. 1 and 2 that Ca II is a good choice of ion for the experimental study of this technology because it allows a large number of operations to be performed with realistic laser stability and ion heating requirements.

The various causes of experimental decoherence which are mentioned above are all the subject of ongoing research. It is not clear, for example, how laser phase fluctuations will affect quantum computations; it may be the case that the laser need be coherent only over the period required to execute each quantum gate operation. Furthermore, the heating rate of the ions' vibrational motion as a function of the number of trapped ions is not known. Other methods of coherent population transfer, which may be less susceptible to the effects of phase fluctuations, for example stimulated Raman adiabatic passage (STIRAP) [15], are being investigated.

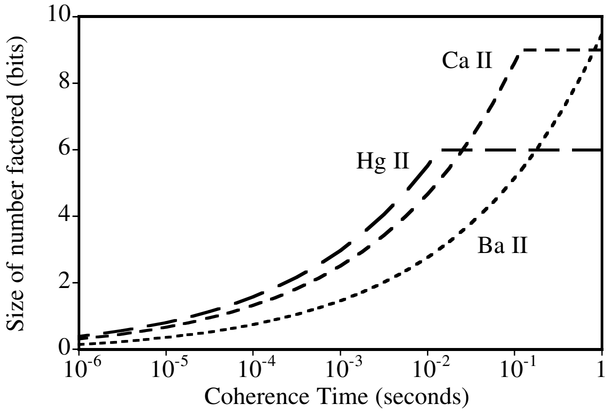

FIG. 2. The variation of the number of bits l in the largest integer that may be factored with the experimental coherence time for the three ions discussed in the text. The maximum values of the computational capacities for the ions Hg II and Ca II are the limits determined by spontaneous emission.

It is clear that if quantum computation is to overcome decoherence and other errors, then error correction must be used extensively. So far, suggestions for error correction have relied either on variations of the "watch dog" effect [16,17], or on exploiting the properties of certain entangled states to reduce the impact of decoherence in a quantum memory [18]. The latter has not yet proven to be practical for use during a quantum computation, primarily because there has not been any analysis of the success of the method under realistic assumptions on operator errors. If operational errors were negligible, the effect of decoherence on quantum memories could be reduced arbitrarily. However some of the "watch dog" methods that have been suggested are quite practical. For example, many computations require the use of ancillary qubits which are periodically returned to the ground state. Measuring these ancillas when they are supposed to be in the ground state can be used to help dissipate errors. Recent simulations [17] indicate that this method is indeed helpful in maintaining the state of the computation. Implementation of the method does require intermediate measurements. In any case the effect of using a "watch dog" method is to stabilize the effective decoherence time τ_e by ensuring less dependence between the errors of successive operations.

In conclusion, we have derived quantitative bounds which show how the computational capabilities of a trapped ion quantum computer depend on the relevant physical parameters and determine the computational "space" (L) and "time" (N_U) combination that should be optimized for the most effective algorithms. The effect of this bound has been illustrated by calculating the size of the largest number that may be factored using a computer based on various species of ion. Our results show there is reason for cautious optimism about the possibility of factoring at least small numbers using a first generation quantum computer design based on cold trapped ions. However, the large number of precise laser operations required and the number of ions involved indicates that even this computationally modest goal will be extremely challenging experimentally.

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