An Improved Lower Bound for Crossing Numbers

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Abstract. Leighton [5] proved that in any n-vertex graph G of bounded degree, the crossing number satisfies $\mathrm{cr}(G) + n = \Omega(\mathrm{bw}^2(G))$, where $\mathrm{bw}(G)$ is the bisection width of G. The lower bound was extended for graphs of arbitrary vertex degrees to $\mathrm{cr}(G) + \frac{1}{16}\sum_{v\in G} d_v^2 = \Omega(\mathrm{bw}^2(G))$ in [6, 7], where d_v is the degree of any vertex v. We improve this bound by showing that the bisection width can be replaced by the cutwidth of the graph. Our result also yields an upper bound for the path-width of G in term of its crossing number.

1 Introduction

The crossing number of a graph $G = (V, E)$, denoted by $\mathrm{cr}(G)$, is the smallest number of edge crossings in any drawing of G in the plane. The bisection width of a graph G is the minimum number of edges whose removal divides G into two parts having at most $2|V|/3$ vertices each. Leighton [5] proved that in any n-vertex graph G of bounded degree, the crossing number satisfies $\mathrm{cr}(G) + n = \Omega(\mathrm{bw}^2(G))$. The lower bound was extended to

$$\mathrm{cr}(G) + \frac{1}{16}\sum_{v\in V} d_v^2 \geq \frac{1}{40}\mathrm{bw}^2(G)$$

in [6, 7], where d_v is the degree of any vertex v. We improve this bound by showing that the bisection width can be replaced by the cutwidth of the graph, defined as follows. Consider an injection of the vertices of G into points of a horizontal line. Draw the edges above the line using semicircles. Find a vertical line between a pair of consecutive points, which cuts the maximal number of edges. Minimize the maximum over all injections. The minmax value is called the cutwidth of G, denoted by $\mathrm{cw}(G)$. We prove

$$\mathrm{cr}(G) + \frac{1}{16}\sum_{v\in G} d_v^2 \geq \frac{1}{1176}\mathrm{cw}^2(G)$$

Note that $\mathrm{cw}(G) \geq \mathrm{bw}(G)$ and there are connected graphs with $\mathrm{bw}(G) = 1$ but with arbitrarily large cutwidth.

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The new crossing number lower bound is tight up to a constant factor. E.g., the cutwidth of the n-cube H_n is $\lceil 2^{n+1}/3 \rceil$, which implies

$$\mathrm{cr}(H_n) + \frac{1}{16} \sum_{v \in H_n} d_v^2 = O(4^n + n^2) = O(\mathrm{cw}^2(H_n)).$$

Moreover, the additive term $\sum_{v \in G} d_v^2$ can not be removed, since the crossing number of any planar graph is 0 and there exists a planar graph P (e.g. the star graph) such that $\mathrm{cw}^2(P) = \Omega(\sum_{v \in P} d_v^2)$. The constant factor in the lower bound is big, improvement on it remains an open problem.

The path-decomposition of a graph G is a sequence $D = X_1, X_2, ..., X_r$ of vertex subsets of G, such that every edge of G has both ends in some set X_i and if a vertex of G occurs in some sets X_i and X_j with $i < j$, then the same vertex occurs in all sets X_k with $i < k < j$. The width of D is the maximum number in any X_i minus 1. The path-width of G, $\mathrm{pw}(G)$, is the minimum width over all path-decompositions of G.

A graph $G = (V, E)$ is k-crossing critical if $\mathrm{cr}(G) = k$ and $\mathrm{cr}(G - e) < \mathrm{cr}(G)$, for all edges $e \in E$. Hliněný [3] proved that $\mathrm{pw(G)} \le 2^{f(k)}$, where $f(k) = O(k^3 \log k)$. This answers an open question of Geelen et al. [2] whether crossing critical graphs with bounded crossing numbers have bounded path-widths. As $\mathrm{pw(G)} \le n-1$ the above upper bound is useful for $k \le \sqrt[3]{\log n}$ only, while the maximal order of crossing numbers is n^4. Our result implies that if $\mathrm{cr}(G) = k$, then $\mathrm{pw(G)} = O\left(\sqrt{k + \sum_{v \in V} d_v^2}\right)$, without the crossing-criticality assumption.

2 A New Lower Bound

We will make use of the following theorem [1].

Theorem 2.1 *Let $G = (V, E)$ be a planar graph with non-negative weights on its vertices that sum up to one and every weight is at most 2/3. Let d_v is the degree of any vertex v. Then there exists at most $(\sqrt{3} + \sqrt{2})\sqrt{\sum_{v \in V} d_v^2/2}$ edges whose removal divides G into disjoint subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that the weight of each is at most 2/3.*

Theorem 2.1 implies an upper bound for the cutwidth of planar graphs which deserves an independent interest.

Theorem 2.2 *For any planar graph $G = (V, E)$*

$$\mathrm{cw}(G) \le \frac{6\sqrt{2} + 5\sqrt{3}}{2} \sqrt{\sum_{v \in V} d_v^2},$$

where d_v is the degree of any vertex v.

Proof.

Apply Theorem 2.1 to G. Assign weights to vertices: $\mathrm{weight}(u) = d_u^2 / \sum_{v \in V} d_v^2$.

1. Assume $\mathrm{weight}(u) \le 2/3$ for all u. By deleting $(\sqrt{3} + \sqrt{2})\sqrt{\sum_{v \in V} d_v^2/2}$ edges we get graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ such that for $i = 1, 2$ $\mathrm{weight}(V_i) \le 2/3$, which implies

$$\sum_{v \in V_i} d_v^2 \le \frac{2}{3} \sum_{v \in V} d_v^2.$$

2. Assume there exists a vertex u such that $\mathrm{weight}(u) > 2/3$. By deleting edges adjacent to u we get disjoint subgraphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, where G_2 is a one vertex graph. We have $\mathrm{weight}(V_1) = 1 - \mathrm{weight}(u) < 2/3$ and

$$\sum_{v \in V_1} d_v^2 < \frac{2}{3} \sum_{v \in V} d_v^2.$$

The number of edges between G_1 and G_2 is

$$d_u \leq \frac{\sqrt{3}+\sqrt{2}}{2} \sqrt{\sum_{v \in V} d_v^2}.$$

Placing the graphs G_1 and G_2 consecutively on the line and adding the deleted edges we obtain the estimation

$$\mathrm{cw}(G) \leq \max\{\mathrm{cw}(G_1), \mathrm{cw}(G_2)\} + \frac{(\sqrt{3}+\sqrt{2})}{2} \sqrt{\sum_{v \in V} d_v^2}.$$

Solving the recurrence we find

$$\mathrm{cw}(G) \leq \frac{\sqrt{3}+\sqrt{2}}{2} \sum_{i=0}^{\infty} \left(\frac{2}{3}\right)^{i/2} \sqrt{\sum_{v \in V} d_v^2} = \frac{6\sqrt{2}+5\sqrt{3}}{2} \sqrt{\sum_{v \in V} d_v^2}. \qquad \square$$

Our main result is

Theorem 2.3 *Let $G = (V, E)$ be a graph. Let d_v denote the degree of any vertex v. Then the crossing number of G satisfies*

$$\mathrm{cr}(G) + \frac{1}{16} \sum_{v \in V} d_v^2 \geq \frac{1}{1176} \mathrm{cw}^2(G).$$

Proof. Consider an embedding of G with $\mathrm{cr}(G)$ crossings. Introducing a new vertex at each crossing results in a plane graph H with $\mathrm{cr}(G)+n$ vertices. By Theorem 2.2 we have

$$\mathrm{cw}(H) \leq \frac{6\sqrt{2}+5\sqrt{3}}{2} \sqrt{\sum_{v \in H} d_v^2} = \frac{6\sqrt{2}+5\sqrt{3}}{2} \sqrt{\sum_{v \in G} d_v^2 + 16\mathrm{cr}(G)}.$$

Finally, note that $\mathrm{cw}(G) \leq \mathrm{cw}(H)$, which proves the claim. $\square$

This result immediately gives an upper bound for the path-width of G in term of its crossing number as the result of Kinnersley [4] implies that $\mathrm{pw}(G) \leq \mathrm{cw}(G)$.

Corollary 2.1 *Let $G = (V, E)$ be a graph. Then*

$$\mathrm{pw}(G) < 9\sqrt{16\mathrm{cr}(G) + \sum_{v \in V} d_v^2}.$$

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